SCARS ON GRAPHS

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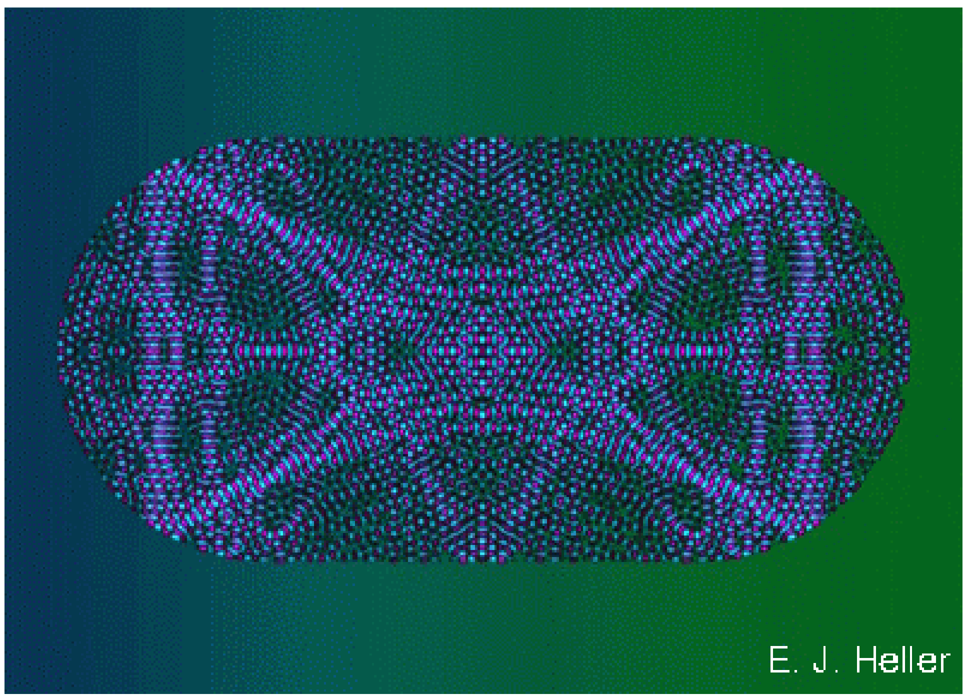

Nichtlineare Dynamik
Universität Göttingen

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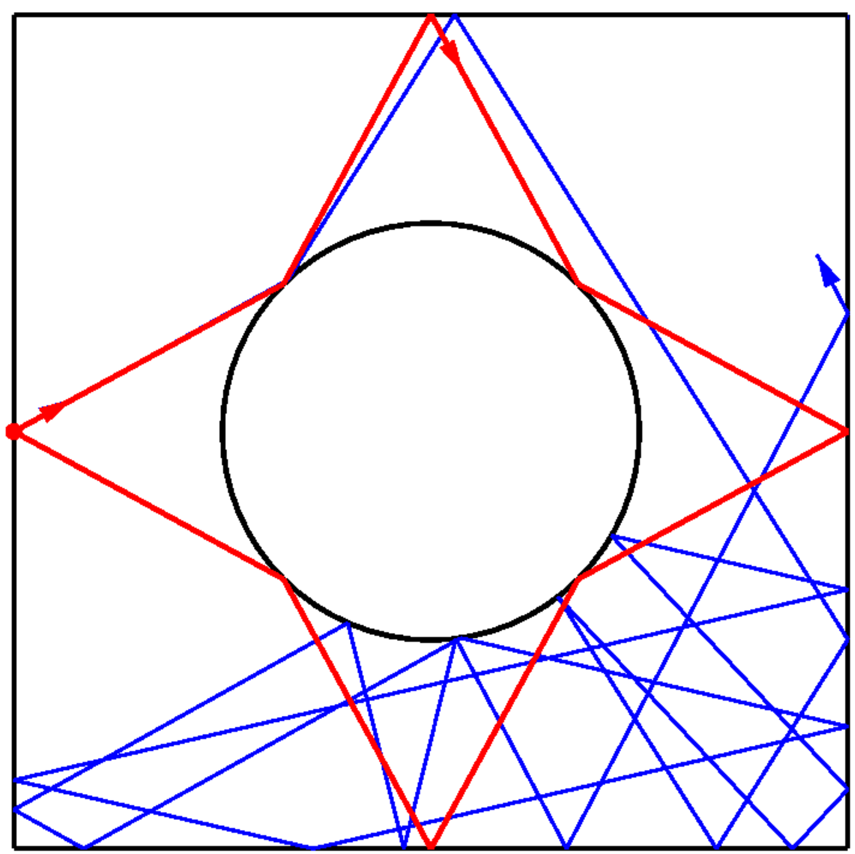
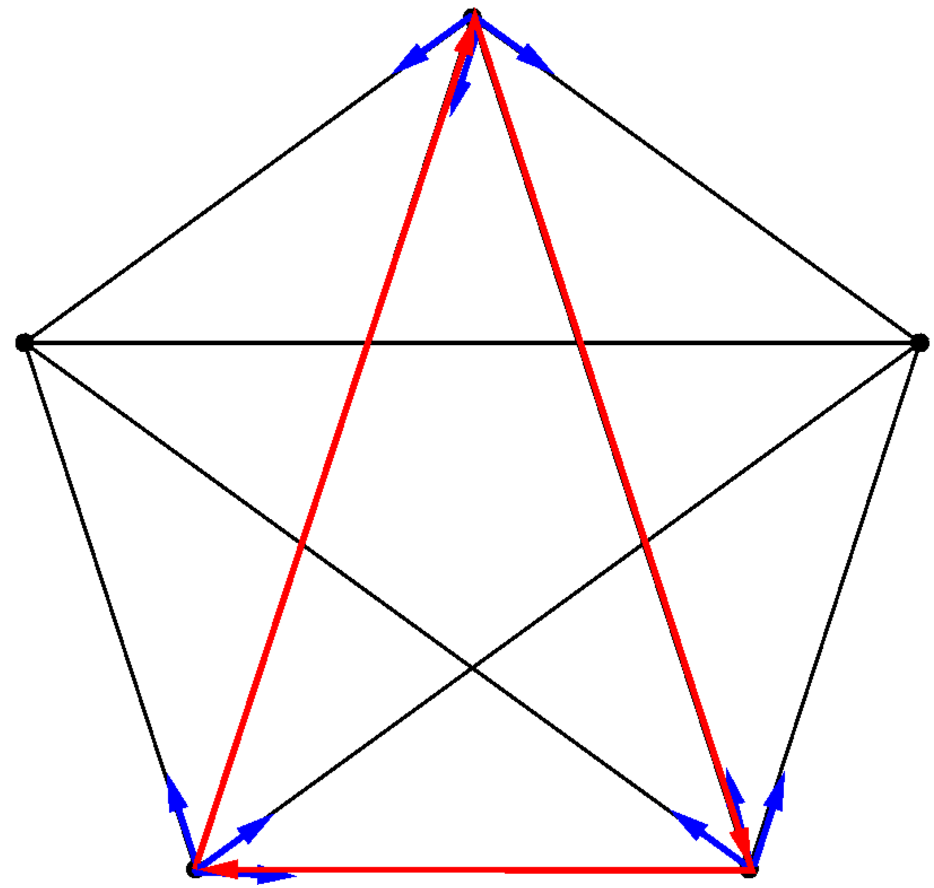
Göttingen

