Wavefunctions in chaotic quantum systems

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Lund, January 2004

Aim

- Overview on properties of eigenfunctions in chaotic systems

Emphasis on

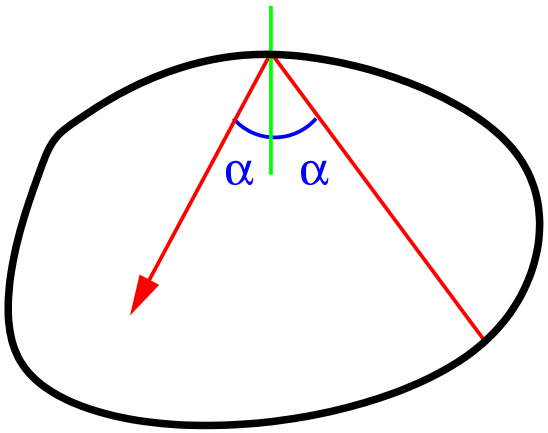
- illustrations
- basic ideas
- no proofs...

Thanks to

- Roman Schubert

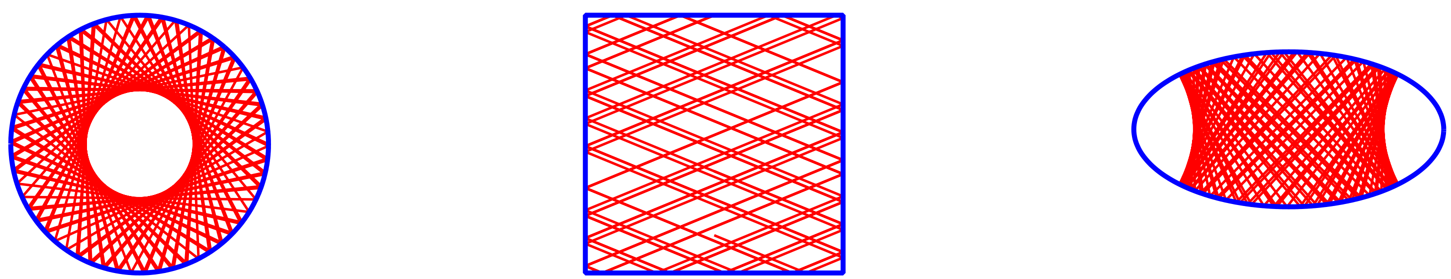
- debian PYTHON Powered NO PATENTS

Free motion of a point particle in some Euclidean domain $\Omega \subset \mathbb{R}^2$ with elastic reflections at the boundary $\partial\Omega$.

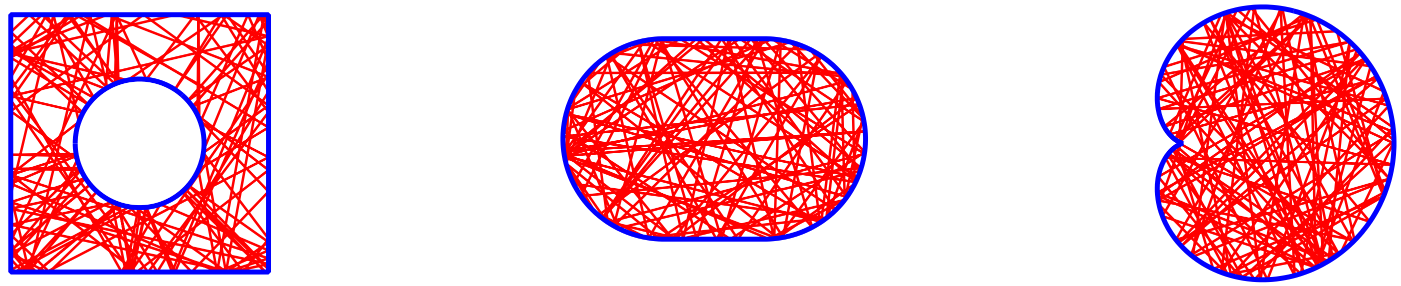


Depending on the boundary one obtains completely *different dynamical behaviour*:

Integrable systems (regular motion)



Chaotic systems

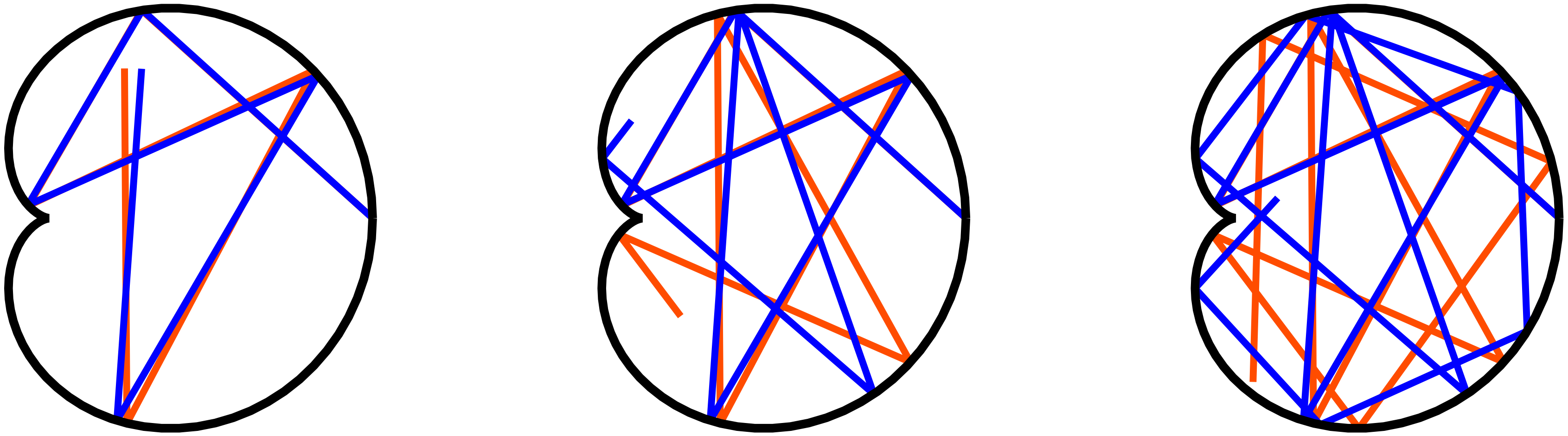


"Chaotic" systems – properties

- ergodicity
- mixing
- K-systems
- Bernoulli

Origin of stochastic properties: **hyperbolicity** —

"Sensitive dependence on the initial conditions"



$t = 10s$ $t = 15s$ $t = 20s$

Phase space

$$T^\star\Omega = \{(\boldsymbol{p}, \boldsymbol{q}) \mid \boldsymbol{p} \in \mathbb{R}^2, \; \boldsymbol{q} \in \Omega\} \quad , \tag{1}$$

with $\boldsymbol{q}$: position and $\boldsymbol{p}$: momentum vector of the particle.

Billiard flow

One-parameter group of automorphisms $\{\Phi^t\}$, acting on $T^\star\Omega$,

$$\Phi^t : T^\star\Omega \mapsto T^\star\Omega, \tag{2}$$

where

$$\Phi^t(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}(t), \boldsymbol{q}(t)) \tag{3}$$

describes the position of the billiard ball at time t.

Two–dimensional billiards are conservative systems.

Thus the motion is restricted to a hypersurface in $\mathbb{R}^4$ of dimension 3.

I.e., the trajectories lie on surfaces of constant energy E

$$\Sigma_E := \left\{ (\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^2 \times \Omega \mid \boldsymbol{p}^2 = E \right\} \quad . \tag{4}$$

Scaling property $\Sigma_E = E^{\frac{1}{2}} \Sigma_1 := \{ (E^{\frac{1}{2}} \boldsymbol{p}, \boldsymbol{q}) \mid (\boldsymbol{p}, \boldsymbol{q}) \in \Sigma_1 \}$.

Thus we can restrict ourselves to $|\boldsymbol{p}| = 1$ and

identify the equi-energy surface Σ_E as follows

$$\Sigma_1 = \{ (\boldsymbol{p}, \boldsymbol{q}) \mid \boldsymbol{q} \in \Omega, \ \boldsymbol{p} \in \mathbb{R}^2, \ |\boldsymbol{p}| = 1 \} \simeq \Omega \times S^1 \quad , \tag{5}$$

where S^1 is the unit circle.

So we have

- $T^\star\Omega$: phase space
- $\{\Phi^t\}$: billiard flow

The invariant measure for the flow is the **Liouville measure**

$$d\nu = \frac{1}{\mathrm{vol}(\Sigma_E)}\delta(E - H(\boldsymbol{p},\boldsymbol{q}))\, d^2p\, d^2q \quad . \qquad (6)$$

Remark: A measure ν is called invariant if $\nu(A) = \nu(\phi^t A)$ for all measurable $A \subset T^\star\Omega$.

Consider a one-parameter group of automorphisms $\{\Phi^t\}$ of a measure space M with invariant probability measure μ.

Definition *The time average $\overset{*}{f}$ of a function of a function $f : M \to \mathbb{R}$ (if it exists) is given by*

$$\overset{*}{f}(X) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\Phi^t X)\, dt \ , \qquad X \in M \ . \tag{7}$$

The spatial average (if it exists) is defined by

$$\bar{f} = \int_M f(X)\, d\mu \ . \tag{8}$$

Remark: The Birkhoff ergodic theorem shows that for $f \in L^1(M, \mu)$ the time average exists (μ a.e.).

Ergodicity

Definition *A system is called ergodic if for any function* $f \in L_1(M, \mu)$ *the time average equals the spatial average*

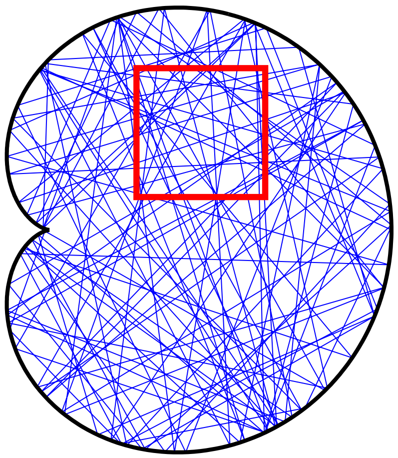
$$\overset{*}{f}(X) = \bar{f} \qquad \text{for almost every point } X \in M \quad . \tag{9}$$

Therefore, ergodicity means that a typical trajectory fills M (eg. the equi–energy surface Σ_E) densely in a uniform way.

However, it does not mean that a typical trajectory hits every point in M.

Classical ergodicity of a flow $\{\phi^t\}$, position space

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \chi_D(\phi^t(\boldsymbol{p}, \boldsymbol{q}))\, \mathrm{d}t = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$



for almost all initial conditions in phase space, $(\boldsymbol{p}, \boldsymbol{q}) \in T^*\Omega$.

Remarks:

- $\chi_D(X)$: characteristic function of D: $\chi_D(X) = \begin{cases} 1 & X \in D \\ 0 & X \notin D \end{cases}$
- "Almost everywhere": Let $\mathcal{S}$ be the set of initial conditions for which the above does not hold.

  Then $\nu(\mathcal{S}) = 0$.

Definition *The time–correlation function of two functions $f_1, f_2 \in L^2(M, \mu)$ is defined by*

$$C(t) = \int_M f_1(\Phi^t X) f_2(X) \, d\mu \quad . \tag{10}$$

Definition *A flow is called mixing if*

$$\lim_{t\to\infty} C(t) = \int_M f_1(X) \, d\mu \int_M f_2(X) \, d\mu = \overline{f}_1 \overline{f}_2 \quad . \tag{11}$$

A classical example (Arnold and Avez '68) is the preparation of cuba libre by mixing 80% cola and 20% rum.

Remarks:
- A mixing system is ergodic
- An ergodic system is not necessarily mixing (Eg.: irrational translations on S^1)

Definition (Periodic orbits)

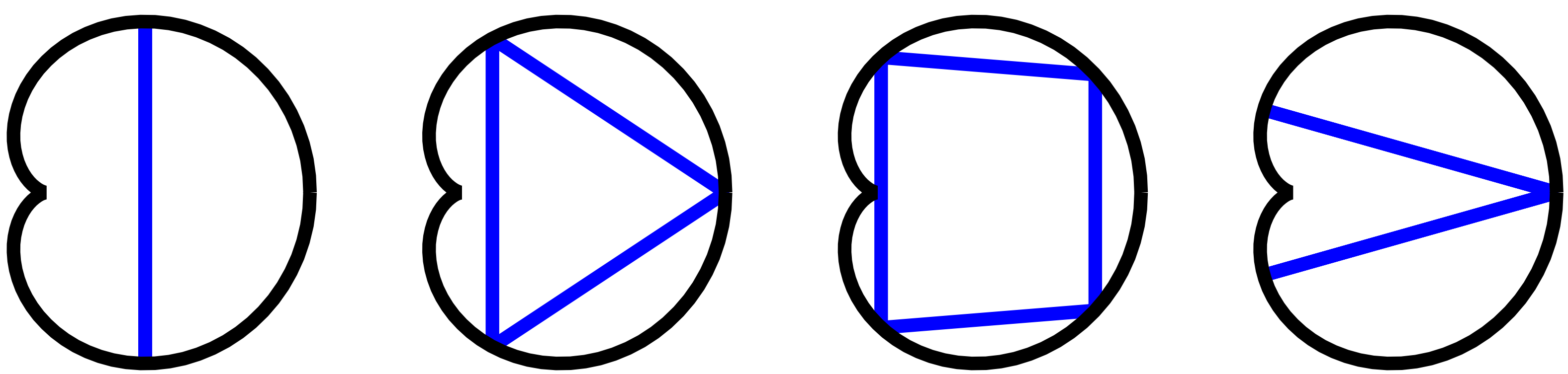
A *periodic orbit* γ *is a trajectory which returns to its initial point in phase space after some time* $t > 0$.

I.e.: for a point $(\boldsymbol{p}, \boldsymbol{q})$ *on a periodic orbit* $\exists t \in \mathbb{R}^+$ *s.t.*

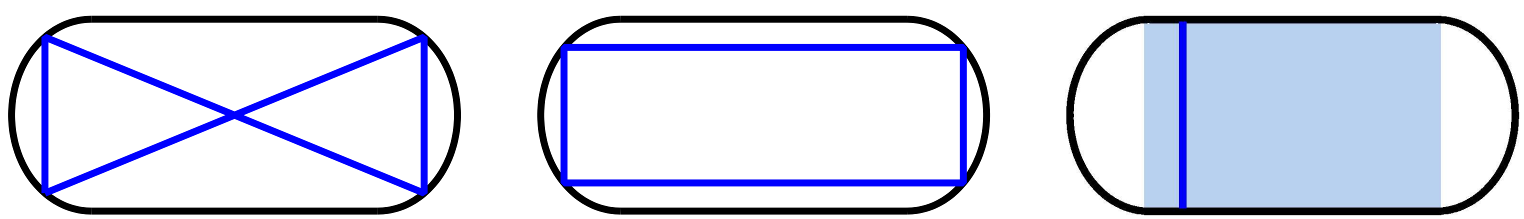
$$\phi^t(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}, \boldsymbol{q}) \tag{12}$$

Examples:

Cardioid:

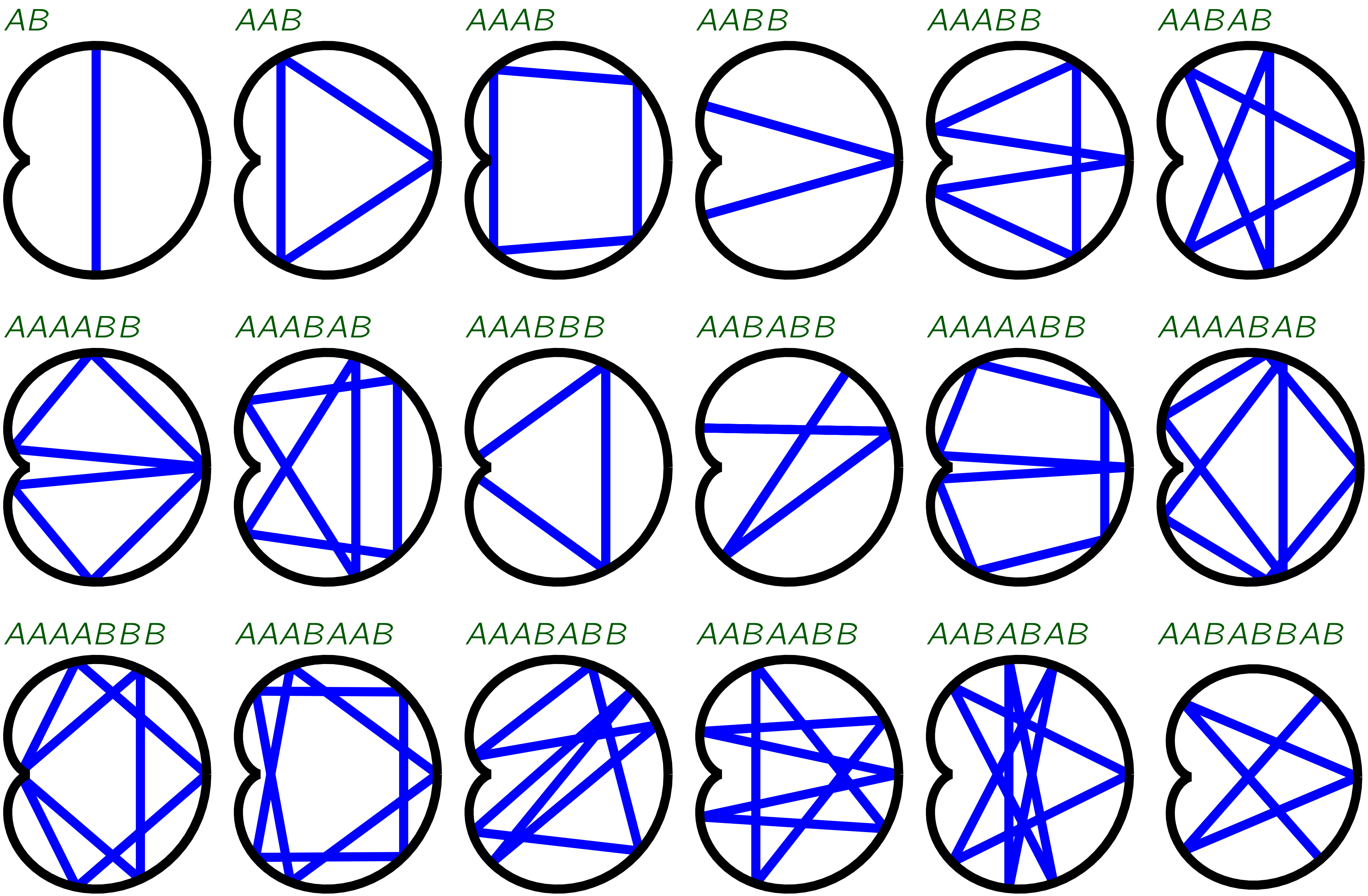


Stadium:



