SS 2001
Problem sheet 1

Abt. Theoretische Physik
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Theory of Quantum Chaos

1. **Action and Period**

   For one-dimensional, autonomous Hamiltonian systems, defined by a Hamiltonian $H(p,q)$, the trajectories in phase space are given by the equienergy contours (why?). Given a periodic solution, we call

   $$S(E) = \oint p \, \mathrm{d}q$$

   the action of the orbit.

   (a) Prove: The period T of the orbit is obtained by differentiation with respect to the energy E, i.e.

   $$T(E) = \frac{\mathrm{d}S}{\mathrm{d}E}(E) \, .$$

   Hint: Recall how to differentiate implicitly defined functions.

   (b) Verify the relation for the harmonic oscillator with period $T = \frac{2\pi}{\omega}$, given by

   $$H(p,q) = \frac{p^2}{2m} + \frac{m}{2}\,\omega^2 q^2 \, .$$

2. **Mathematical Pendulum**

   Consider a planar pendulum, given by a massless bar of length l and a point mass m. We choose an angle ϕ such that the Lagrangian is given by (make a thumbsketch!)

   $$L(\phi,\dot\phi) = \frac{m}{2} l^2 \dot\phi^2 + mgl(\cos\phi - 1) \, .$$

   The gravitational acceleration g is assumed to be constant.

   (a) Calculate the canonical momentum conjugate to ϕ, the Hamiltonian and the equations of motion.

   (b) Sketch trajectories in phase space for different initial conditions. Distinguish librations and rotations and determine the corresponding energy ranges and mark the rest positions.

   (c) Calculate action and period of the *rotation*. Express the result in terms of the following elliptic integrals:

   $$E(x) = \int_0^{\frac{\pi}{2}} \sqrt{1 - x^2 \sin^2\xi} \, \mathrm{d}\xi \, , \quad K(x) = \int_0^{\frac{\pi}{2}} \left(1 - x^2 \sin^2\xi\right)^{-1/2} \, \mathrm{d}\xi \, .$$

3. **Kicked Rotator / Standard Map**

The periodically kicked rotator is defined by the Hamiltonian

$$H(p,q) = \frac{p^2}{2m} + k\cos(q)\sum_{n\in\mathbb{Z}}\delta(t-n)\,, \quad q\in[0,2\pi)\,.$$

(a) Compare to the Hamiltonian of the mathematical pendulum and describe the physical situation.

(b) Determine the equations of motion and sketch p and q as functions of the time t (qualitatively). Hint: Consider

$$\lim_{\epsilon\to 0+}\int_{n-\epsilon}^{n+\epsilon}\dot{p}\,\mathrm{d}t\,.$$

(c) Obviously, the dynamics between the kicks is not very interesting, but rather the change of (p,q) at times $t=n$. Therefore, one often studies the map

$$F:\begin{pmatrix}p_n\\q_n\end{pmatrix}\mapsto\begin{pmatrix}p_{n+1}\\q_{n+1}\end{pmatrix}\quad\text{with}\quad\begin{pmatrix}p_n\\q_n\end{pmatrix}:=\lim_{\epsilon\to 0+}\begin{pmatrix}p(n+\epsilon)\\q(n+\epsilon)\end{pmatrix}\,,$$

i.e. we consider the system immediately after the kicks. This dynamical system is called the standard map; determine F.

(d) By calculating

$$F\begin{pmatrix}p+2\pi\nu\\q\end{pmatrix}\,,\quad \nu\in\mathbb{Z}\,,$$

we convince ourselves that the structures in phase space are also periodic in p. Thus, in the following we only consider $p \mod 2\pi$. In addition let $m=1$.
Determine all fixed points of the map F and all periodic points of period 2 with $p=\pi$, i.e. find the solutions of

$$F\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}p\\q\end{pmatrix}\quad\text{and}\quad F^2\begin{pmatrix}\pi\\q\end{pmatrix}=\begin{pmatrix}\pi\\q\end{pmatrix}\,,$$

respectively.

(e) With growing k the dynamics of the standard map undergoes a transition from regular to chaotic behaviour. Illustrate this statement with the help of a computer programme, iterating different initial conditions for different values of the parameter k (e.g. $k=0.0, 1.0, 2.5, 5.0, 7.0$).