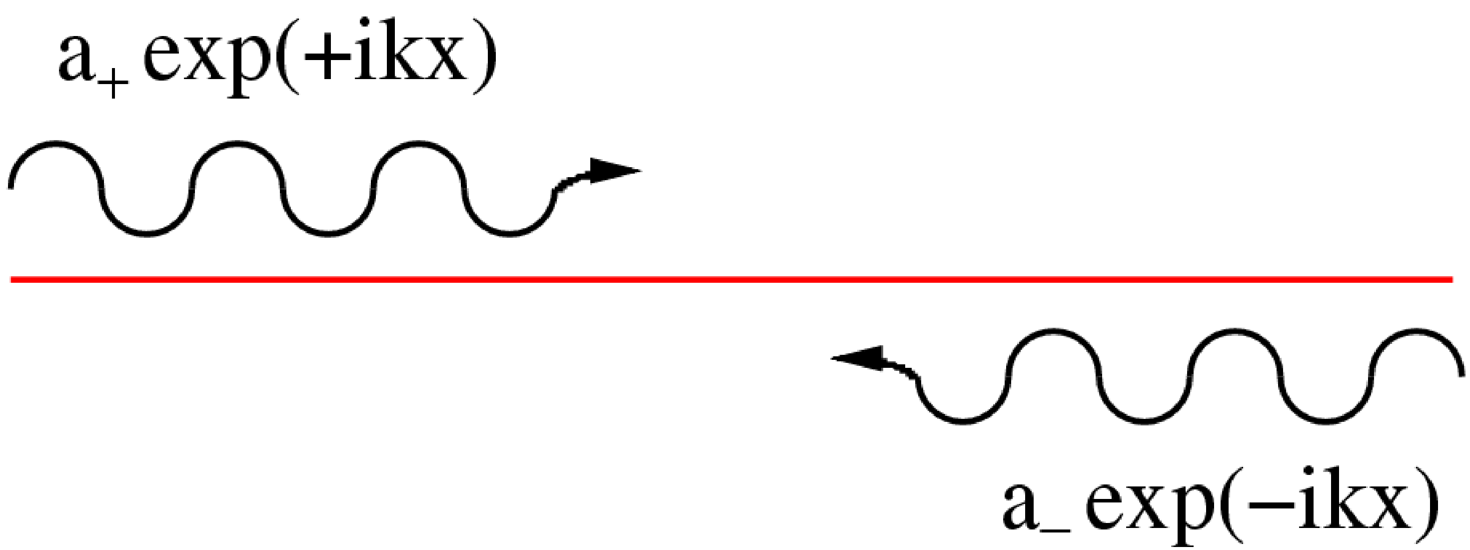
Deterministic Chaos vs Random Walk on a Graph

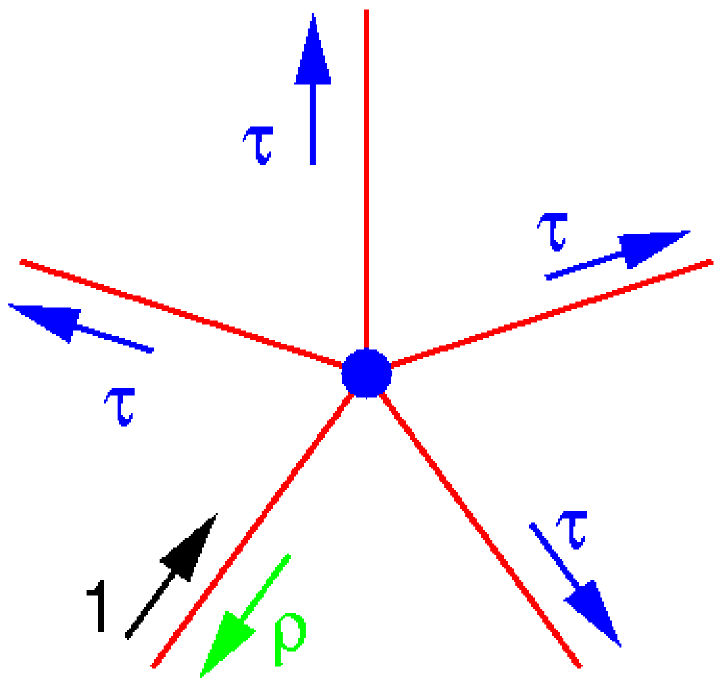




$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x) = E\,\Psi(x)$$

$$k = \sqrt{2mE/\hbar^2}$$





$$\mathbf{a}_- = \sigma\, \mathbf{a}_+$$

- current conservation $\sigma\sigma^\dagger = \mathsf{I}$
- continuity of wavefunction

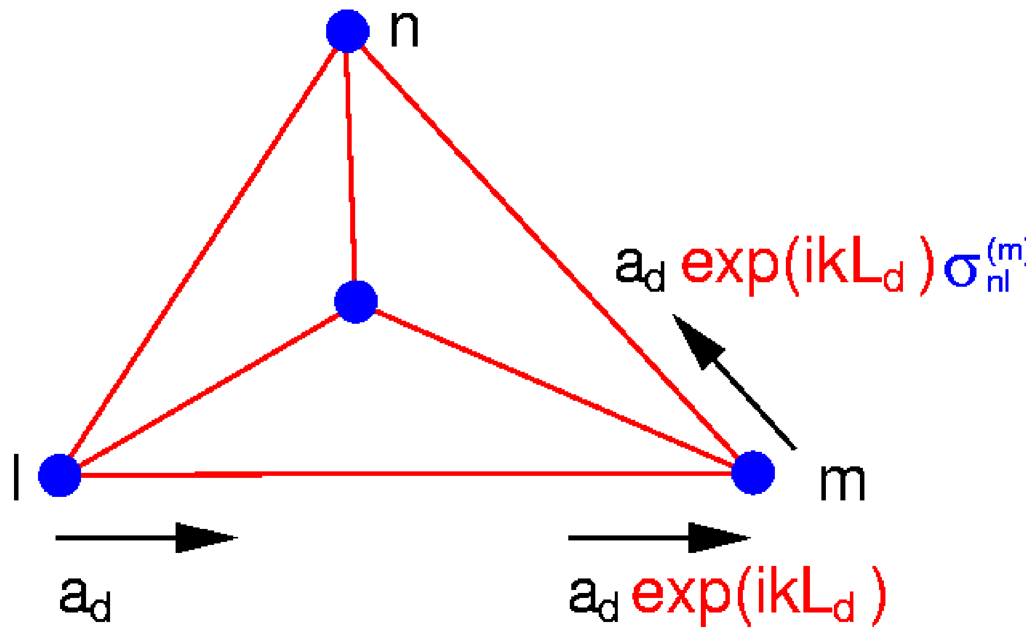
Neumann b.c.:

$$\sigma = \begin{pmatrix} \rho & \tau & \tau & \\ \tau & \rho & \tau & \cdots \\ \tau & \tau & \rho & \\ & \cdots & & \cdots \end{pmatrix}$$

$$\tau = \frac{2}{\mathsf{v}} \qquad \ll \; 1 \qquad (\mathsf{v} \to \infty)$$

$$\rho = \frac{2}{\mathsf{v}} - 1 \;\sim -1 \qquad (\mathsf{v} \to \infty)$$

Quantum graphs: *3. Network*



$$U_{d',d}(k) = \exp(ikL_{lm})\,\sigma_{ln}^{(m)}$$

$d = [l \to m]$ $\qquad$ $d' = [m \to n]$

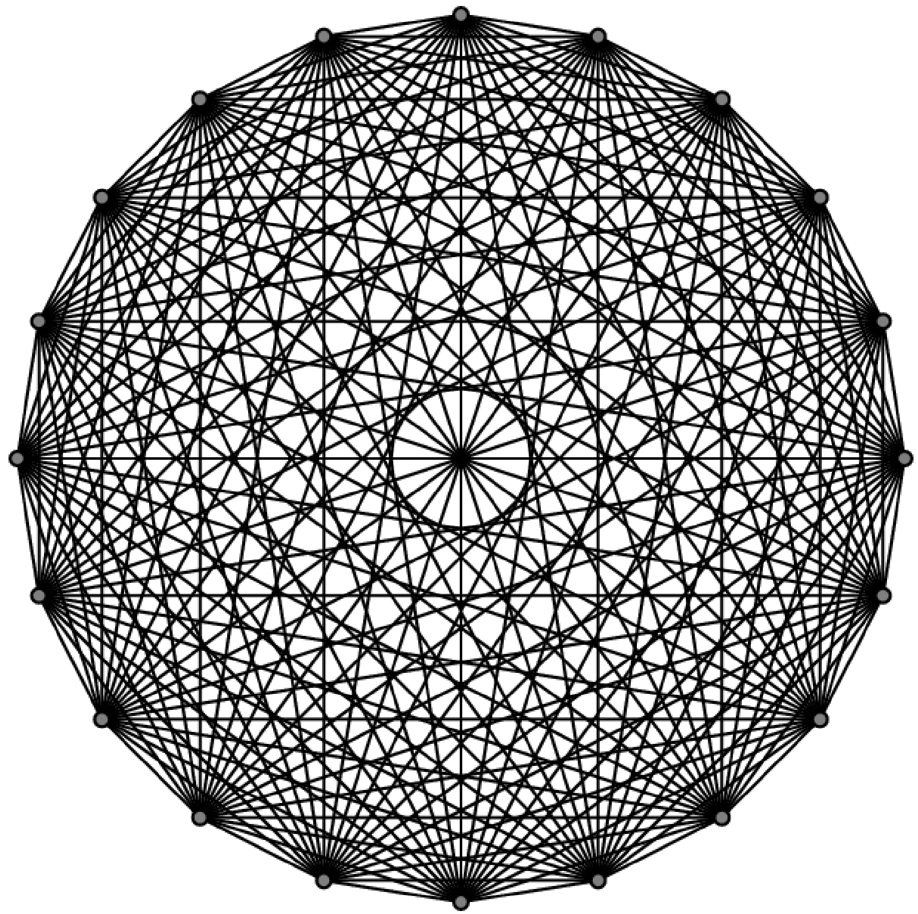
interpretation:
discrete time evolution

$$|\psi(t)\rangle = U^t(k)|\,\psi(0)\rangle$$