II Classical billiards — periodic orbits

To compute periodic orbits systematically: **symbolic dynamics**



Stationary Schrödinger equation (in units $\hbar = 2m = 1$)

$$-\Delta\psi_n(\boldsymbol{q}) = E_n\psi_n(\boldsymbol{q}) \ , \qquad \boldsymbol{q} \in \Omega \tag{13}$$

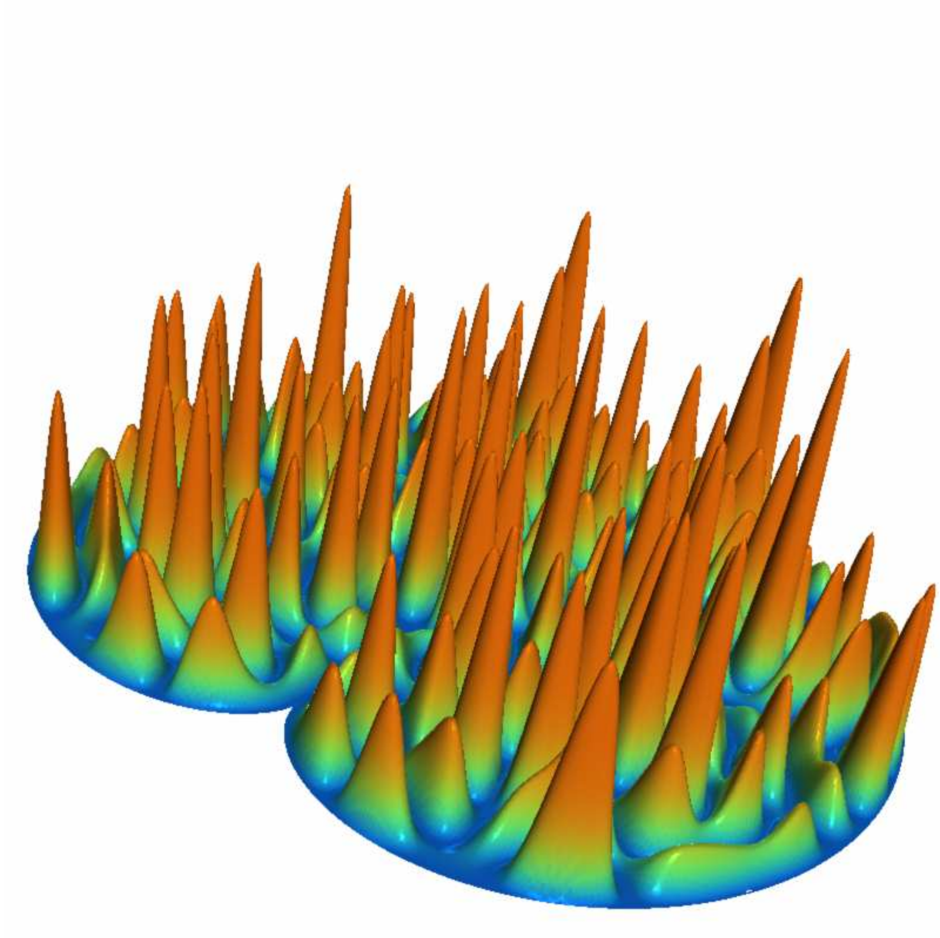
with (for example) Dirichlet boundary conditions i.e. $\psi_n(\boldsymbol{q}) = 0$ for $\boldsymbol{q} \in \partial\Omega$.

For compact Ω: discrete spectrum $\{E_n\}$ with associated eigenfunctions ψ_n.

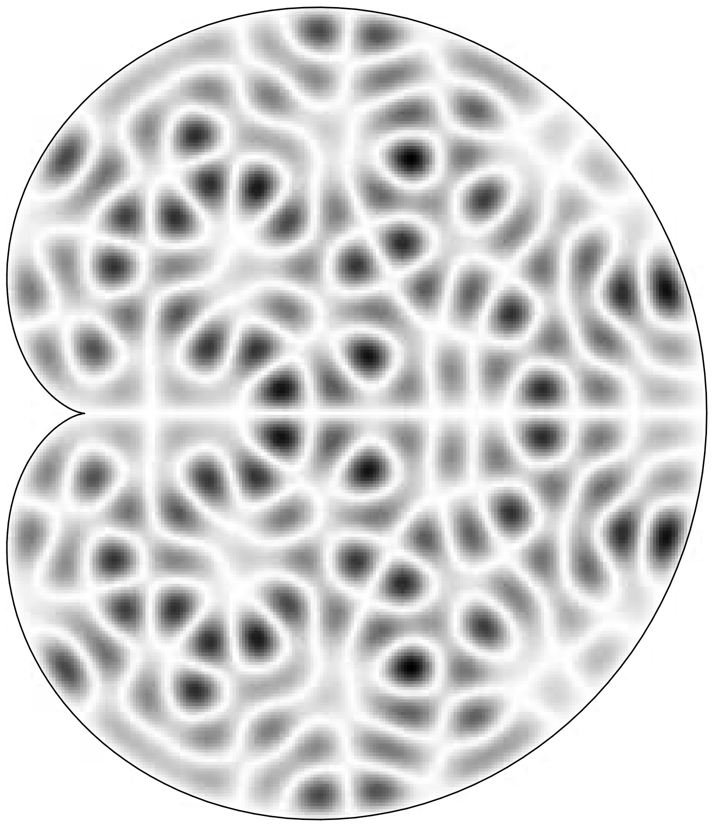
Interpretation of ψ_n: $\displaystyle\int_D |\psi_n(\boldsymbol{q})|^2 \, \mathrm{d}^2 q$ is the probability of finding the particle inside the domain $D \subset \Omega$.

Question: What is the behaviour of the eigenvalues and eigenfunctions in the *semiclassical limit* $E \to \infty$?

3D Plot of $|\psi_n(\boldsymbol{q})|^2$



Density plot of $|\psi_n(\boldsymbol{q})|^2$



Remark: Numerical computations via boundary integral method.

Integrable systems: eigenstates rectangular billiard

For a box $[0, 2\pi]^2$ with Dirichlet boundary conditions

$$\psi_{kl}(x, y) = \frac{1}{\pi} \sin(kx) \sin(ly) \qquad \text{with } k, l \in \mathbb{N} \tag{14}$$

Using this ansatz in the Schrödinger equation

$$-\Delta\psi(\boldsymbol{q}) = E\psi(\boldsymbol{q}) \ , \qquad \boldsymbol{q} \in \Omega \tag{15}$$

gives

$$(k^2 + l^2)\psi_{kl}(x, y) = E_{k,l}\psi_{kl}(x, y) \tag{16}$$

I.e. the eigenvalues are

$$E_{k,l} = k^2 + l^2 \qquad k, l \in \mathbb{N} \tag{17}$$

Integrable systems: eigenstates circular billiard

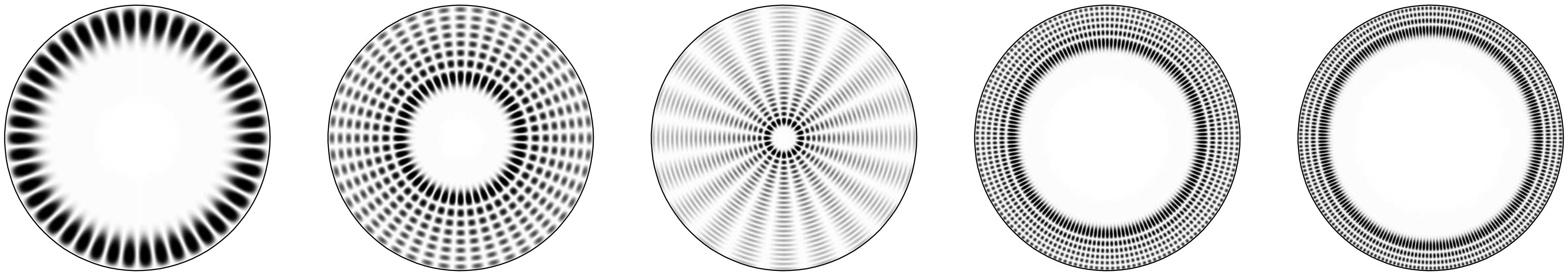
The eigenfunctions are given in polar coordinates by

$$\psi_{kl}(r,\phi) = J_k(j_{kl}r)\begin{cases}\cos(k\phi), & \text{even:} \quad k = 0,1,2,\ldots \\ \sin(k\phi), & \text{odd:} \quad k = 1,2,\ldots\end{cases}, \tag{18}$$

where j_{kl} is the l-th zero of the Bessel function $J_k(x)$.

The boundary condition $\psi^{\pm}_{kl}(1,\varphi) = 0$ leads

to the eigenvalues $E_{k,l} = j^2_{k,l}$.



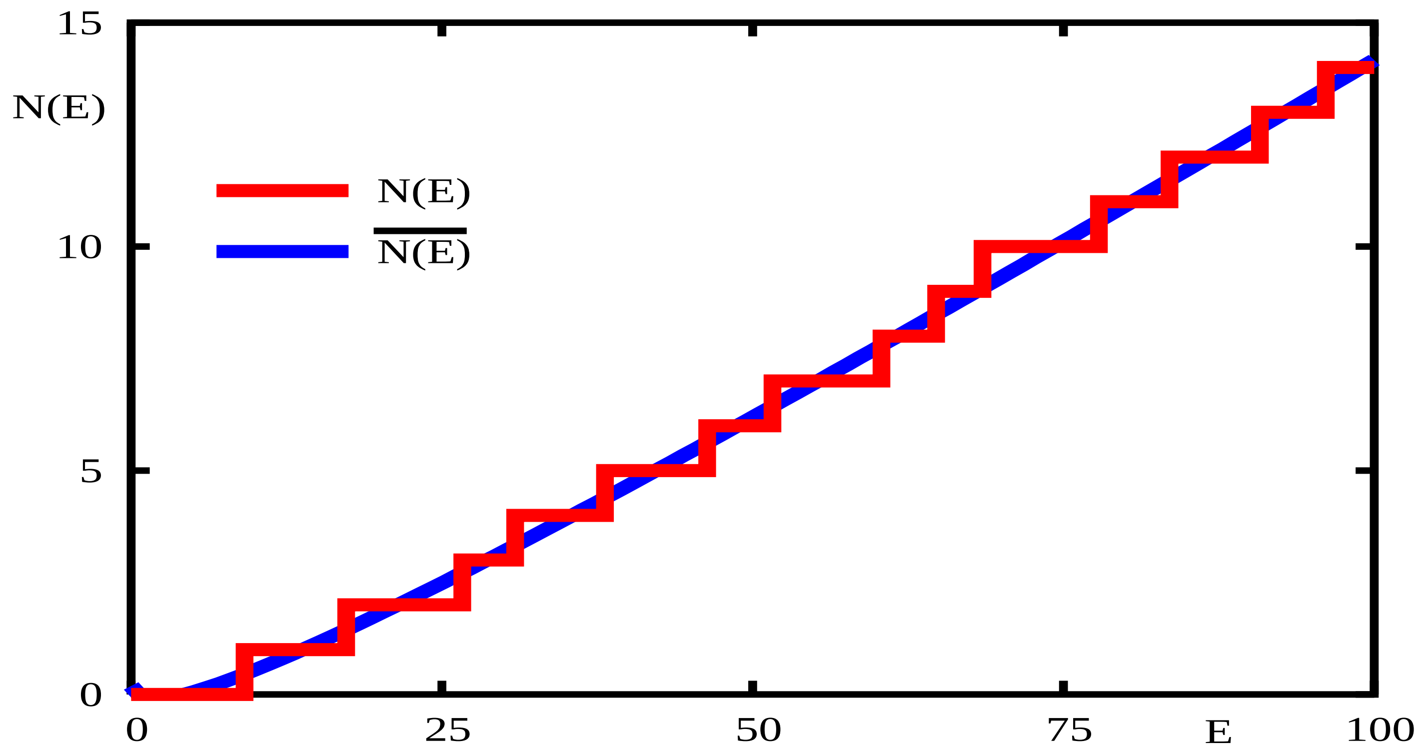
$n = 100$ $n = 400$ $n = 1000$ $n = 1500$ $n = 2000$

Spectral staircase function (integrated density of states)

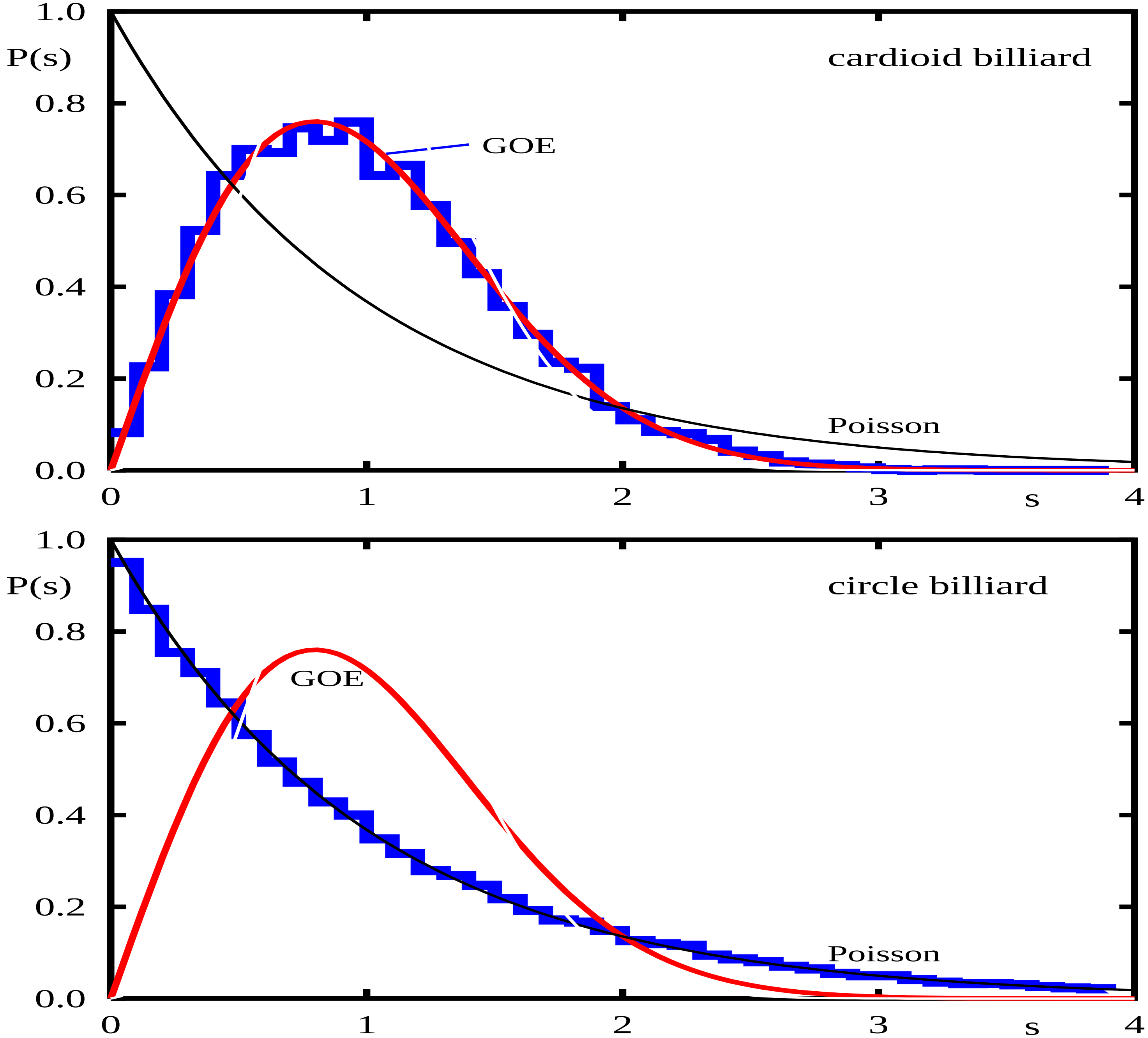
$$N(E) = \#\{n \mid E_n \leq E\} \tag{19}$$

