Abt. Theoretische Physik
J. Bolte and S. Keppeler

Theory of Quantum Chaos

13. **Level spacings – Poisson**
Let $\{x_n\}$, $x_n \in \mathbb{R}$, $x_n < x_{n+1}$ be a set of uncorrelated equally distributed random numbers with mean spacing one, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (x_{n+1} - x_n) = 1\,.$$

Calculate the level spacing density $P(s)$ which is defined by

$$\int_a^b P(s)\mathrm{d}s = \lim_{N\to\infty} \frac{\#\{x_n \,|\, a < x_{n+1} - x_n < b\,,\ n \leq N\}}{N}\,.$$

Thus, $P(s)\,\mathrm{d}s$ is the probability of finding a spacing, $x_{n+1} - x_n$, between s and $s + \mathrm{d}s$.
`Hint`: Split the interval $(0, s)$ in N subintervals of length s/N. The probability p_N of finding no x_n in a single, small subinterval is determined by the condition of having a mean density of one. The probability of finding no x_n in any subinterval is given by the product of the single probabilities since the $\{x_n\}$ are uncorrelated. The level spacing density $P(s)$ is obtained in the limit $N \to \infty$. Calculate also the level spacing distribution

$$I(s) := \int_0^s P(s')\,\mathrm{d}s'\,.$$

14. **Wigner's surmise for the GOE**
Let $A \in \mathbb{R}^{2,2}$ be a matrix with uncorrelated random entries,

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

The probability density for each single entry be given by a Gaussian

$$P_{\text{Gauß}}(x) = \tilde{N} \exp\left(-\frac{x^2}{2\sigma^2}\right)\,, \qquad x = a, b, c, d\,,$$

with the same variance σ. Write down the joint probability density for a, b, c and d.

   (a) Now we choose symmetric matrices, i.e. $c = b$. Show that the joint probability density for a, b and d is given by[1]

$$P_A(a, b, d) = N \exp\left(-\frac{1}{2\sigma^2}\mathrm{Tr}A^2\right)\,,$$

   and determine the normalisation constant N from

$$\int_{\mathbb{R}^3} P_A(a, b, d)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}d \stackrel{!}{=} 1\,.$$

[1] This is a trick! A real justification for this formula will be given in the lectures.

(b) Determine the probability density $P_E(E_1, E_2)$ for the distribution of the eigenvalues $\lambda_{1,2}$ of the matrix A from

$$P_E(E_1, E_2) = \int_{\mathbb{R}^3} \delta(\lambda_1 - E_1)\, \delta(\lambda_2 - E_2)\, P_A(a, b, d)\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}d\,.$$

`Hint`: There is an orthogonal matrix

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, \pi] \quad \text{with} \quad U^\top A U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Change variables from a, b, d to $\lambda_1, \lambda_2, \theta$. The result is

$$P_E(E_1, E_2) = C\, |E_1 - E_2| \exp\left(-\frac{E_1^2 + E_2^2}{2\sigma^2}\right),$$

with some constant C.

(c) Calculate the level spacing density

$$P(s) := \int_{\mathbb{R}^2} \delta(|E_1 - E_2| - s)\, P_E(E_1, E_2)\, \mathrm{d}E_1 \mathrm{d}E_2\,.$$

Check that $P(s)$ is normalised and fix σ such that the mean spacing is one, i.e.

$$\int_0^\infty P(s)\, \mathrm{d}s = 1 \overset{!}{=} \int_0^\infty s\, P(s)\, \mathrm{d}s\,.$$

Determine also the level spacing distribution $I(s)$.

15. **Level spacings for the quantum kicked rotator**
The propagator U of the quantum kicked rotator is given by the matrix elements (cf. problem 12)

$$u_{jl} = \frac{1}{\sqrt{\mathrm{i}N}} \exp\left[\mathrm{i}\frac{N}{2\pi}\left(\frac{1}{2}\left(2\pi\frac{j-l}{N}\right)^2 - k\cos\left(\frac{2\pi j}{N}\right)\right)\right].$$

(a) Show that its eigenvalues λ_n are given by $\mathrm{e}^{\mathrm{i}\phi_n}$ with $\phi_n \in [0, 2\pi)$. `Hint`: U is unitary.

(b) We want to analyse the level spacing distribution of the spectrum $\{\phi_n\}$. To this end rescale ("unfold") the ϕ_n such that they have mean density one, i.e. define some suitable $x_n = g(\phi_n)$.

(c) On `www.physik.uni-ulm.de/theo/qc/kep/chaos.html` we provide a C++ program which calculates the spectrum of the quantum kicked rotator and its level spacing distribution $I(s)$ and density $P(s)$. Make yourself familiar with the program. Compare the data to the distributions calculated in the previous two exercises for different values of the kicking strength k, e.g. $k = 0.1, 0.5, 1.0, 7.0, 100000.0$. How do you interpret the result?