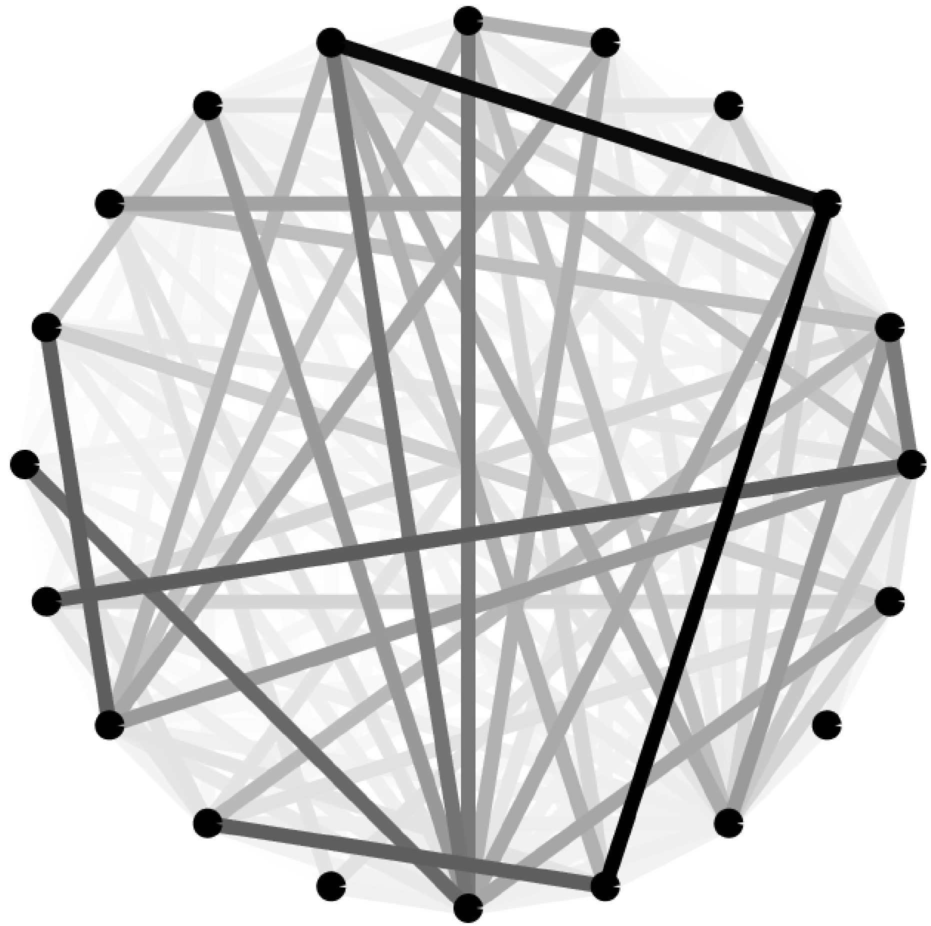
classical analogue:
Markov chain

$$M_{d'd} = |U_{d'd}|^2$$

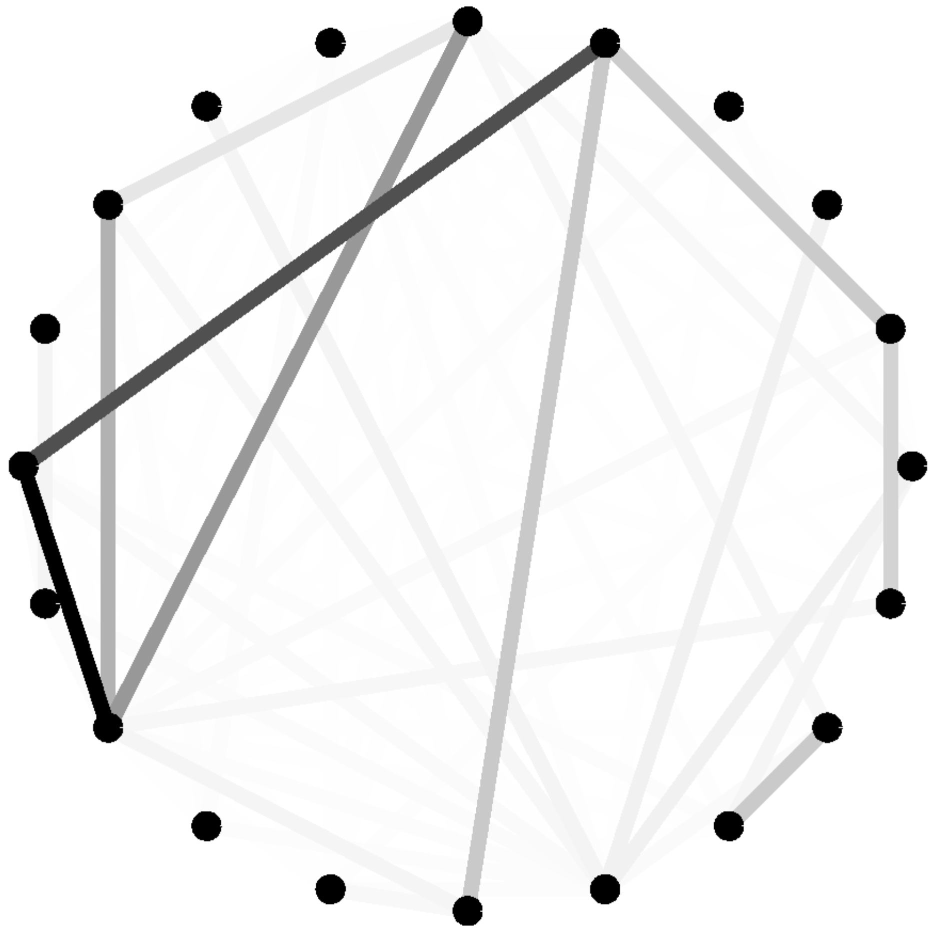
A complete Graph



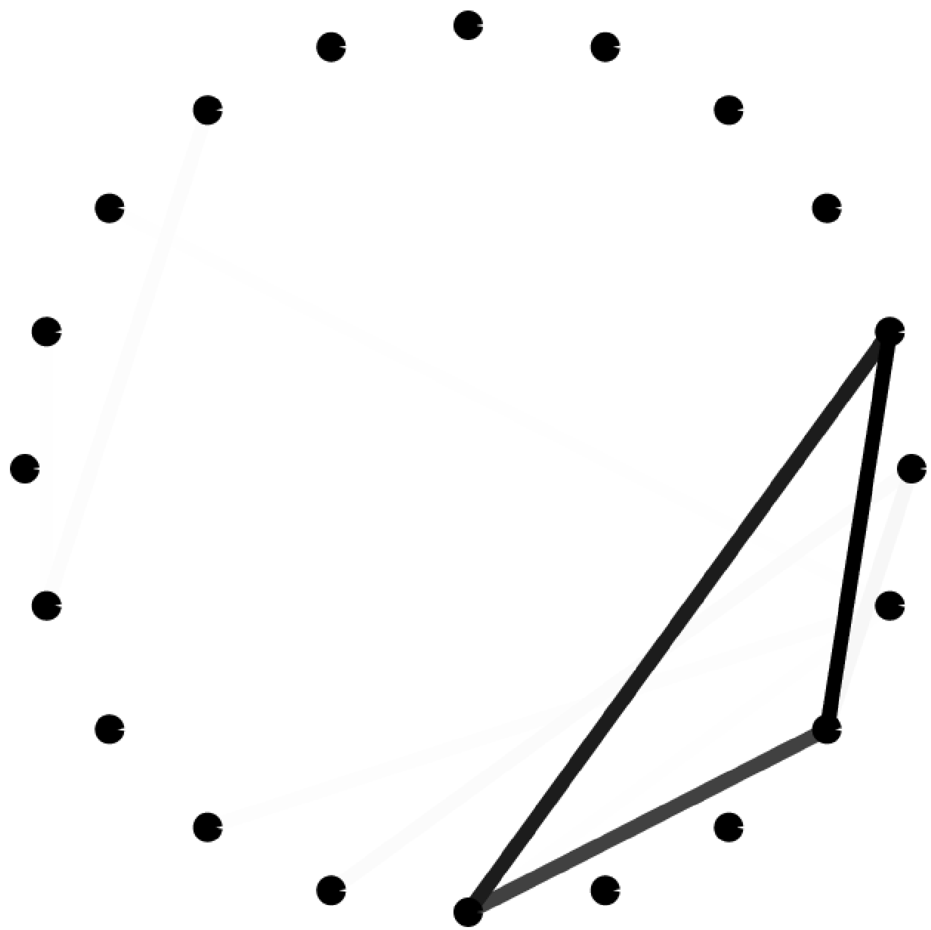
An "ergodic" eigenstate



A typical eigenstate

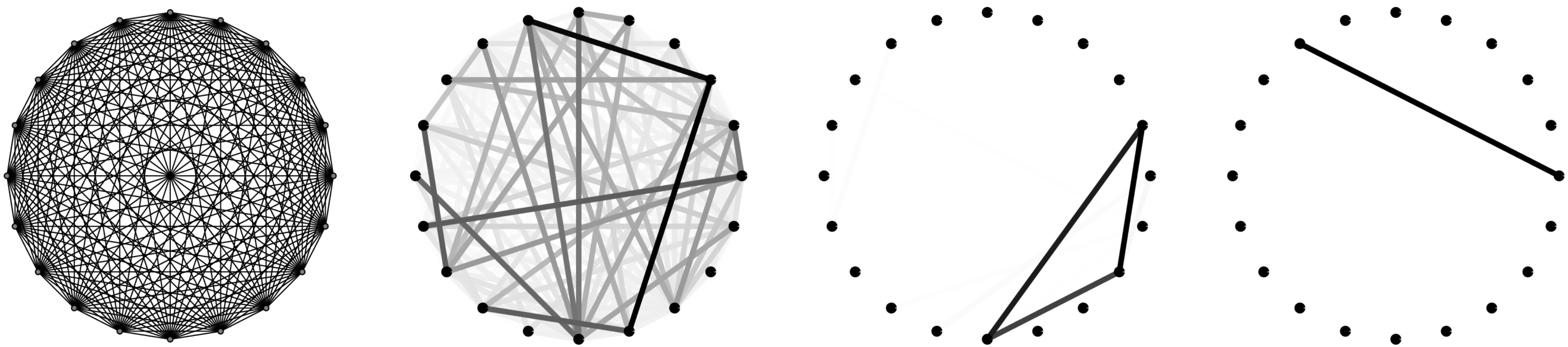


A scar on a graph



The inverse participation number

$$\sum_{d=1}^{2B} |a_d|^2 = 1 \qquad I = \sum_{d=1}^{2B} |a_d|^4 < 1$$



$\frac{1}{2B} = \frac{1}{380}$ $\quad 0.0096 \approx \frac{1}{104}$ $\quad 0.1516 \approx \frac{1}{6}$ $\quad \frac{1}{2}$