Mean behaviour (Weyl formula)

$$\overline{N}(E) = \frac{\mathcal{A}}{4\pi}E - \frac{\mathcal{L}}{4\pi}\sqrt{E} + \mathcal{C} \tag{20}$$

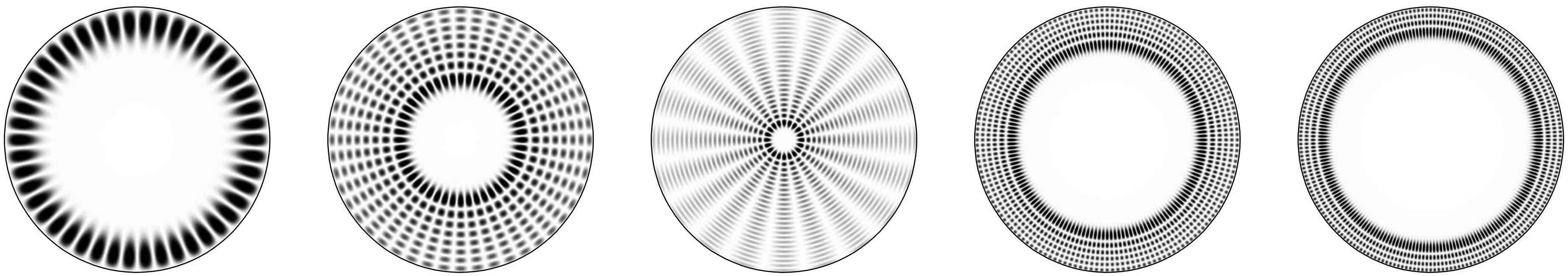


Spectral statistics – level spacing distribution

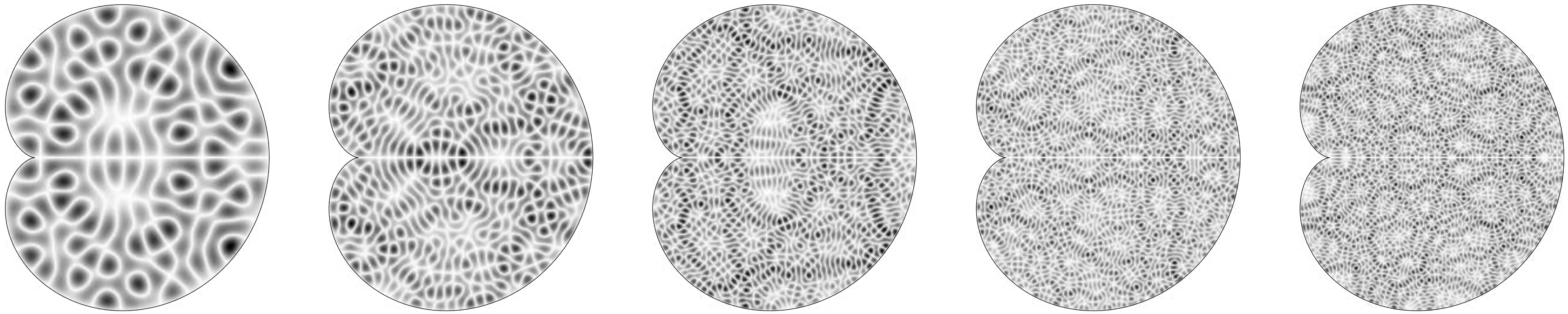


Eigenstates circular billiard

$n = 100$ $\quad$ $n = 400$ $\quad$ $n = 1000$ $\quad$ $n = 1500$ $\quad$ $n = 2000$



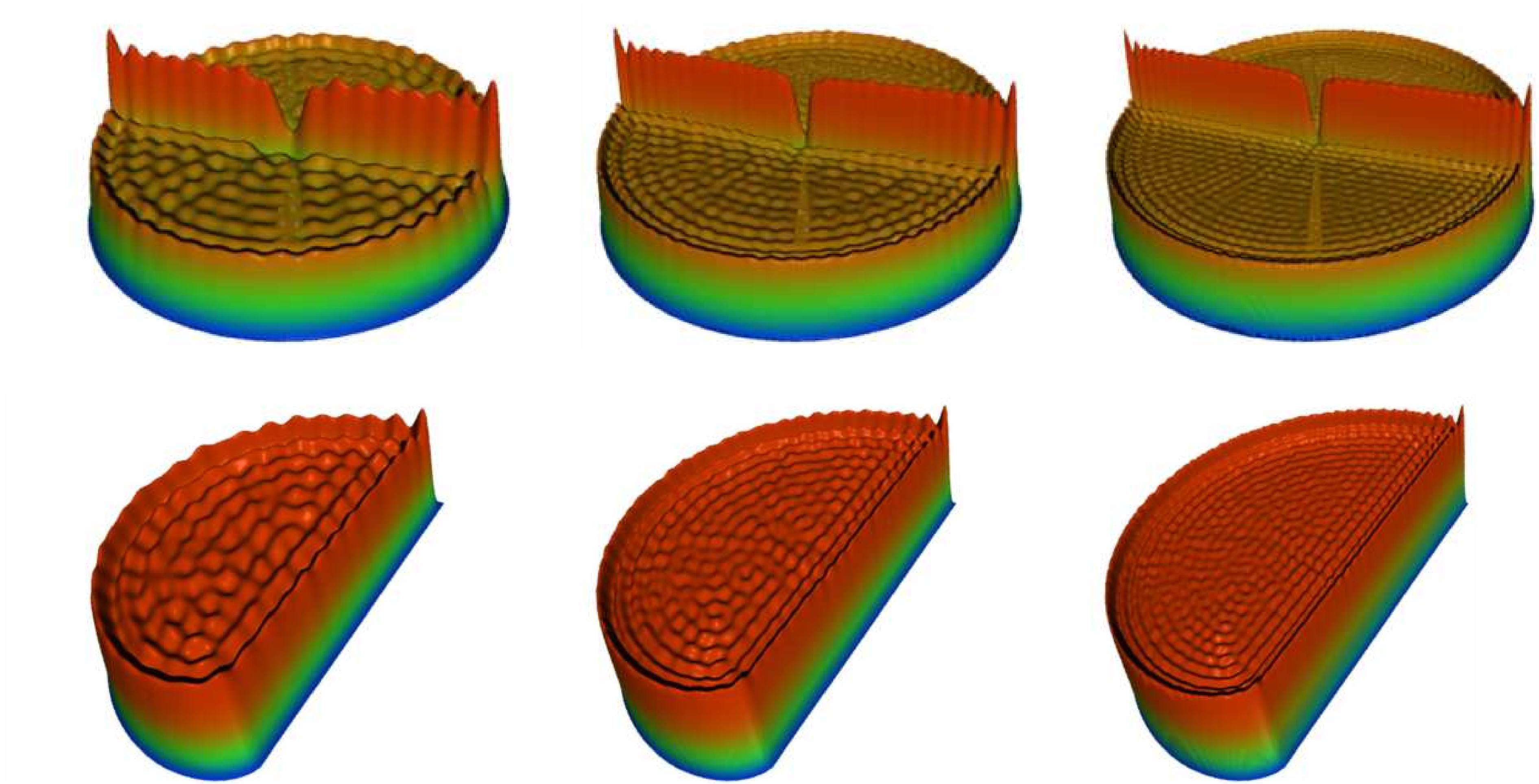
Eigenstates cardioid billiard



What is the asymptotic mean behaviour of eigenstates?

Consider

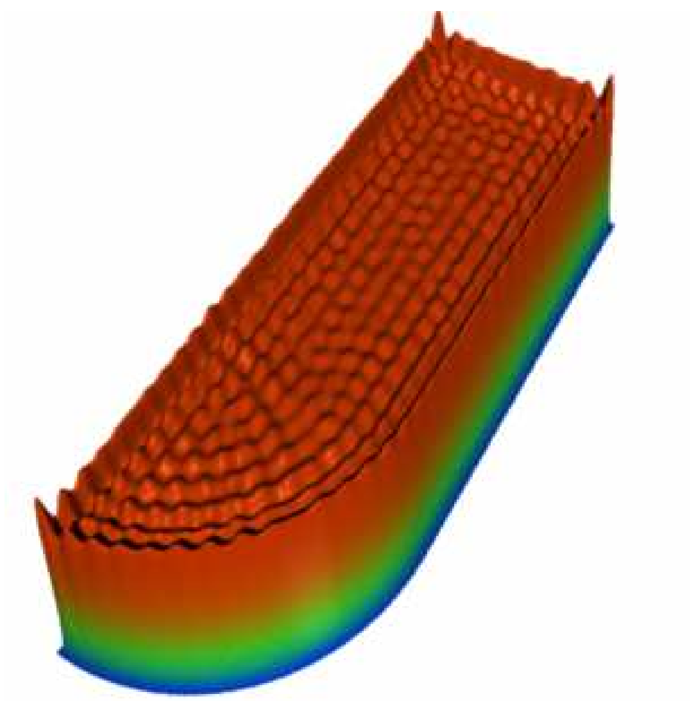
$$\frac{1}{N(E)} \sum_{E_n \leq E} \left|\psi_n(q)\right|^2 \qquad (21)$$

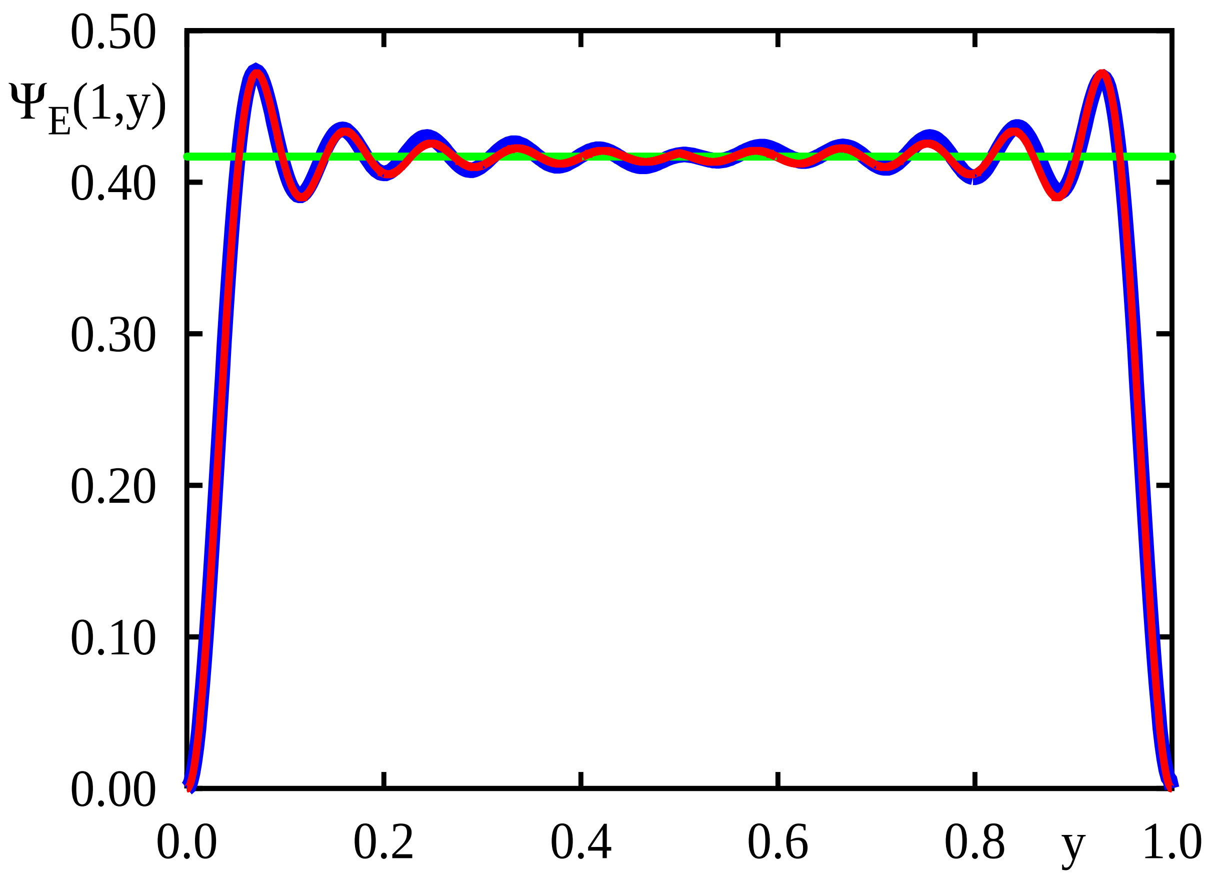


One has ([Hörmander '85])

$$\sum_{E_n \le E} |\psi_n(q)|^2 \sim \frac{1}{4\pi} E - \frac{1}{4\pi} \frac{J_1\left(2d(q)\sqrt{E}\right)}{d(q)} \sqrt{E} \,, \qquad (22)$$

where $d(q)$ is the distance of the point $q \in \Omega$ to the boundary.





Conjecture (Random wave model [Berry '77]):

For **chaotic systems** eigenfunctions behave locally like a superposition of plane waves with random amplitude, phase and direction.

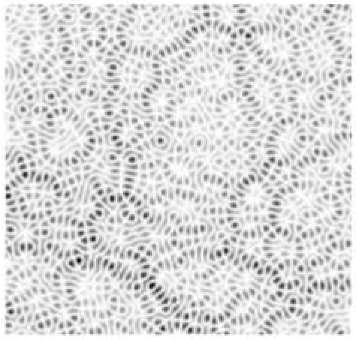
On a domain $\Omega \subset \mathbb{R}^2$ a random wave may be written as

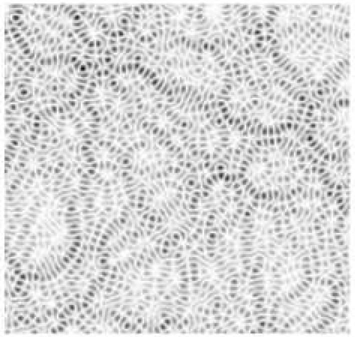
$$f(\boldsymbol{q}) = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} a_n \cos(\boldsymbol{k}_n \boldsymbol{q} + \varepsilon_n) \quad , \tag{23}$$

where:

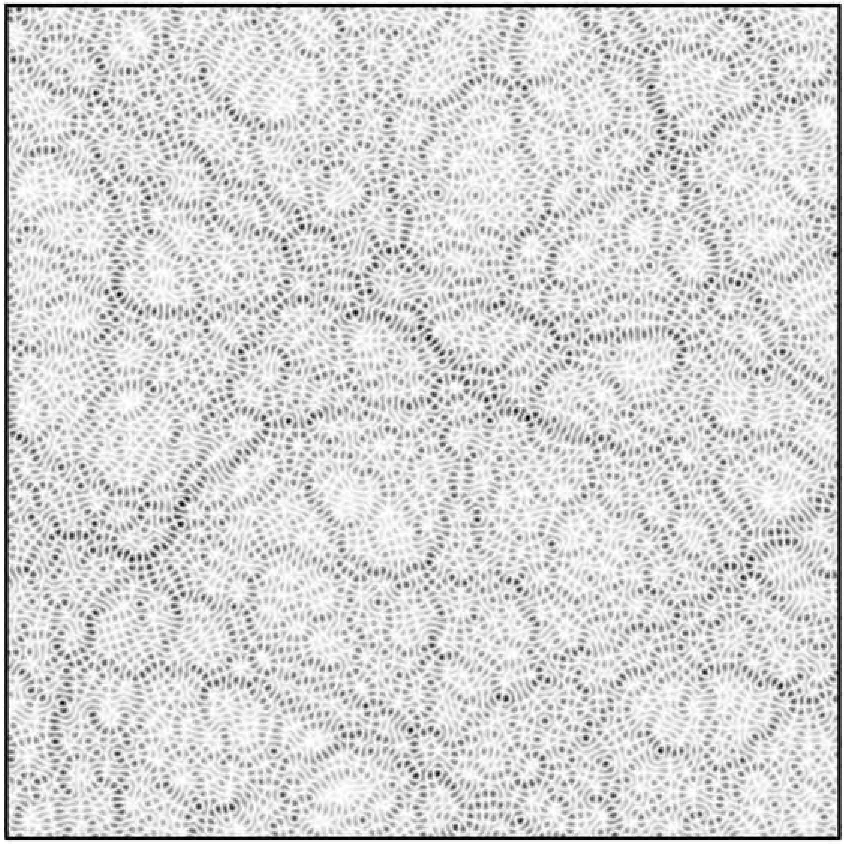
- $a_n \in \mathbb{R}$ are independent Gaussian random variables,
- momenta $\boldsymbol{k}_n \in \mathbb{R}^2$ are randomly equidistributed on the circle of radius $\sqrt{E}$, i.e. $|\boldsymbol{k}_n| = \sqrt{E}$,
- ε_n are equidistributed random variables on $[0, 2\pi[$,
- f is a normalized random function on D when $\mathrm{vol}(D) = 1$.