IPN from return probability

Heller '84: scars $\Leftrightarrow$ short-time dynamics:

$$\langle d|U^t|d\rangle = \sum_m \langle d|m\rangle\, e^{-i\epsilon_m t}\, \langle m|d\rangle$$

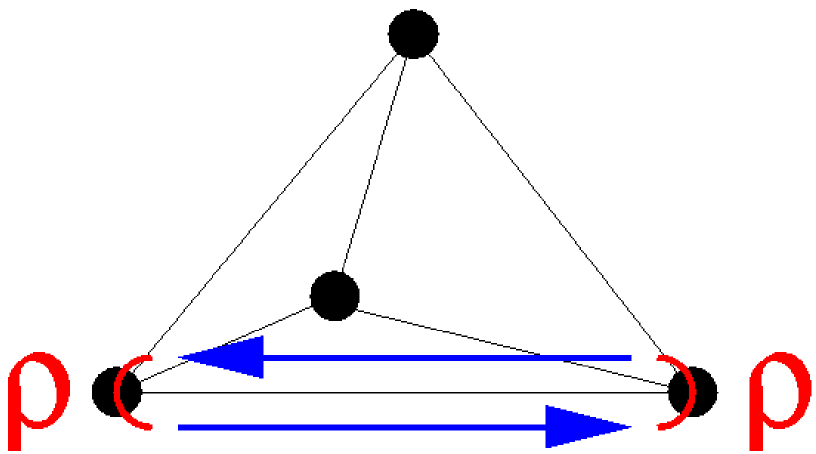
$$\langle P_d(t)\rangle_t = \sum_{m,n} |\langle m|d\rangle|^2\, |\langle n|d\rangle|^2\, \underbrace{\langle e^{i(\epsilon_m-\epsilon_n)t}\rangle_t}_{\delta_{m,n}}$$

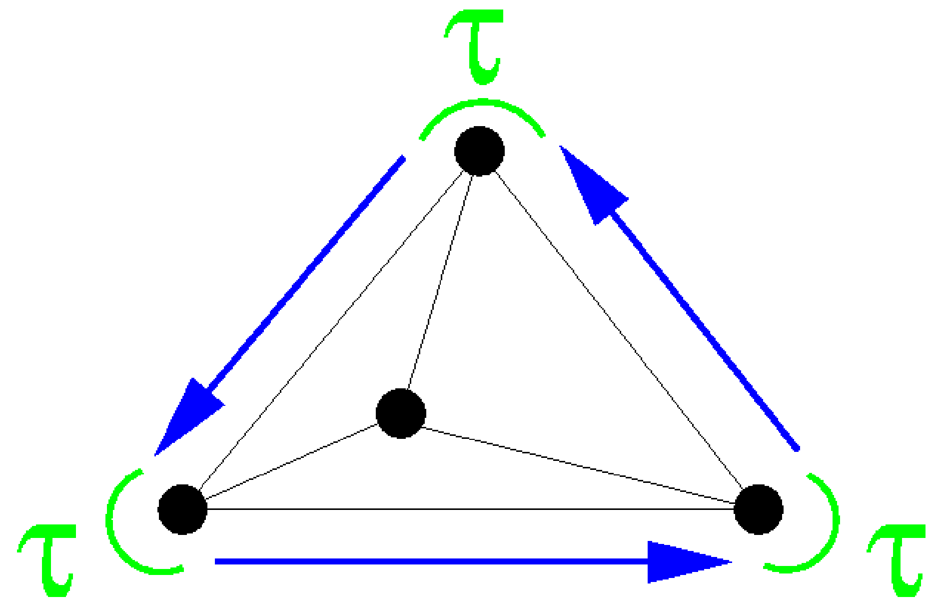
$$\boxed{\langle P_d(t)\rangle_{t,d} = \langle I_m\rangle_m}$$

$\Rightarrow$ average localization of eigenstates

$\Delta\epsilon \sim \frac{2\pi}{2B} \quad \Rightarrow \quad \langle\cdot\rangle_t \sim \frac{1}{2B}\sum_{t=1}^{2B}$

Many ways to return ...



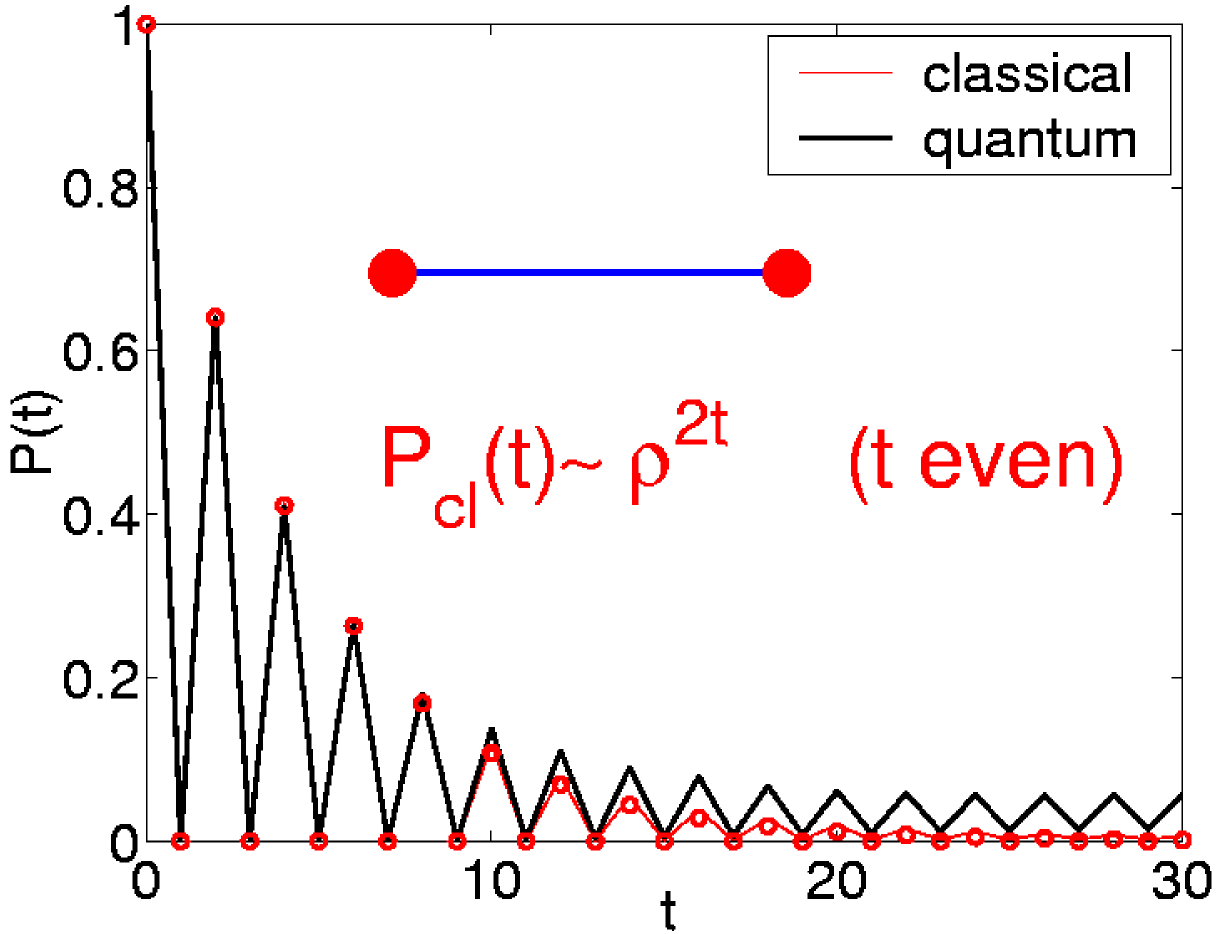


$e^{ikL_1}\,\rho\,e^{ikL_1}\,\rho$ $\qquad$ $e^{ikL_1}\,\tau\,e^{ikL_2}\,\tau\,e^{ikL_3}\,\tau$