IV SoE – Random wave model – chaotic systems

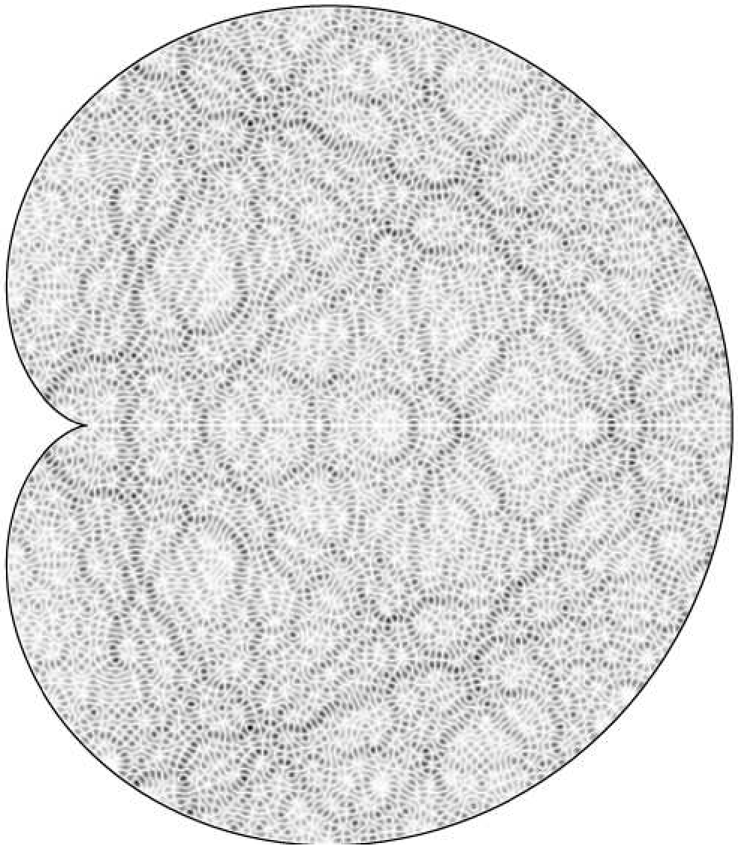




IV SoE – Random wave model – chaotic systems



Random wave



6000^{th} eigenfunction, cardioid billiard

Amplitude distribution is Gaussian

For the amplitude distribution $P_n(\psi)$ of an eigenfunction $\psi_n(\boldsymbol{q})$

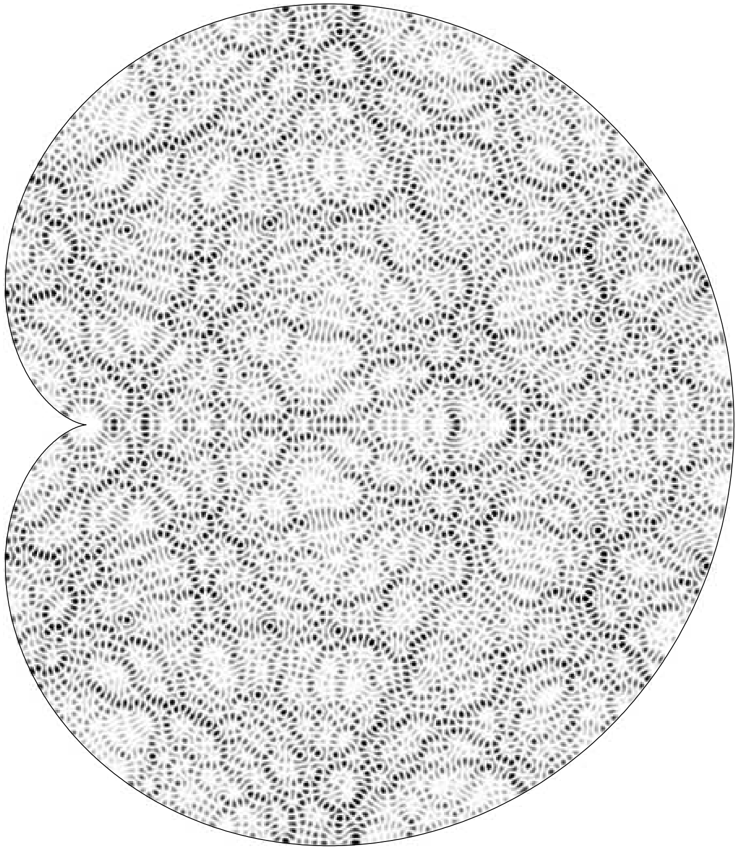
$$\frac{\mathrm{vol}(\{\boldsymbol{q} \in \Omega \mid \psi_n(\boldsymbol{q}) \in [a, b] \subset \mathbb{R}\})}{\mathrm{vol}(\Omega)} =: \int_a^b P_n(\psi)\, \mathrm{d}\psi \quad . \tag{24}$$

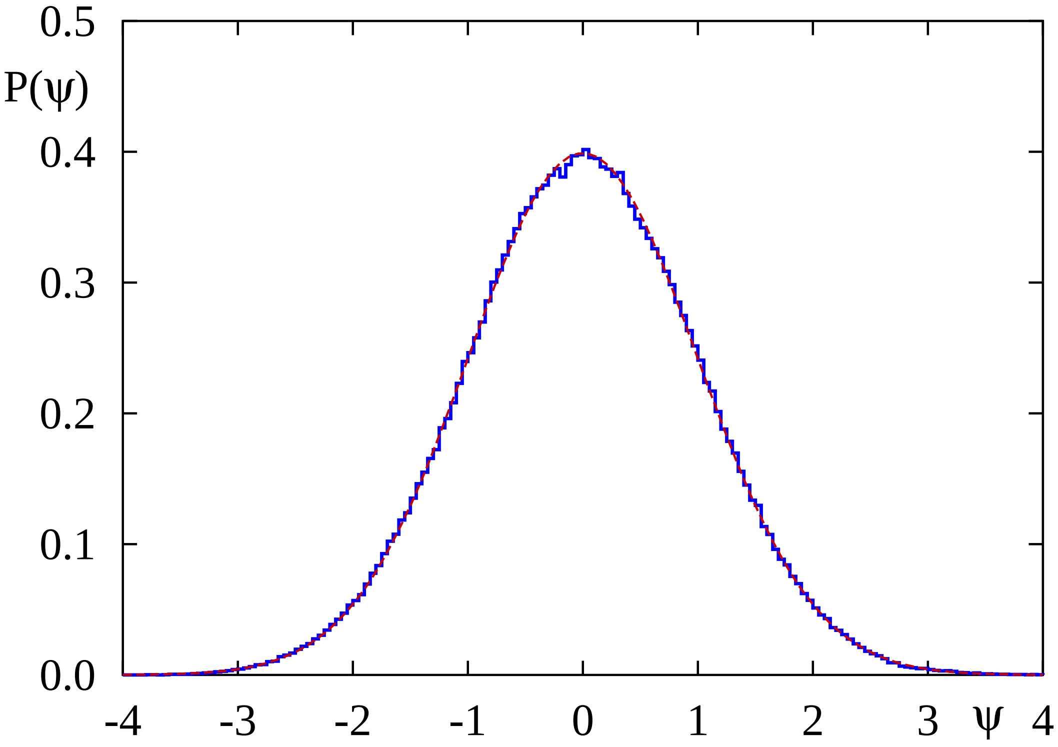
the random wave model implies

$$P(\psi) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{\psi^2}{2\sigma^2}\right) \quad . \tag{25}$$

with variance $\sigma^2 = 1/\,\mathrm{vol}(\Omega)$.

Amplitude distribution – example





Bound on the growth of eigenfunctions

For random waves one has with probability one

(see ([R. Aurich, AB, R. Schubert, and M. Taglieber '99]))

$$\limsup_{E\to\infty} \frac{\max_{x\in\Omega} |f(x)|}{\sqrt{\ln E}} \leq 3\sqrt{2} \; . \tag{26}$$

In contrast to the general result ([Seeger, Sogge '89; Grieser '97])

$$||\psi_n||_\infty \; < \; c\, E_n^{1/4} \quad , \tag{27}$$

which is sharp (e.g. sphere S^2, circle billiard).

- **Conjecture** [Sarnak 95, Iwaniec and Sarnak 95]: for surfaces of constant negative curvature:

$$||\psi_n||_\infty < c_\varepsilon E_n^\varepsilon \quad , \quad \forall \varepsilon > 0 \quad . \tag{28}$$

(related to Lindelöf hypothesis)

- Arithmetic surfaces [Iwaniec and Sarnak 95]: for a Hecke basis

$$||\psi_n||_\infty < c_\varepsilon E_n^{\frac{5}{24}+\varepsilon} \quad , \quad \forall \varepsilon > 0 \quad .$$

$$||\psi_{n_j}||_\infty \geq c\sqrt{\ln \ln E_{n_j}} \; , \quad \text{for a subsequence.}$$

- Arithmetic three manifolds [Rudnick and Sarnak 94, Koyama 95]: there exist

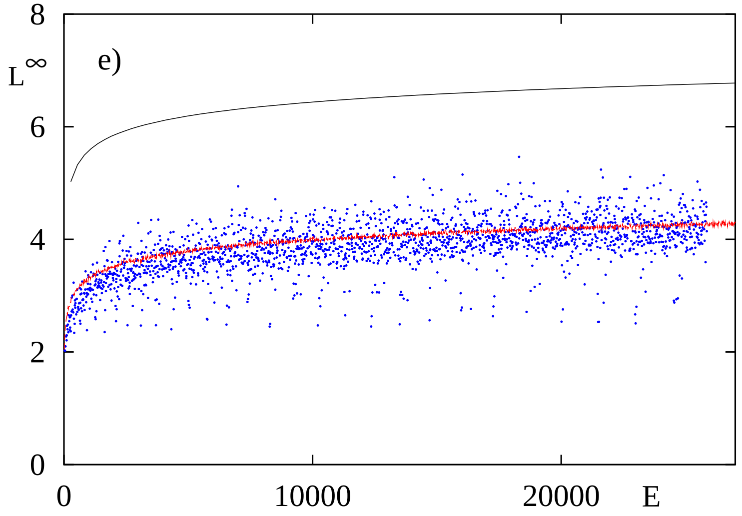
$$\text{systems with} \qquad ||\psi_n||_\infty < c_\varepsilon E_n^{37/76+\varepsilon} \; ,$$

$$\text{and a system with} \qquad ||\psi_{n_j}||_\infty > c E_{n_j}^{1/4} \; ,$$

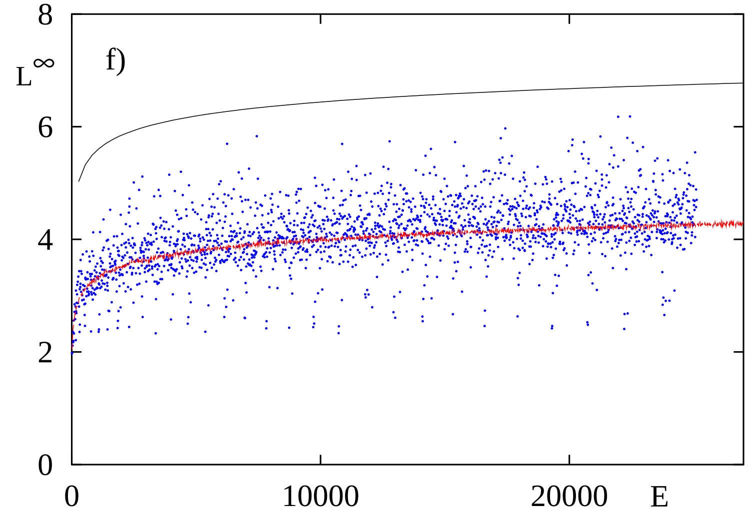
for Hecke eigenfunctions.

Stadium billiard

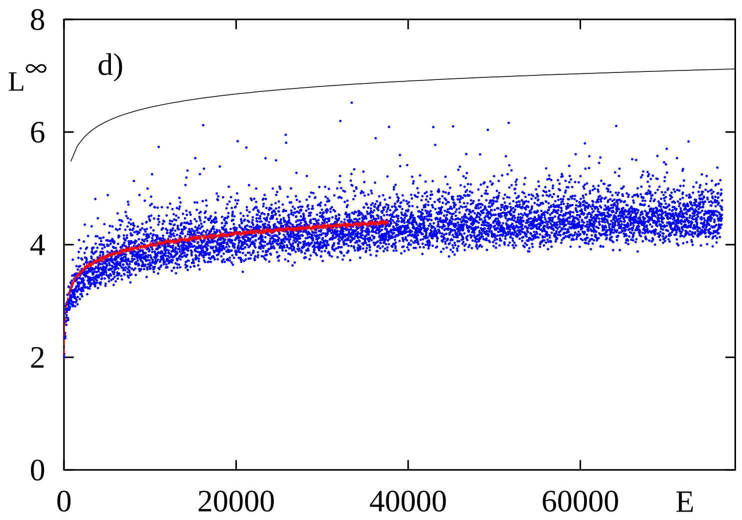
2000 odd-odd eigenfunctions



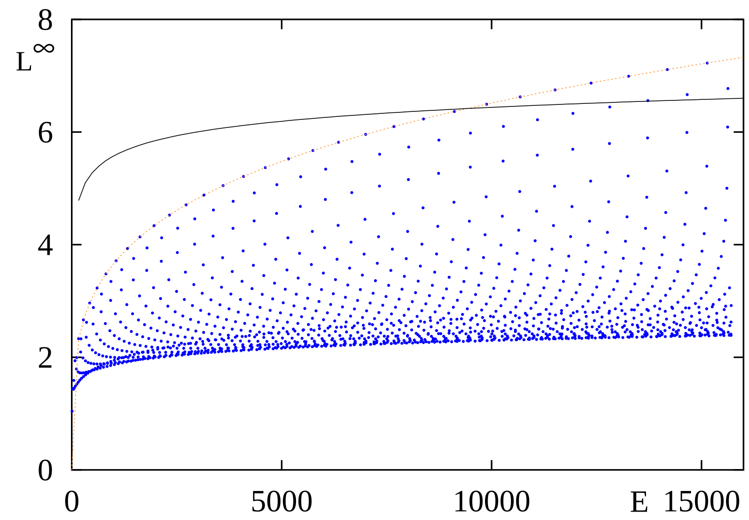
2000 even-even eigenfunctions



Cardioid: 6000 odd eigenfunctions

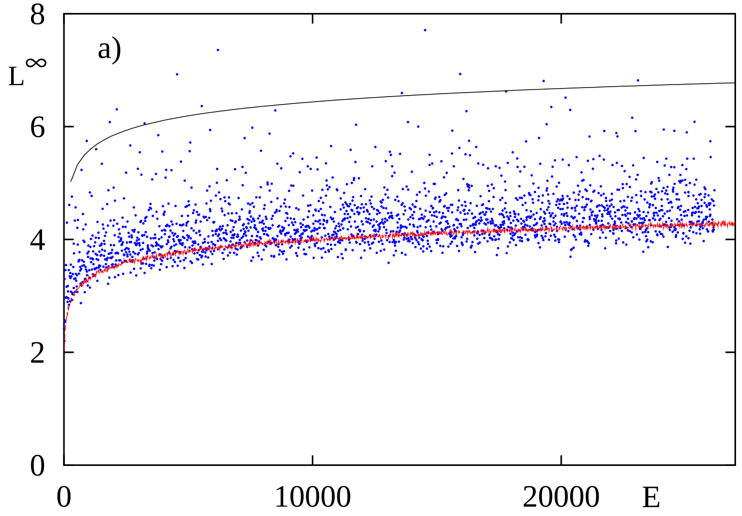


Circle billiard: 1244 eigenfunctions

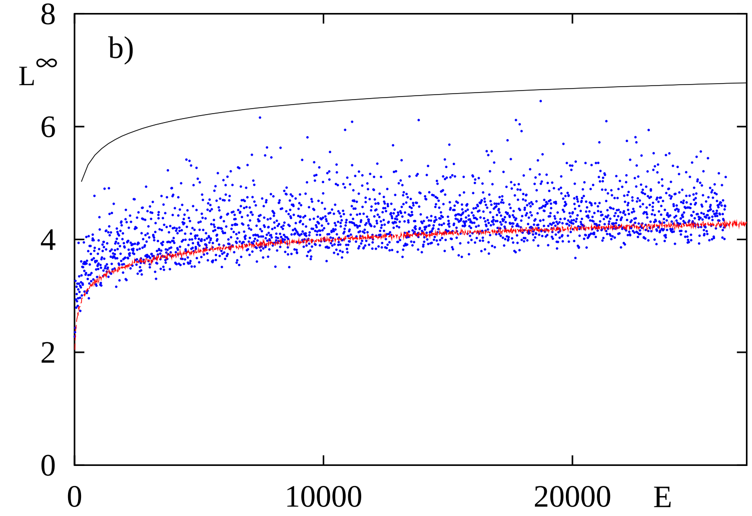


IV SoE – Maximum norms – constant negative curvature

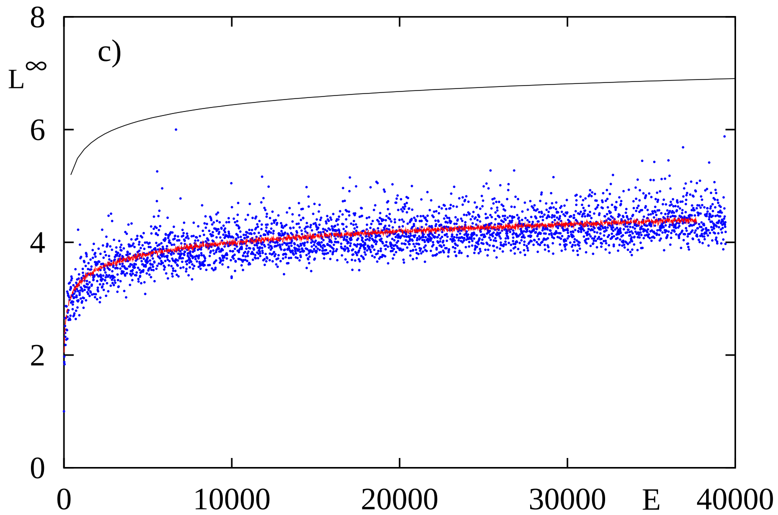
Arithmetic triangle: 2099 functions

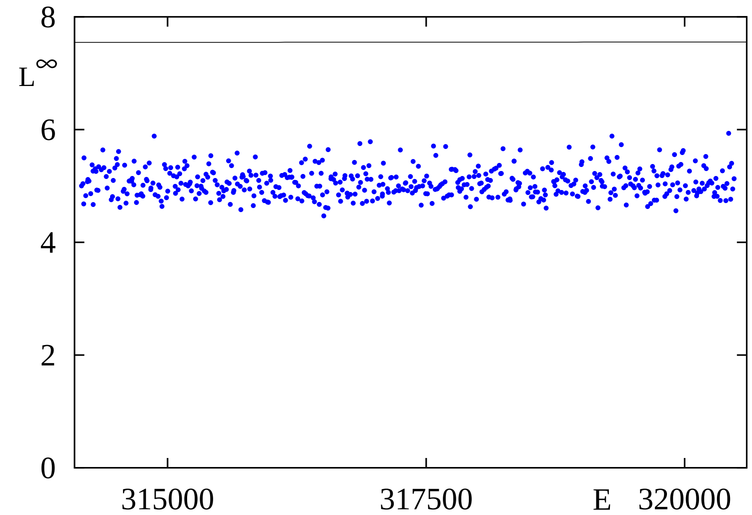


Non-arithmetic triangle: 2092 functions



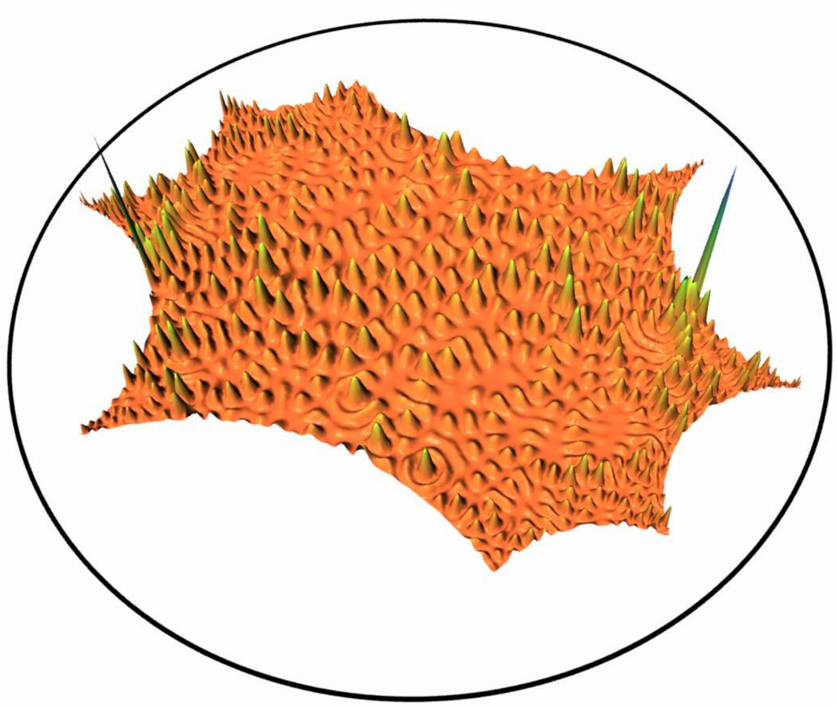
3139 eigenfunctions Octagon 500 eigenfunctions

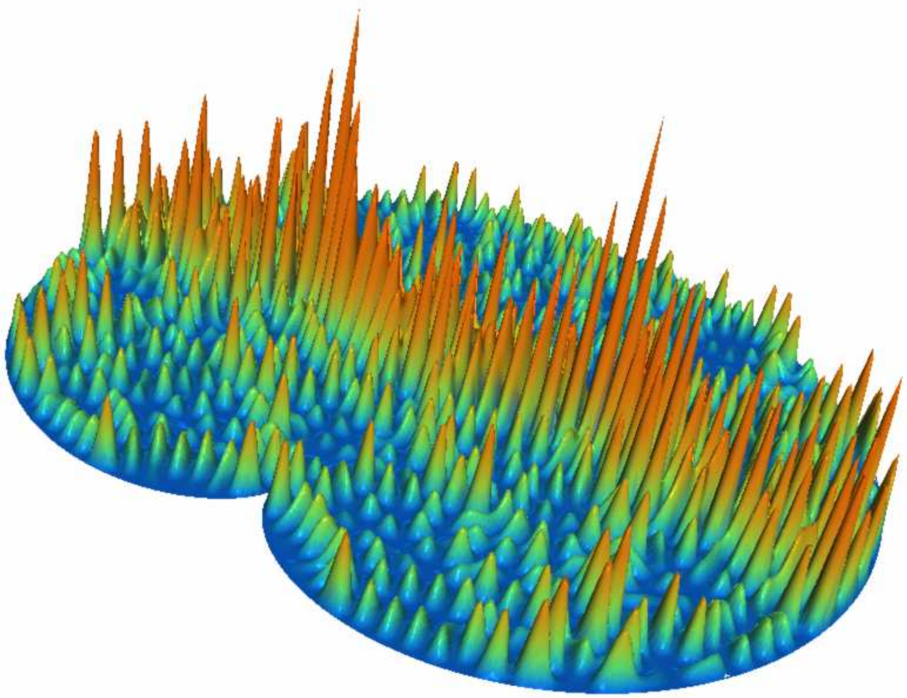




blue – maxima of eigenfunctions

red – mean of maxima of 200 random waves





Semiclassical eigenfunction hypothesis [Berry '77, '83, Voros '79]

The Wigner function

$$W_n(\boldsymbol{p},\boldsymbol{q}) := \frac{1}{(2\pi)^2}\int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{q}'}\psi_n^*(\boldsymbol{q}-\boldsymbol{q}'/2)\psi_n(\boldsymbol{q}+\boldsymbol{q}'/2)\,\mathrm{d}q' \ ,$$

semiclassically concentrates on those regions in phase space, which a typical orbit explores in the long time limit $t \to \infty$.

Implications for

- integrable systems
- chaotic systems

Consequences of the semiclassical eigenfunction hypothesis:

- For **integrable systems**: Localization on invariant tori

$$W(\boldsymbol{p},\boldsymbol{q}) \sim \frac{\delta(I(\boldsymbol{p},\boldsymbol{q})-I)}{(2\pi)^2} \tag{29}$$

(here: $I(\boldsymbol{p},\boldsymbol{q})$: action variable)

- For **chaotic systems**

$$W_n(\boldsymbol{p},\boldsymbol{q}) \to \frac{1}{\mathrm{vol}(\Sigma_E)}\delta(H(\boldsymbol{p},\boldsymbol{q})-E) \quad , \tag{30}$$

i.e. semiclassical condensation on the energy surface Σ_E.

Remark: for ergodic systems one can show ([AB, RS, PS '98])

QET implies the semiclassical eigenfunction hypothesis
(when restricted to a subsequence of density one).

Classical observables are functions on phase space $\mathbb{R}^2 \times \Omega$,

The *mean value* of an observable $a(\boldsymbol{p}, \boldsymbol{q})$ at energy E is given by

$$\overline{a}^E = \frac{1}{\mathrm{vol}(\Sigma_E)} \int\limits_{\Sigma_E} a(\boldsymbol{p}, \boldsymbol{q})\, \mathrm{d}\nu \quad . \tag{31}$$

Weyl symbol $\mathrm{W}[A]$: To an operator A associate

$$\mathrm{W}[A](\boldsymbol{p}, \boldsymbol{q}) := \int\limits_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\boldsymbol{q}'\boldsymbol{p}}\, K_A\left(\boldsymbol{q} - \frac{\boldsymbol{q}'}{2}, \boldsymbol{q} + \frac{\boldsymbol{q}'}{2}\right) \mathrm{d}^2 q' \quad , \tag{32}$$

where K_A is the Schwarz kernel, $A\psi(\boldsymbol{q}) = \int\limits_{\Omega} K_A(\boldsymbol{q}, \boldsymbol{q}')\psi(\boldsymbol{q}')\, \mathrm{d}^2 q'$.

A is called a **pseudodifferential operator**, $A \in S^m(\Omega)$, if its Weyl symbol belongs to a certain class of functions $S^m(\mathbb{R}^2 \times \Omega) \subset C^\infty(\mathbb{R}^2 \times \Omega)$.

Weyl quantization: $a \mapsto \mathrm{A}$

To any function $a \in S^m(\mathbb{R}^2 \times \Omega)$ one can associate an operator $\mathrm{Op}[a] \in S^m(\Omega)$,

$$\mathrm{Op}[a]f(\boldsymbol{q}) := \frac{1}{(2\pi)^2} \iint\limits_{\Omega\times\mathbb{R}^2} \mathrm{e}^{\mathrm{i}(\boldsymbol{q}-\boldsymbol{q}')\boldsymbol{p}}\, a\left(\boldsymbol{p}, \frac{\boldsymbol{q}+\boldsymbol{q}'}{2}\right) f(\boldsymbol{q}')\, \mathrm{d}^2q'\mathrm{d}^2p$$

such that its Weyl symbol is a, i.e. $\mathrm{W}[\mathrm{Op}[a]] = a$.

Classical symbols $S^m_{\mathrm{cl}}(\mathbb{R}^2 \times \Omega) \subset C^\infty(\mathbb{R}^2 \times \Omega)$:

have an asymptotic expansion in homogeneous functions in $\boldsymbol{p}$,

$$a(\boldsymbol{p}, \boldsymbol{q}) \sim \sum_{k=0}^{\infty} a_{m-k}(\boldsymbol{p}, \boldsymbol{q}), \quad \text{with} \quad a_{m-k}(\lambda\boldsymbol{p}, \boldsymbol{q}) = \lambda^{m-k} a_{m-k}(\boldsymbol{p}, \boldsymbol{q})$$

Classical pseudodifferential operators : S^m_{cl}: corresponding class of pseudodifferential operators

$m \in \mathbb{R}$: order of the pseudodifferential operator.

Principal symbol: For $A \in S^m_{\mathrm{cl}}(\Omega)$ and $\mathrm{W}[A] \sim \sum_{k=0}^{\infty} a_{m-k}$

the leading term $a_m(\boldsymbol{p}, \boldsymbol{q})$ is called the *principal symbol* of A

The *principal symbol* denoted by $\sigma(A)(\boldsymbol{p}, \boldsymbol{q}) := a_m(\boldsymbol{p}, \boldsymbol{q})$.

For details see e.g.:

- [AB, R. Schubert, P.Stifter '98]
- [R. Schubert 2001]

The *Wigner function* of a state $|\psi\rangle$ is given as the Weyl symbol of the corresponding projection operator $|\psi\rangle\langle\psi|$

$$\mathrm{W}\left[|\psi\rangle\langle\psi|\right](\boldsymbol{p},\boldsymbol{q}) = \int\limits_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\boldsymbol{q}'\boldsymbol{p}}\, \psi^\star\left(\boldsymbol{q}-\frac{\boldsymbol{q}'}{2}\right)\psi\left(\boldsymbol{q}+\frac{\boldsymbol{q}'}{2}\right)\,\mathrm{d}^2q' \ . \tag{33}$$

From the Wigner function one can recover $|\psi(\boldsymbol{q})|^2$ by

$$|\psi(\boldsymbol{q})|^2 = \frac{1}{(2\pi)^2}\int\limits_{\mathbb{R}^2} \mathrm{W}\left[|\psi\rangle\langle\psi|\right](\boldsymbol{p},\boldsymbol{q})\,\mathrm{d}^2p \ . \tag{34}$$

For the expectation value $\langle\psi, A\psi\rangle$ we have

$$\langle\psi, A\psi\rangle = \frac{1}{(2\pi)^2}\iint\limits_{\Omega\times\mathbb{R}^2} \mathrm{W}[A](\boldsymbol{p},\boldsymbol{q})\,\mathrm{W}\left[|\psi\rangle\langle\psi|\right](\boldsymbol{p},\boldsymbol{q})\,\mathrm{d}^2p\,\mathrm{d}^2q \ .$$

QET [Shnirelman '74, Colin de Verdière '85, Zelditch '87, Zelditch/Zworski '96, ...]

For ergodic systems there exists a subsequence $\{n_j\}$ of density one such that

$$\lim_{j\to\infty} \langle \psi_{n_j}, A\psi_{n_j}\rangle = \overline{\sigma(A)} \ , \qquad (35)$$

for every classical pseudodifferential operator A of order zero.

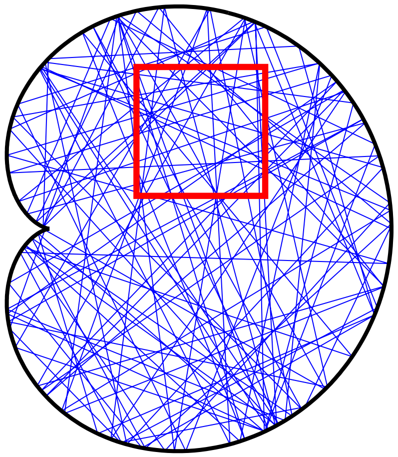
Here $\sigma(A)$ is the principal symbol of A.

And $\overline{\sigma(A)}$ is its classical expectation value,

$$\overline{a} = \frac{1}{\mathrm{vol}(\Sigma_1)} \iint_{\mathbb{R}^2\times\Omega} a(\boldsymbol{p}, \boldsymbol{q})\, \delta(\boldsymbol{p}^2 - 1)\, \mathrm{d}p\, \mathrm{d}q \ . \qquad (36)$$

A subsequence $\{n_j\} \subset \mathbb{N}$ has density one if $\lim_{E\to\infty} \frac{\#\{n_j \mid E_{n_j} < E\}}{N(E)} = 1$,
where $N(E) := \#\{n \mid E_n < E\}$ is the spectral staircase function.

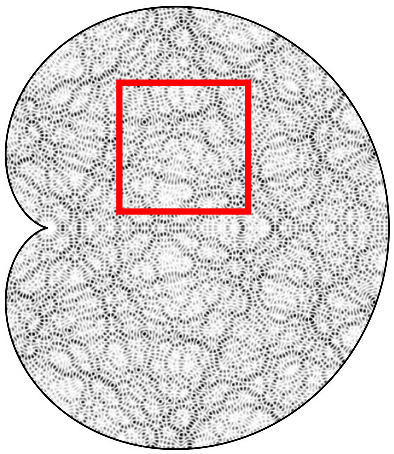
Classical ergodicity of a flow $\{\phi^t\}$



$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \chi_D(\phi^t(\boldsymbol{p},\boldsymbol{q})\, \mathrm{d}t = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

for almost all initial conditions
in phase space, $(\boldsymbol{p},\boldsymbol{q}) \in T^*\Omega$.