Neumann b.c., large graphs: $\tau = 2/v \quad \rightarrow \quad 0$

$\rho = \tau - 1 \quad \rightarrow \quad -1$

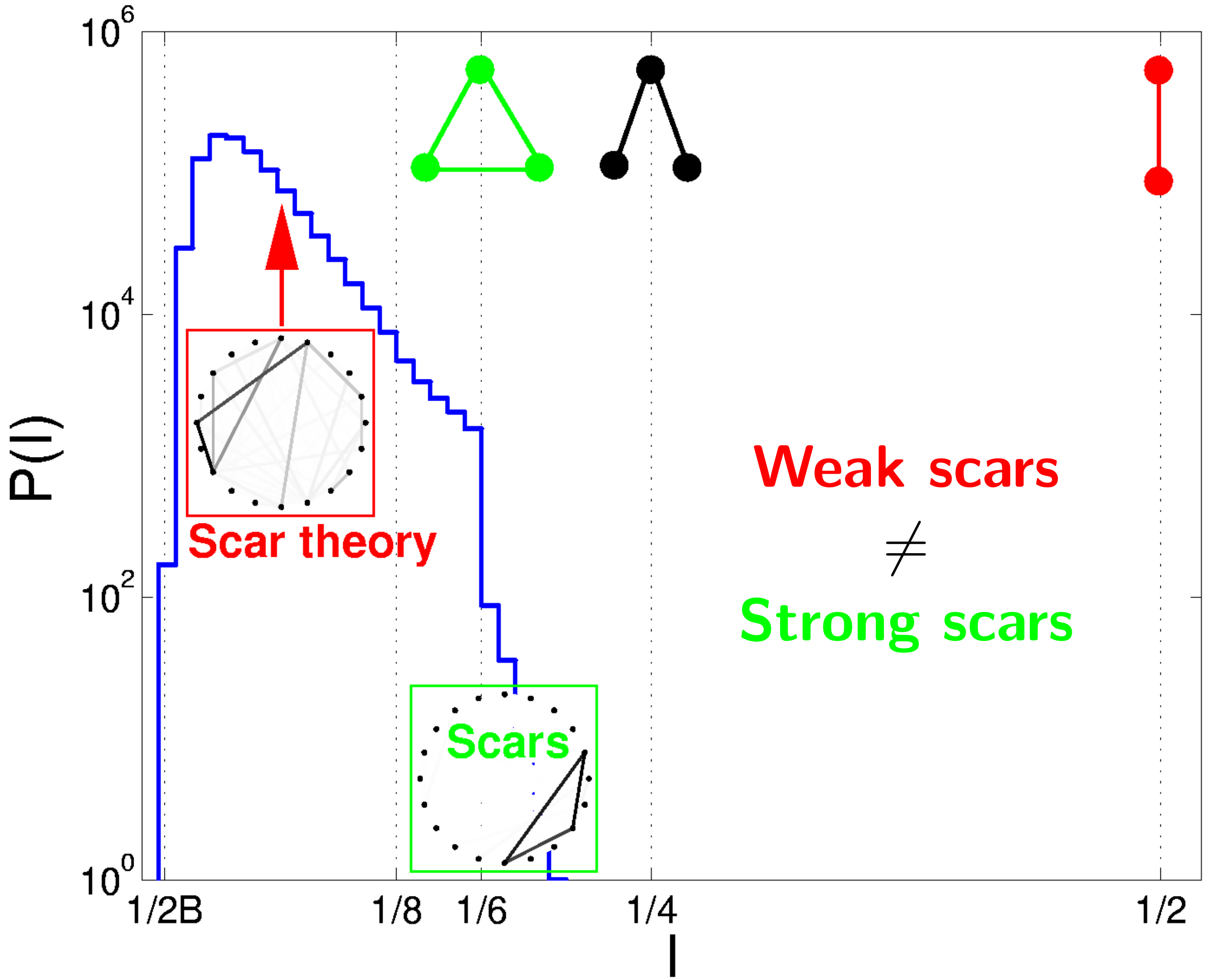
... *but only one is important*



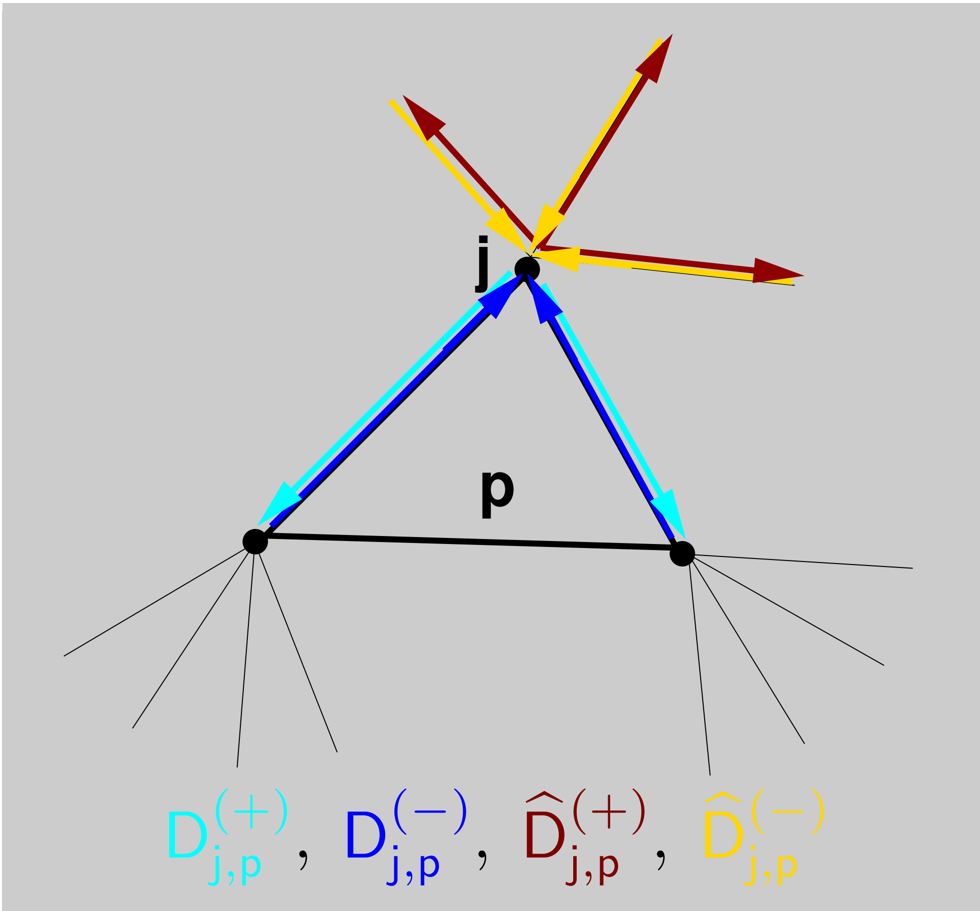
Kaplan 2001: period-two orbits $\Rightarrow \langle \mathrm{I} \rangle \sim \mathrm{v} \times \mathrm{I}_{\mathrm{RMT}}$

The shortest and most stable orbits cause enhanced localization.

Take-home message



Which orbits can scar?

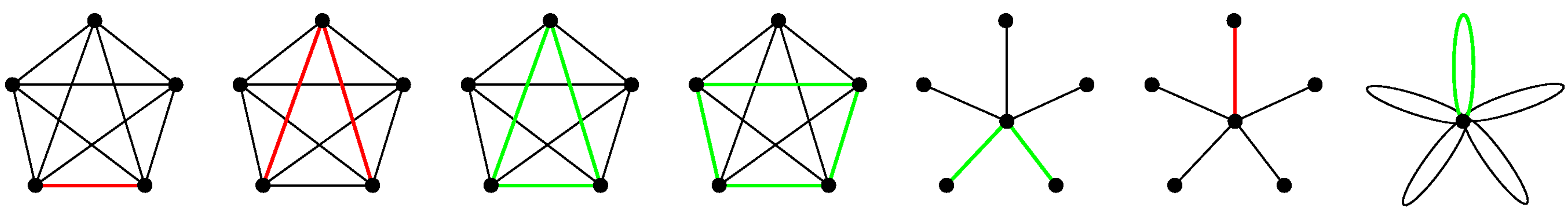


perfect scars:

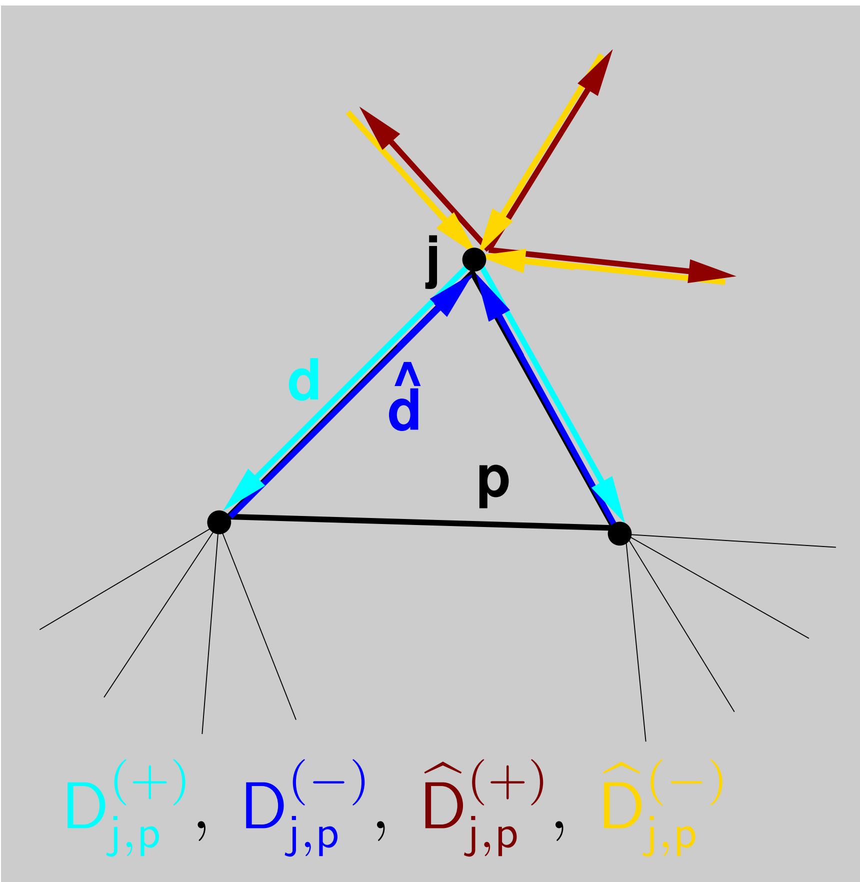
$$0 = 0 + \tau \sum_{d \in D_{j,p}^{(-)}} e^{ikL_d} a_d$$

$$v_{j,p} \geq 2 - \delta_{v_j,1} \quad (\forall j \in p)$$

Stability is irrelevant!



Energies of scars?



$$a_d = \underbrace{\sum_{d' \in D^{(-)}_{j,p}} e^{ikL_{d'}} a_{d'} \, (\tau}_{0} + \delta_{d'\hat{d}} \underbrace{[\rho - \tau]}_{-1})$$