Quantum ergodicity in position space



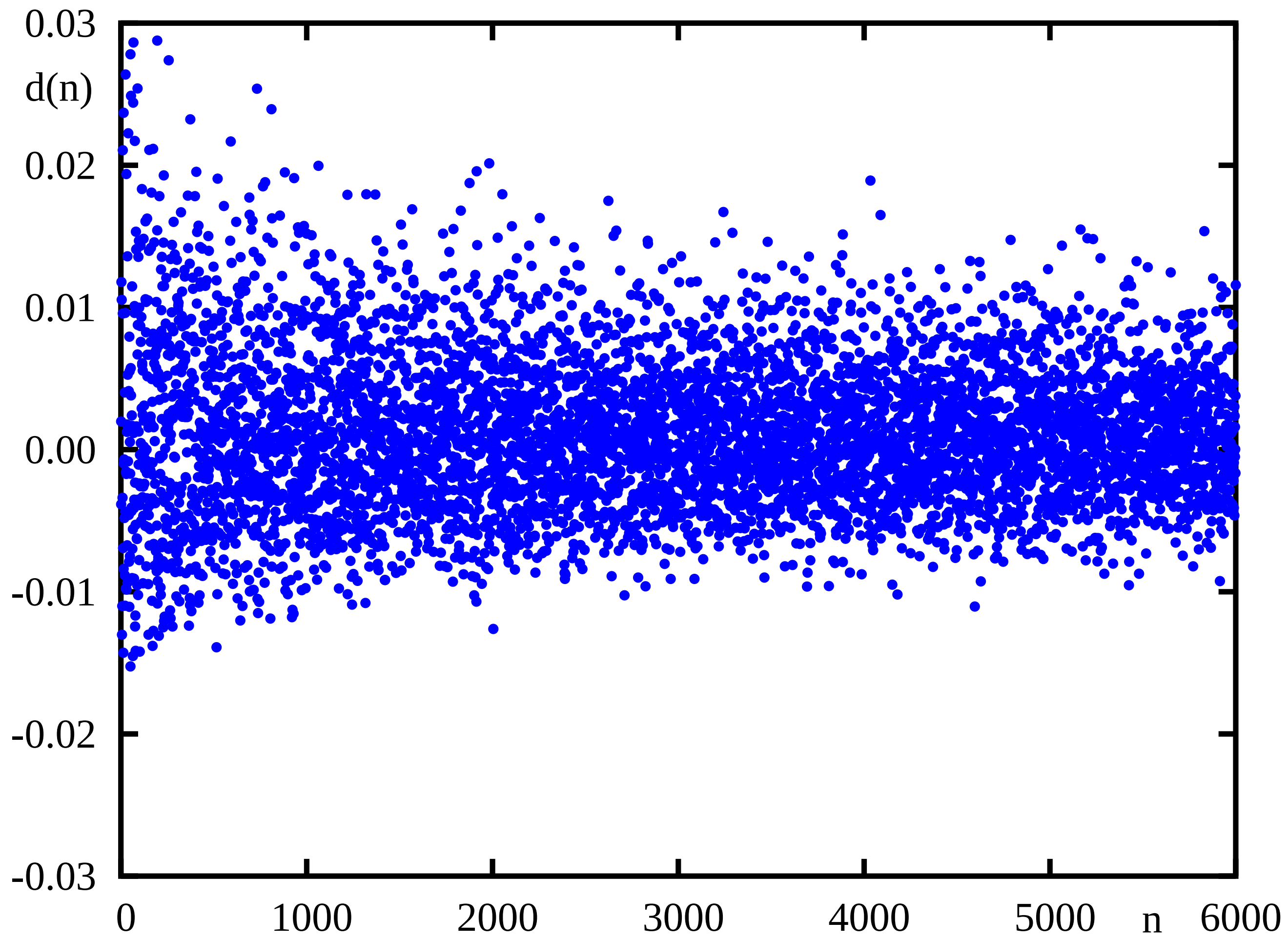
$$\lim_{j\to\infty} \int_{\Omega} \chi_D(\boldsymbol{q})\, |\psi_{n_j}(\boldsymbol{q})|^2 \, \mathrm{d}^2 q = \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)}$$

for a subsequence of density one.

Quantum ergodicity theorem makes statement about sequences of eigenfunctions (weak limit!).

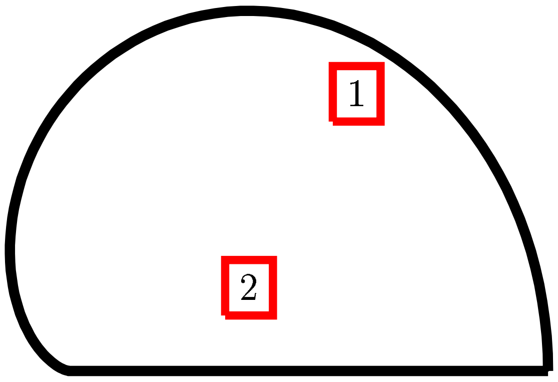
Consider as observable $A = \chi_D(\boldsymbol{q})$ and plot

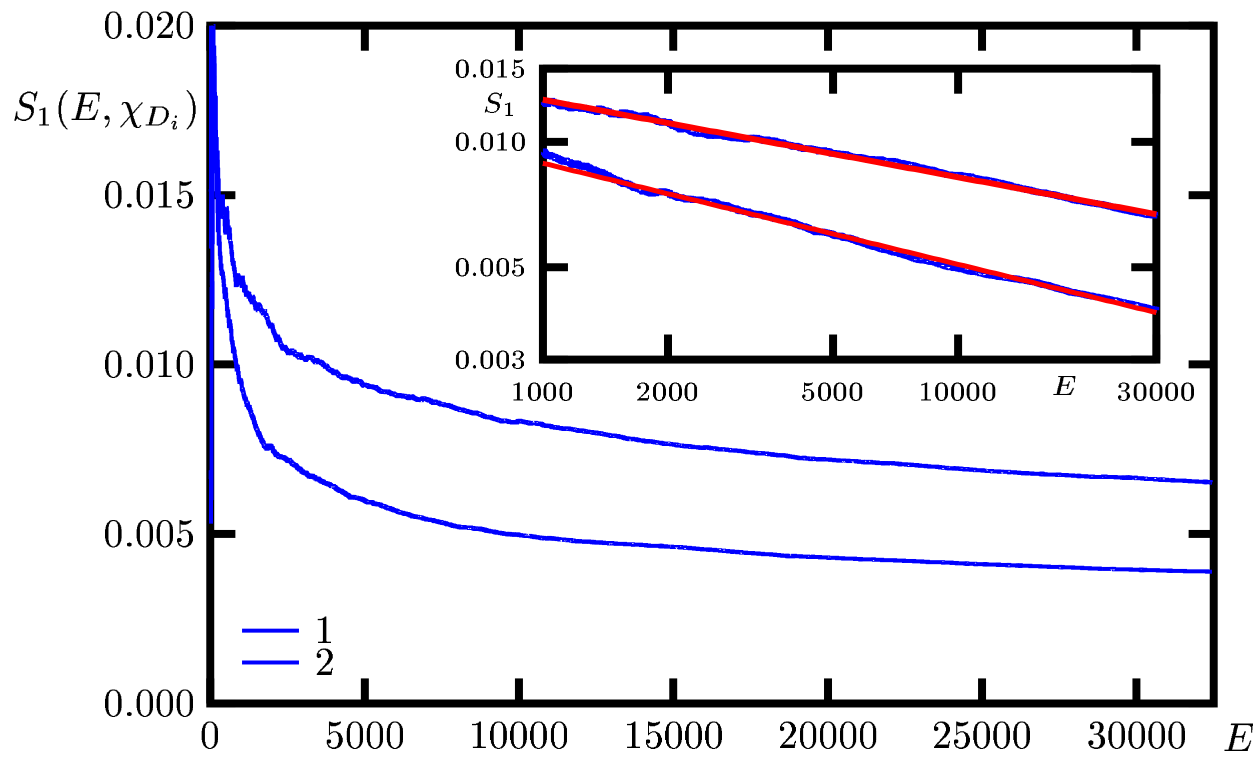
$$\int_\Omega \chi_D(\boldsymbol{q})\, |\psi_{n_j}(\boldsymbol{q})|^2 \, \mathrm{d}^2 q - \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)} \tag{37}$$



Thus consider cumulative differences

$$S_1(E,A) := \frac{1}{N(E)} \sum_{n:E_n \le E} \left| \int_\Omega \chi_D(\boldsymbol{q}) \, |\psi_{n_j}(\boldsymbol{q})|^2 \, \mathrm{d}^2 q - \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)} \right| .$$



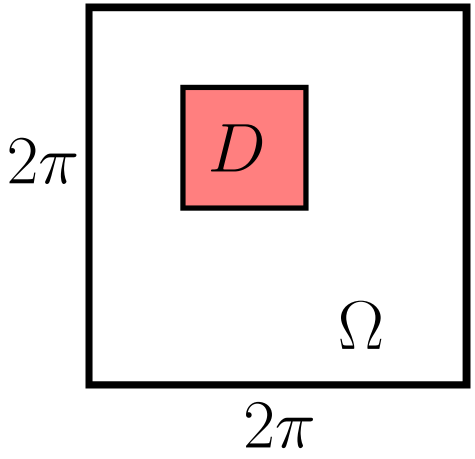


Remark: QET is equivalent to $S_1(E,A) \to 0$ as $E \to \infty$.

Example: Square billiard (to confuse you ... ;-):

$$\psi_{kl}(x,y) = \frac{1}{\pi}\sin(kx)\sin(ly) \quad k,l \in \mathbb{N} \tag{38}$$

Then one gets



$$\iint_D |\psi_{kl}(x,y)|^2 \, \mathrm{d}x\mathrm{d}y$$

$$= \frac{1}{\pi^2}\int_{x_0}^{x_1}\mathrm{d}x\int_{y_0}^{y_1}\mathrm{d}y\,\sin^2(kx)\sin^2(ly) \tag{39}$$

$$\rightarrow \frac{(x_1-x_0)(y_1-y_0)}{\pi^2} \equiv \frac{\mathrm{vol}(D)}{\mathrm{vol}(\Omega)} \tag{40}$$

for a subsequence of density one.

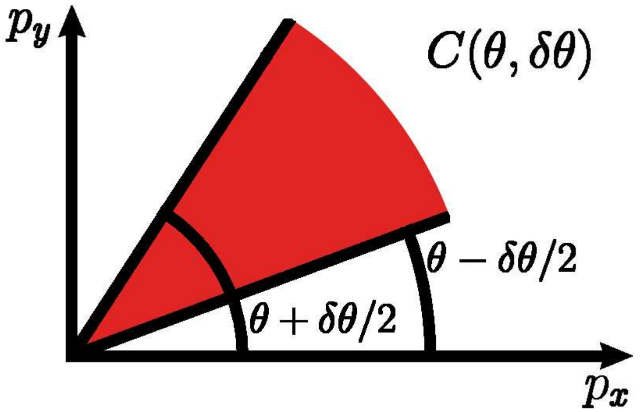
Consider the observable $a(\boldsymbol{p}, \boldsymbol{q}) = a(\boldsymbol{p})$. Then

$$\langle \psi_n, A\psi_n \rangle = \int_{\mathbb{R}^2} |\widehat{\psi}_n(\boldsymbol{p})|^2 a(\boldsymbol{p}) \, \mathrm{d}^2 p \quad , \tag{41}$$

with

$$\widehat{\psi}_n(\boldsymbol{p}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{q}} \psi(\boldsymbol{q}) \, \mathrm{d}^2 q \tag{42}$$

Characteristic function in momentum space



$$a(\boldsymbol{p}) = \chi_{C(\theta,\delta\theta)}(\boldsymbol{p})$$

where

$$C(\theta, \delta\theta) := \left\{ (r, \varphi) \;\middle|\; r \in \mathbb{R}^+, \varphi \in [\theta - \delta\theta/2, \theta + \delta\theta/2] \right\} \tag{43}$$

QET implies for a subsequence of density one:

$$\lim_{n_j \to \infty} \iint_{C(\theta,\delta\theta)} |\widehat{\psi}_{n_j}(\boldsymbol{p})|^2 \, \mathrm{d}^2 p = \frac{\delta\theta}{2\pi} \tag{44}$$

Example: Circle billiard (to confuse you even more ... ;-):

$$\psi_{kl}(r,\phi) = J_k(j_{kl} r) \cos(k\phi) \tag{45}$$

One can show that a subsequence of density one of eigenfunctions is quantum ergodic in momentum space.

Several interesting questions

- Do exceptional eigenfunctions exist ?
  E.g.: scars, bouncing ball modes, ...
  (quantum limit has to be invariant under the flow!)
- If yes, how many are there ?
  The quantum ergodicity theorem implies

$$\lim_{E\to\infty} \frac{N_{\text{exceptional}}(E)}{N(E)} = 0 \quad .$$

  Can one say more about $N_{\text{exceptional}}(E)$?
- How fast do quantum expectation values tend to the corresponding classical limit ?
  I.e., what is the rate of quantum ergodicity ?

Quantum limits (in position space)

Consider the sequence of probability measures on Ω

$$\mathrm{d}\mu_n := |\psi_n(\boldsymbol{q})|^2 \, \mathrm{d}^2 q \tag{46}$$

Definition *A measure μ^{ql} is called quantum limit if a subsequence of the μ_n converges to μ^{ql}.*

QET: for a subsequence of density one the quantum limit is

$$\mathrm{d}\mu = \mathrm{d}^2 q \quad . \tag{47}$$

What quantum limits can occur?

They have to be invariant under the flow!

For a quantum limit μ^{ql} consider

$$\langle \psi_{n_j}, A\psi_{n_j} \rangle = \int_\Omega a(\boldsymbol{q}) |\psi_{n_j}|^2 \, \mathrm{d}^2 q \to \int_\Omega a(\boldsymbol{q}) \mathrm{d}\mu^{\mathrm{ql}} \tag{48}$$

We have (where U_t is the time evolution operator)

$$\langle \psi_{n_j}, A\psi_{n_j} \rangle = \langle \psi_{n_j}, U_{-t} A U_t \psi_{n_j} \rangle \tag{49}$$

Next we use

Theorem (Egorov, special case)
Under certain assumptions

$$\sigma \left(U_t^* A U_t \right) = \sigma(A) \circ \phi^t \tag{50}$$

I.e.: time evolution for finite times and quantization commute in the semiclassical limit.

For

$$\langle \psi_{n_j}, A\psi_{n_j} \rangle = \langle \psi_{n_j}, U_{-t} A U_t \psi_{n_j} \rangle \qquad (51)$$

the Egorov theorem gives

$$\langle \psi_{n_j}, U_{-t} A U_t \psi_{n_j} \rangle = \langle \psi_{n_j}, \mathrm{Op}(a \circ \phi_t), \psi_{n_j} \rangle + \text{lower order terms}$$

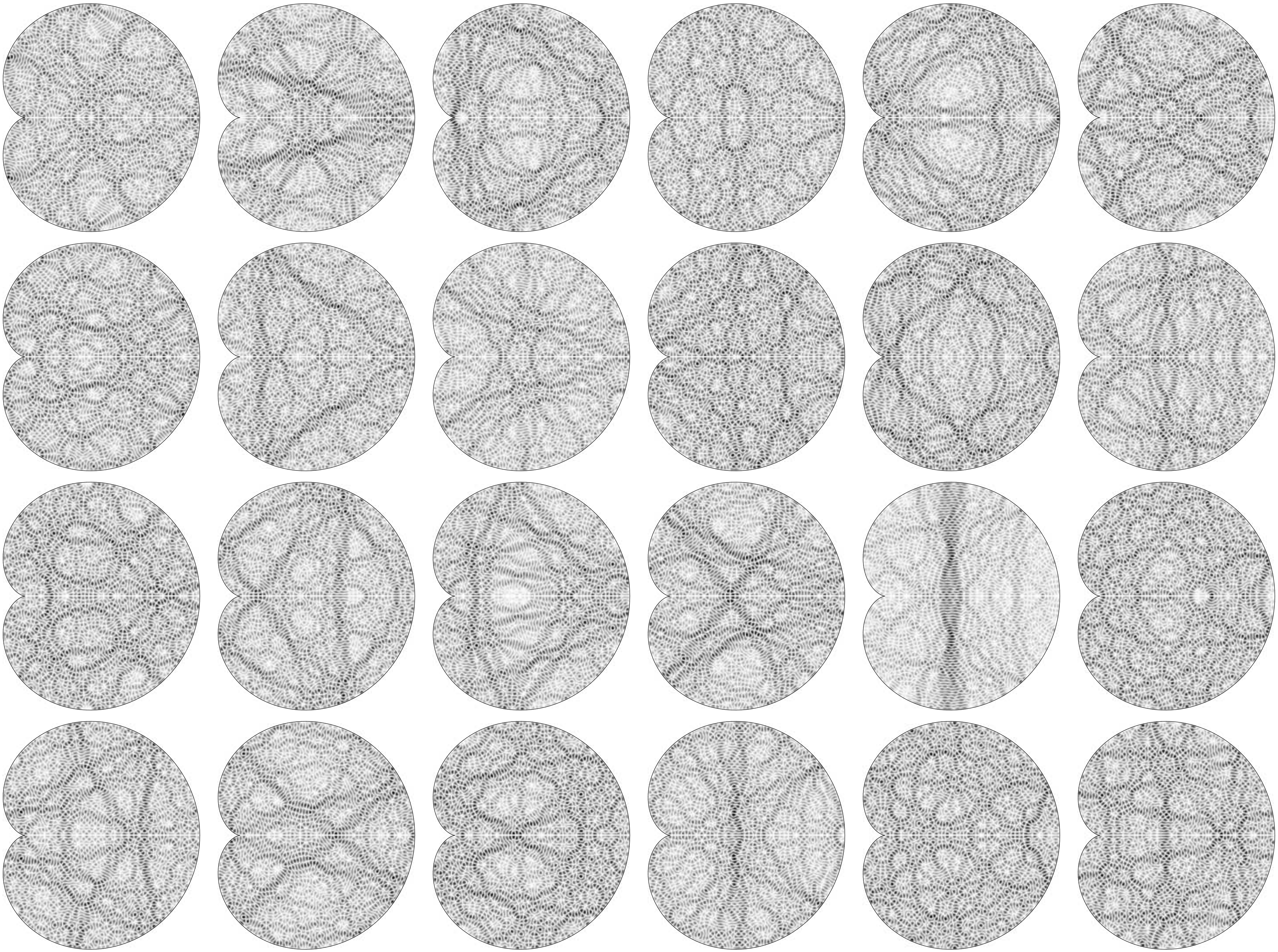
From this: quantum limits are invariant under the flow.

Possible examples:

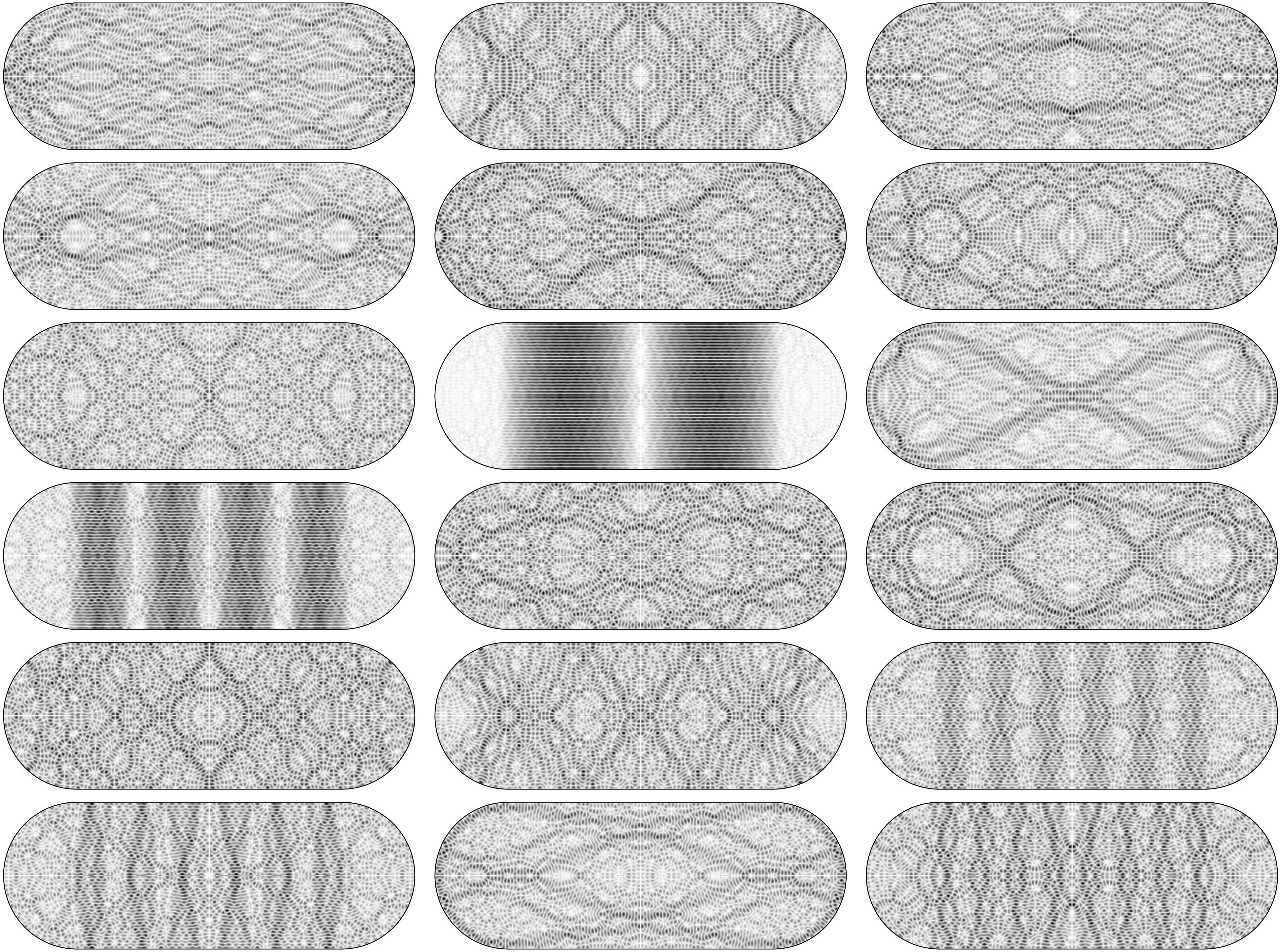
- Liouville measure
- unstable periodic orbits
- collection of finitely/countably many unstable periodic orbits
- marginally stable orbits (eg stadium billiard)

  and: combinations of these

V QET – eigenfunctions cardioid

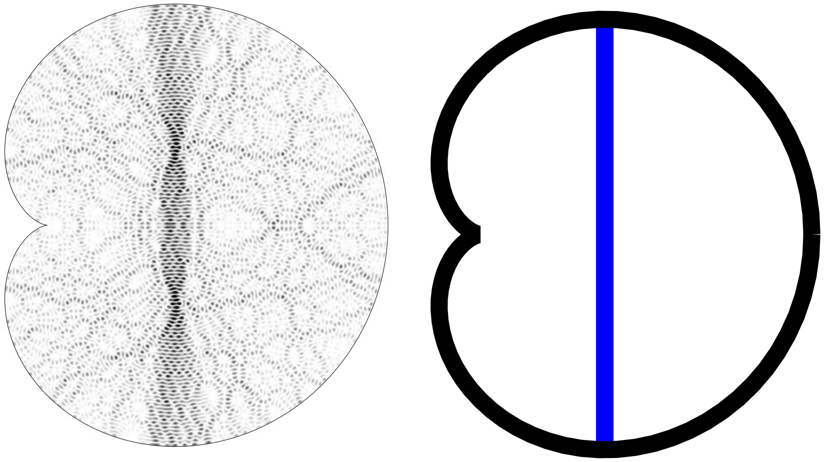


V QET – eigenfunctions stadium



V QET – exceptional eigenfunctions

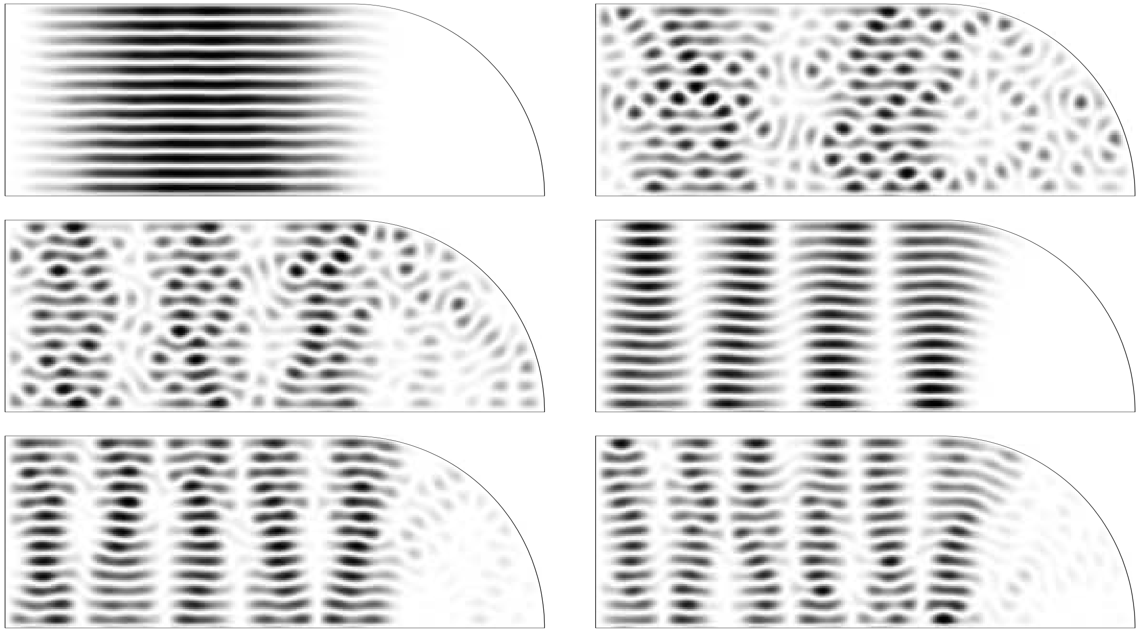
Look at sequence of eigenfunctions in the cardioid billiard . . .



. . . there are states, localizing around unstable periodic orbits (“scars”)

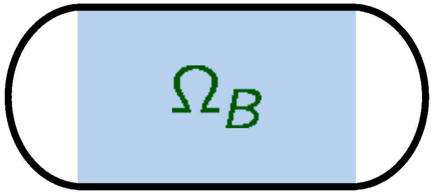
And for the stadium billiard . . .

. . . “Bouncing–Ball–Modes”:



V QET – BBMs – Counting function

Quantum limit for bouncing ball modes:



In position space

$$\lim_{n_j \to \infty} \mathrm{supp}(\psi_{n_j}) \subset \Omega_B \tag{52}$$

and in momentum space

$$\lim_{n_j \to \infty} |\widehat{\psi}_{n_j}|^2 = \delta(p_x) \frac{\delta(p_y - 1) + \delta(p_y + 1)}{2} \tag{53}$$

Consider counting function

$$N_{\mathrm{bb}}(E) := \{ n \mid \psi_n \text{ is a bouncing ball mode} \} \tag{54}$$

The QET implies for $E \to \infty$

$$\frac{N_{\mathrm{bb}}(E)}{N(E)} \to 0 \tag{55}$$

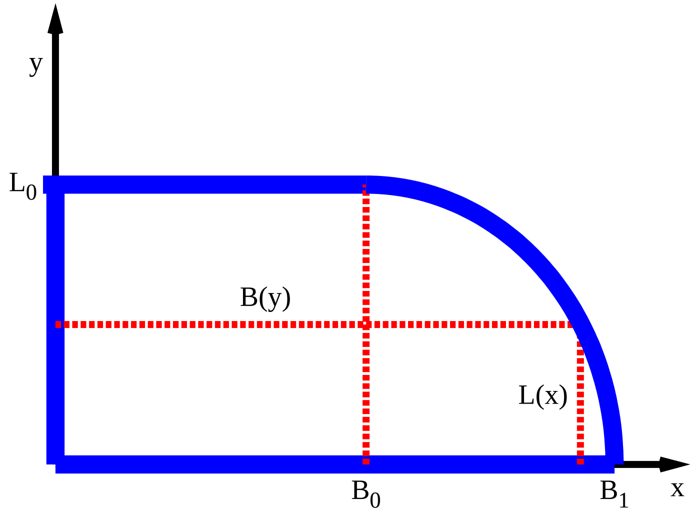
One can show ([G. Tanner '97], [AB, R. Schubert, P. Stifter '97])

- Stadium billiard

$$N_{\mathrm{bb}}(E) \sim cE^{3/4}$$

- Cosine billiard

$$N_{\mathrm{bb}}(E) \sim cE^{9/10}$$



$$L(x) \sim L_0 - C(B_0 + x)^{\gamma}$$

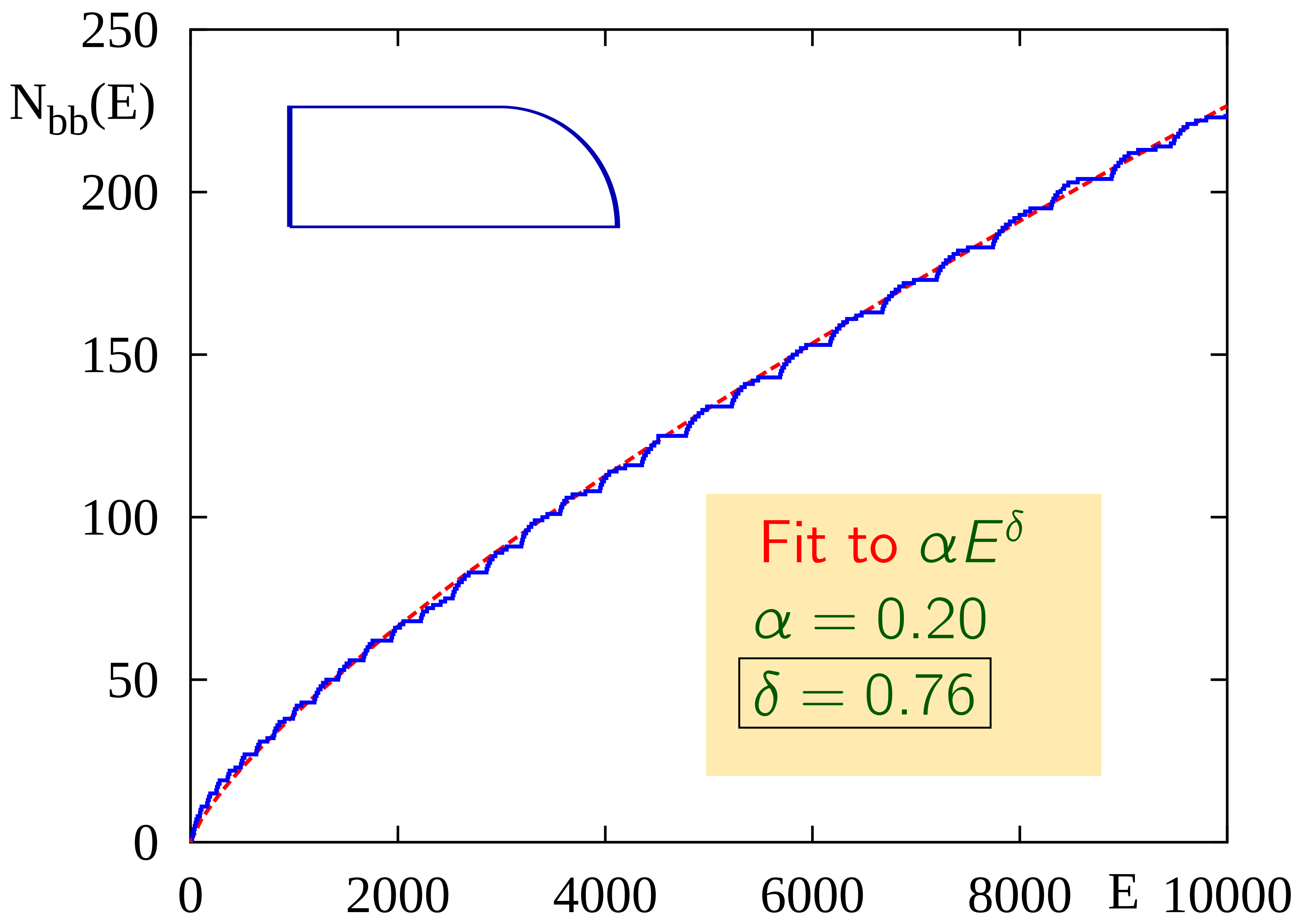
$$\delta = \frac{1}{2} + \frac{1}{2+\gamma} .$$

Remark:

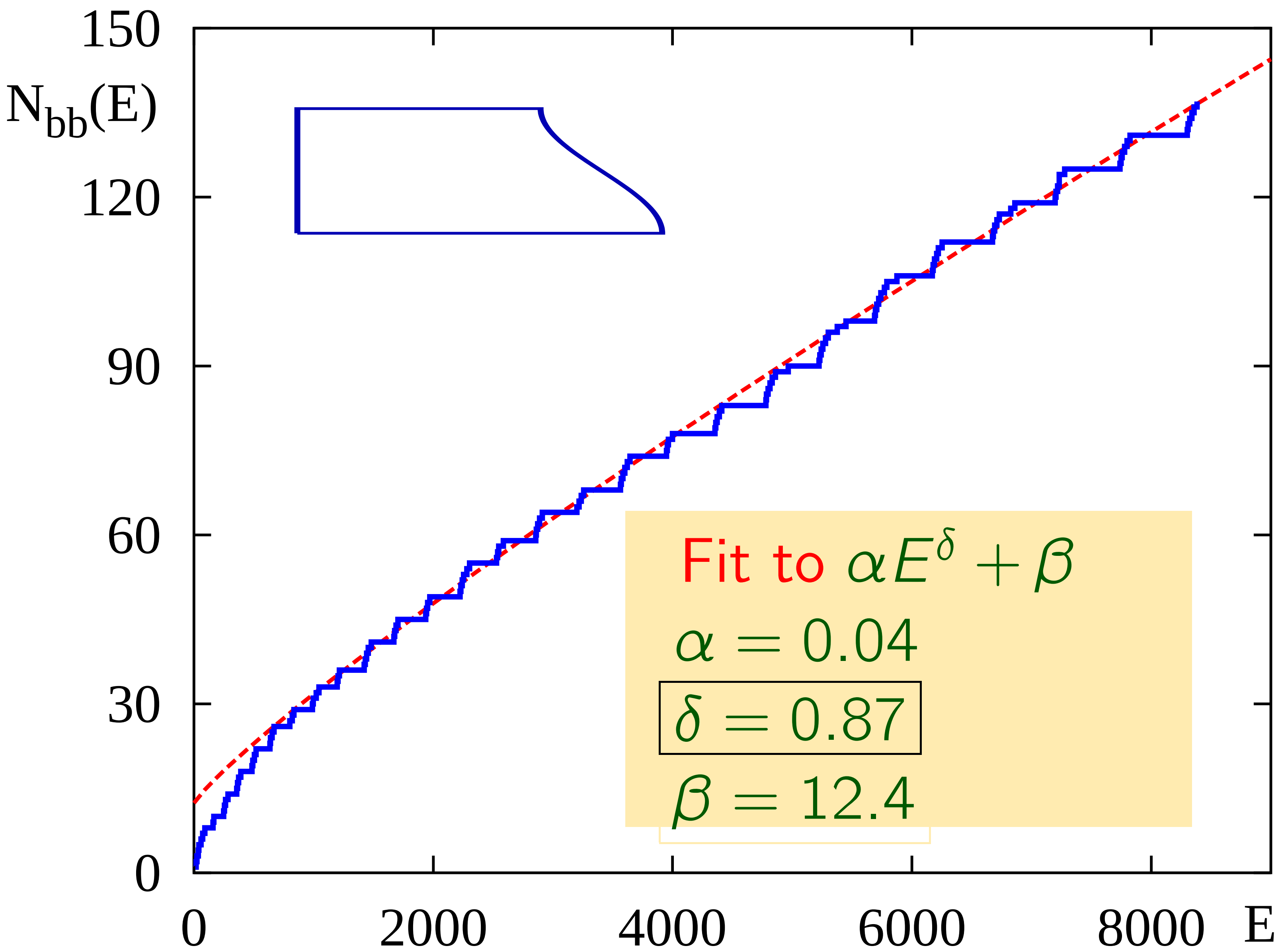
- For every $\frac{1}{2} < \delta < 1$ one can find an ergodic Sinai billiard, s.t. $N_{\mathrm{bb}}(E) \sim cE^{\delta}$.
  This suggests: **the QET is sharp**