$$a_d = -e^{ikL_d} a_{\hat{d}}$$

$$a_d = +e^{2ikL_d} a_d$$

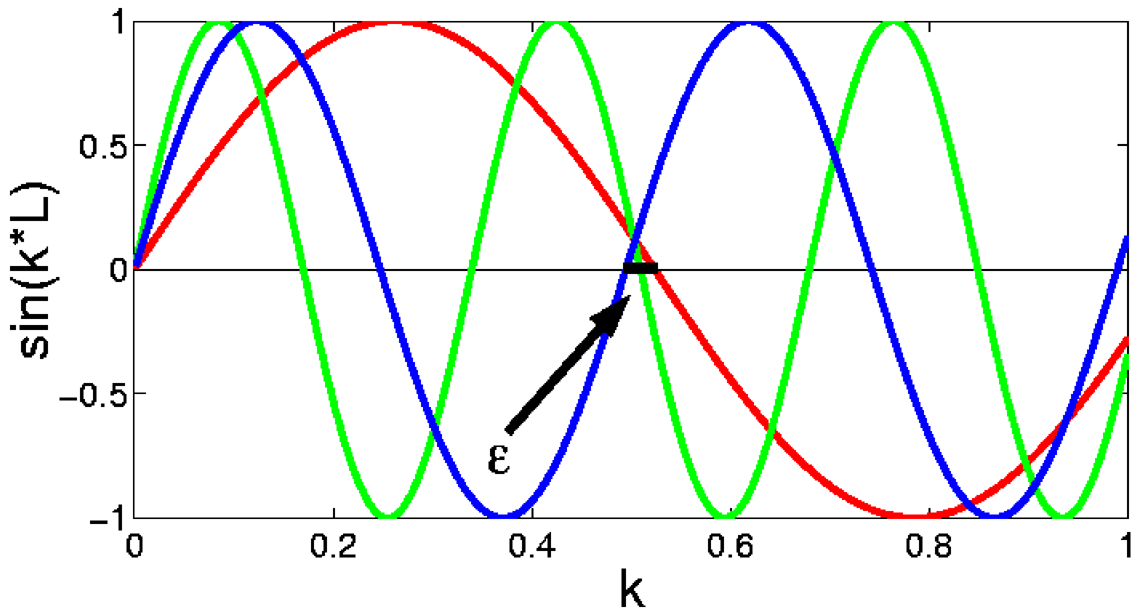
$$kL_d = m_d \pi \qquad \forall\, d \in p$$

$\Rightarrow$ commensurate bond lengths

$$L_d / L_{d'} = m_d / m_{d'}$$

No perfect scars for generic graphs!

Perturbation theory for the scar quality



$$P(\varepsilon \to 0) \sim \varepsilon^{N-2}$$

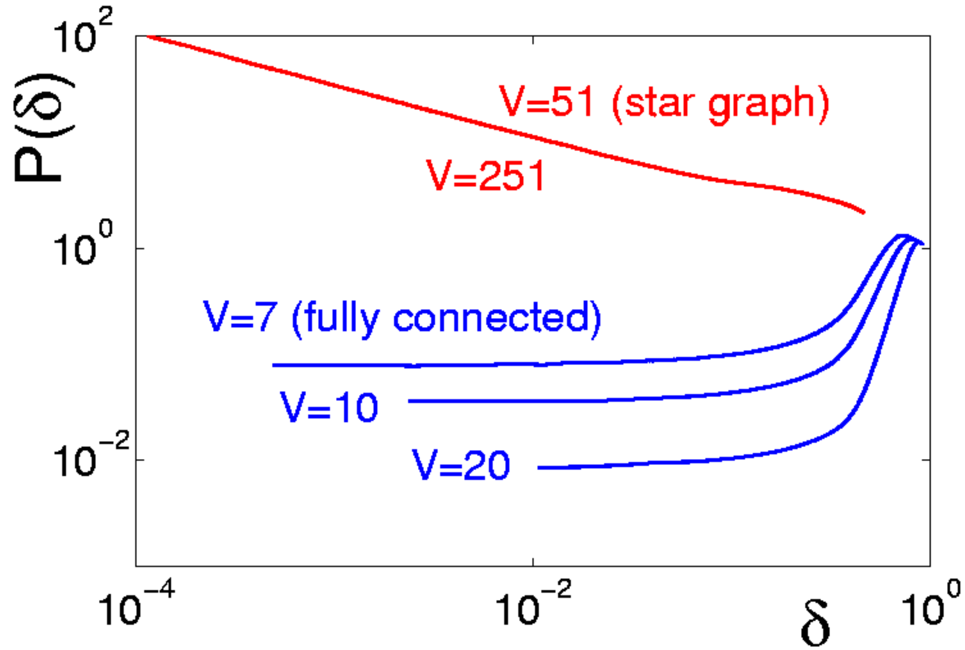
$$|a_{\mathrm{scar}}\rangle = |a^{(0)}_{\mathrm{scar}}\rangle + \varepsilon \sum_{m \neq \mathrm{scar}} \frac{\langle a^{(0)}_m|\hat{\Phi}|a^{(0)}_{\mathrm{scar}}\rangle}{1 - \exp(\mathrm{i}[\lambda^{(0)}_m - \lambda^{(0)}_{\mathrm{scar}}])} |a^{(0)}_m\rangle$$

Scar quality:

$$\delta_p = \sum_{d \notin p} |a_d|^2$$

$(0 \leq \delta \leq 1, \quad \delta_p = 0$: perfect scar)

Distribution of scars



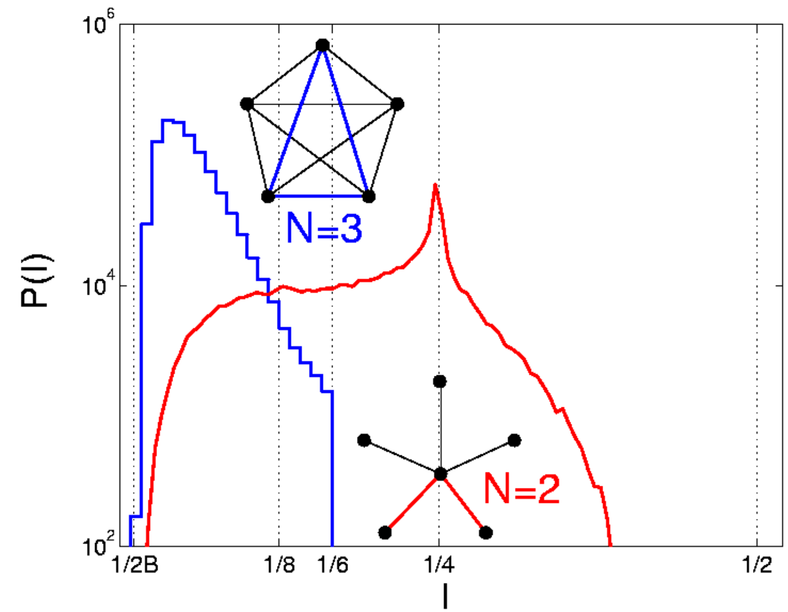
$$\mathcal{P}^{(N)}(\delta \to 0) = C\,\delta^{(N-3)/2}$$

$\mathcal{P}^{(2)}(\delta \to 0) \sim \delta^{-1/2}$

$\mathcal{P}^{(3)}(\delta \to 0) = C$

$\mathcal{P}^{(4)}(\delta \to 0) \sim \delta^{+1/2}$

$I \geq 1/6 - \delta/3$



cf Berkolaiko et al. (2003): No quantum ergodicity for star graphs.

Conclusions

- The scar theory of Heller et al. applies to graphs, ...
- ... but it does not describe the scars ...
- ... because strong and weak scarring are unrelated phenomena.
- A detailed understanding of strong scars was achieved, ...
- ... but the method does not (immediately) generalize to other systems.

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