- Recent results: [Burq,Zworski 2003], [Zelditch 2003]

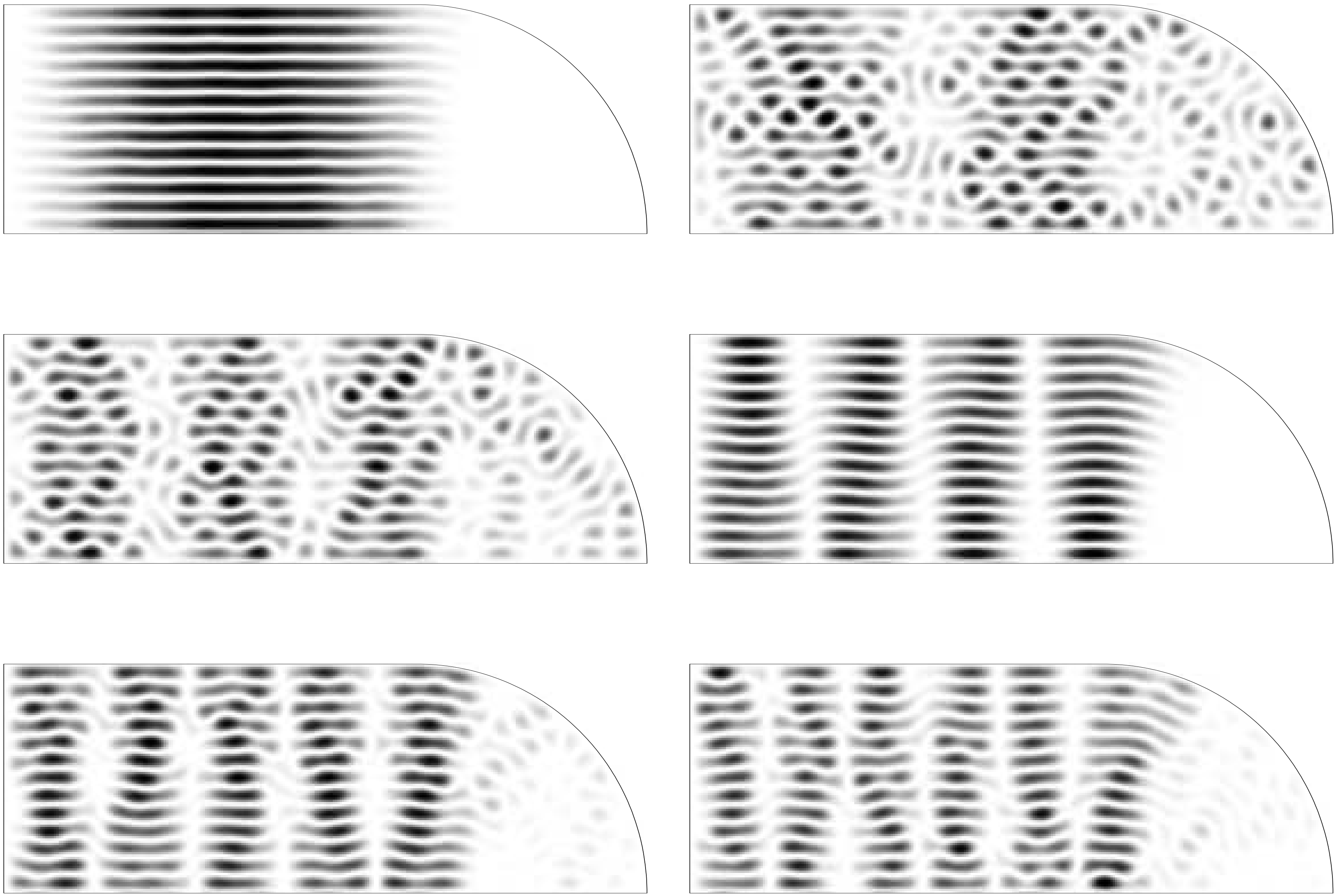
Counting function for bouncing ball modes, stadium billiard



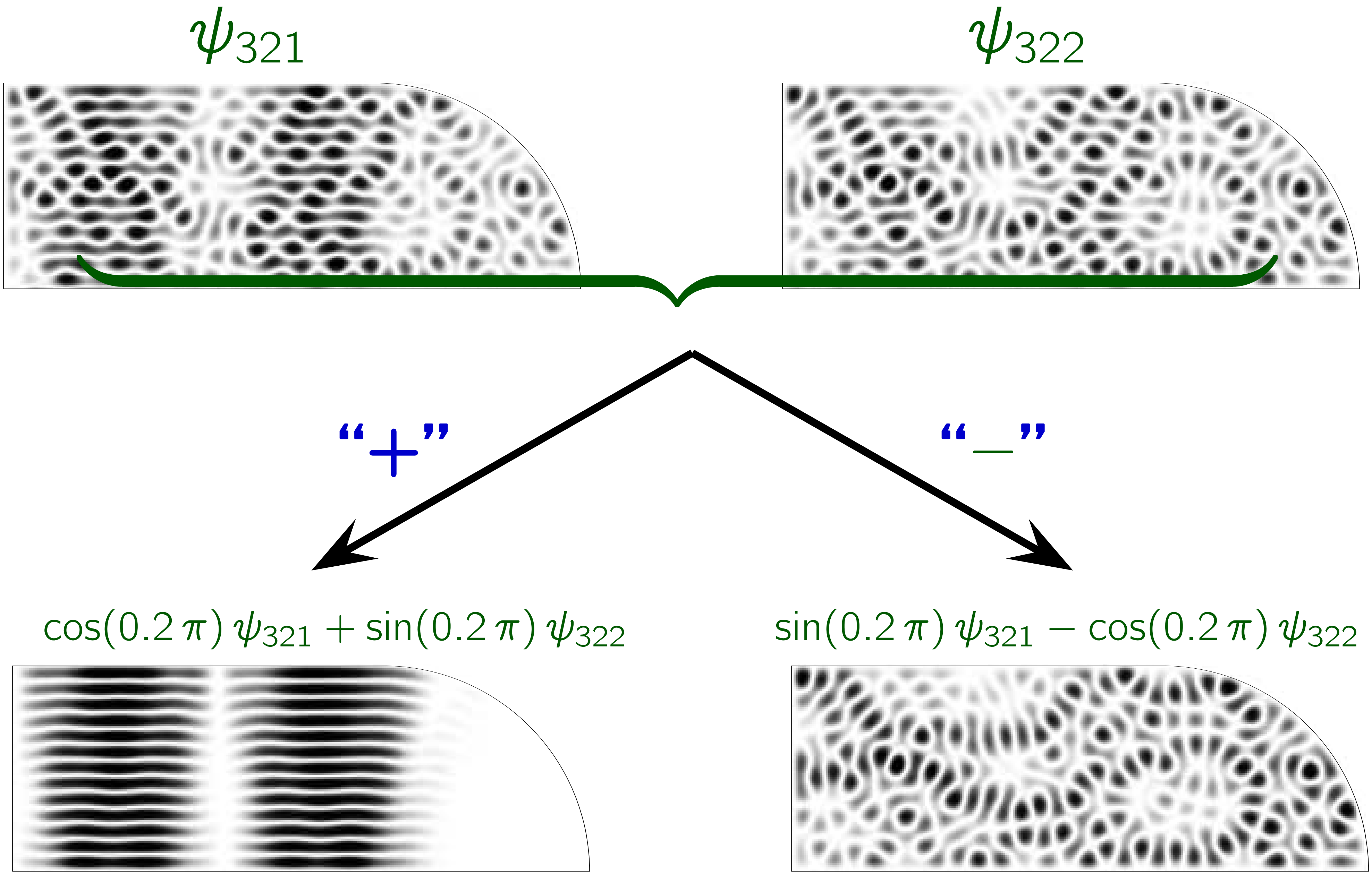
Counting function for bouncing ball modes, cosine billiard



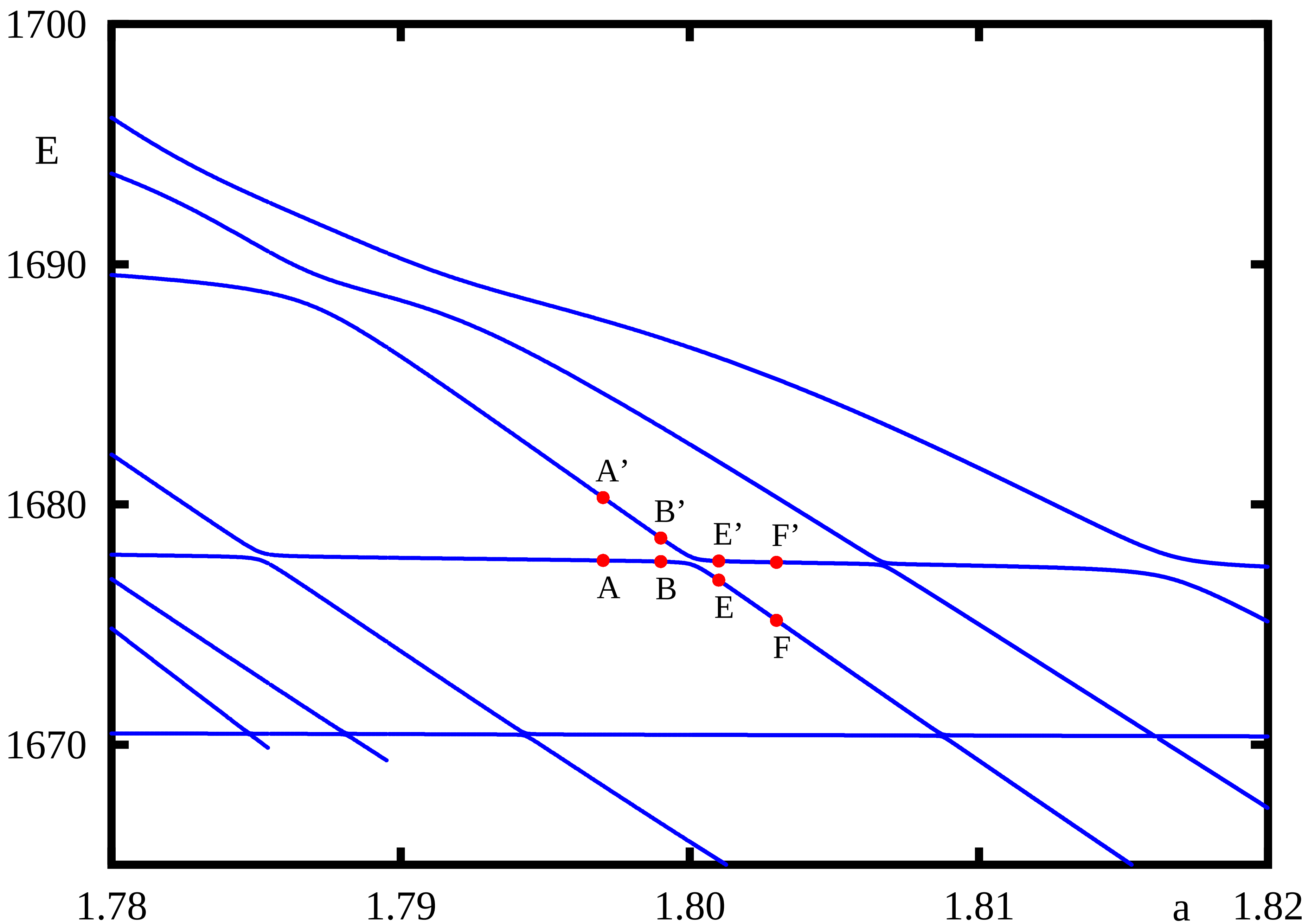
A second look at a sequence of bbm's:



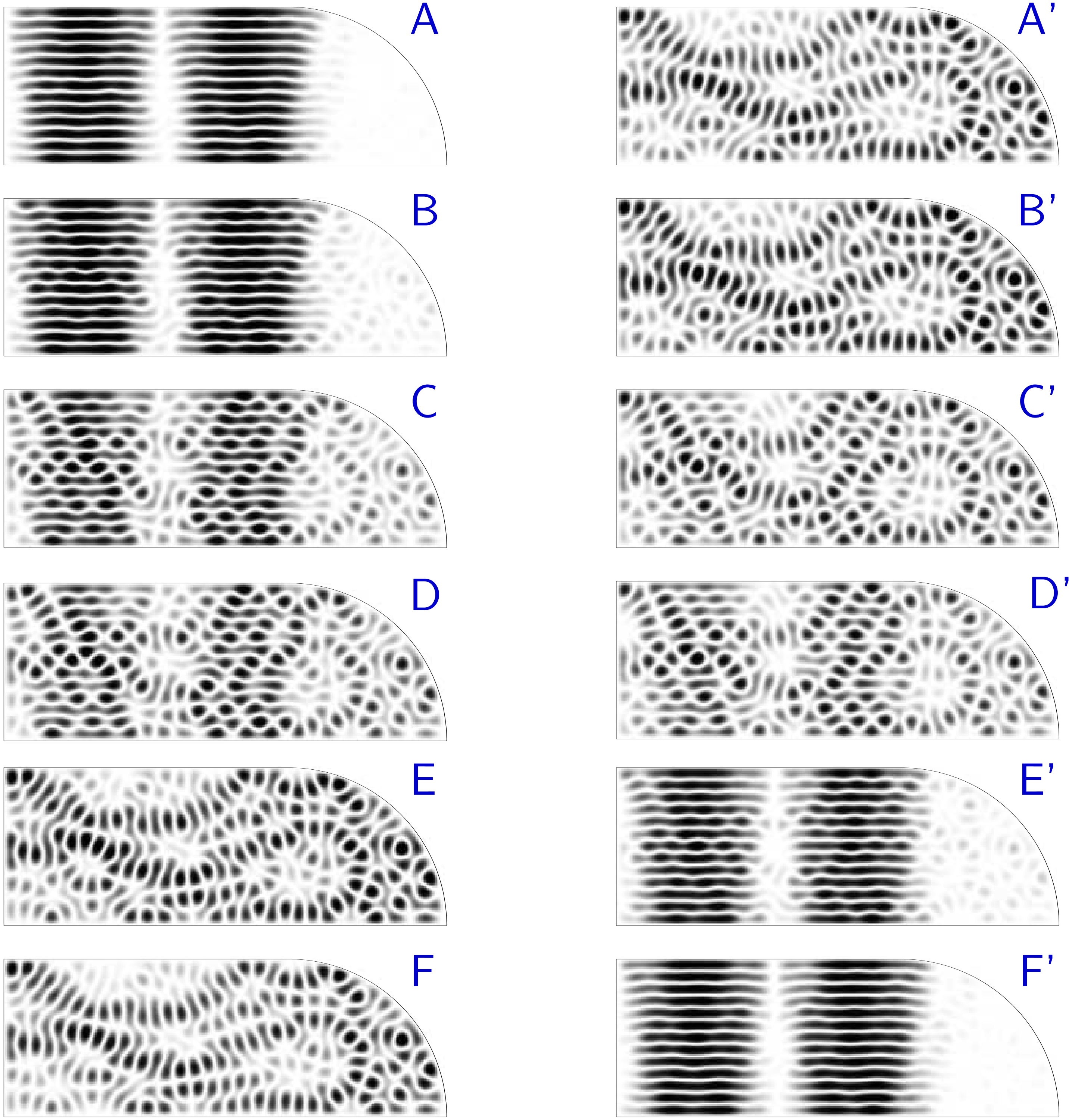
Linear superposition



Parameter variation



V BBMs – Counting function



Definition *A pair* $(\tilde{\psi}, \tilde{E})$, *where* $\tilde{\psi} : \Omega \to \mathbb{R}$ *and* $\tilde{E} \in \mathbb{R}$, *is called quasimode with discrepancy* ϵ *if* $||\Delta\tilde{\psi} + \tilde{E}\tilde{\psi}|| < \epsilon$, *where* $|| \cdot || := \sqrt{\langle \cdot, \cdot \rangle}$

Proposition (Lazutkin '93)

- *The interval* $[\tilde{E} - \epsilon, \tilde{E} + \epsilon]$ *contains at least one eigenvalue of* $-\Delta$.
- *If there is only one eigenvalue* E_n *with eigenfunction* ψ_n *in this interval, then* $||\tilde{\psi} - \psi_n|| < C\epsilon$.
  I.e. the quasimode is an approximate eigenfunction.
- *If there is more than one eigenvalue in this interval*

$$\tilde{\psi}(\boldsymbol{q}) \approx \sum_{E_n \in [\tilde{E}-\epsilon, \tilde{E}+\epsilon]} a_n \psi_n(\boldsymbol{q}) \quad . \tag{56}$$

Basic idea:

Scars are eigenfunctions showing an enhanced density around an unstable periodic orbit

Theoretical studies:

- Heller '84, Kaplan/Heller '98
- Bogomolny '88, Berry '89
- Ozorio de Almeida '98
- ... many others ...

Problems

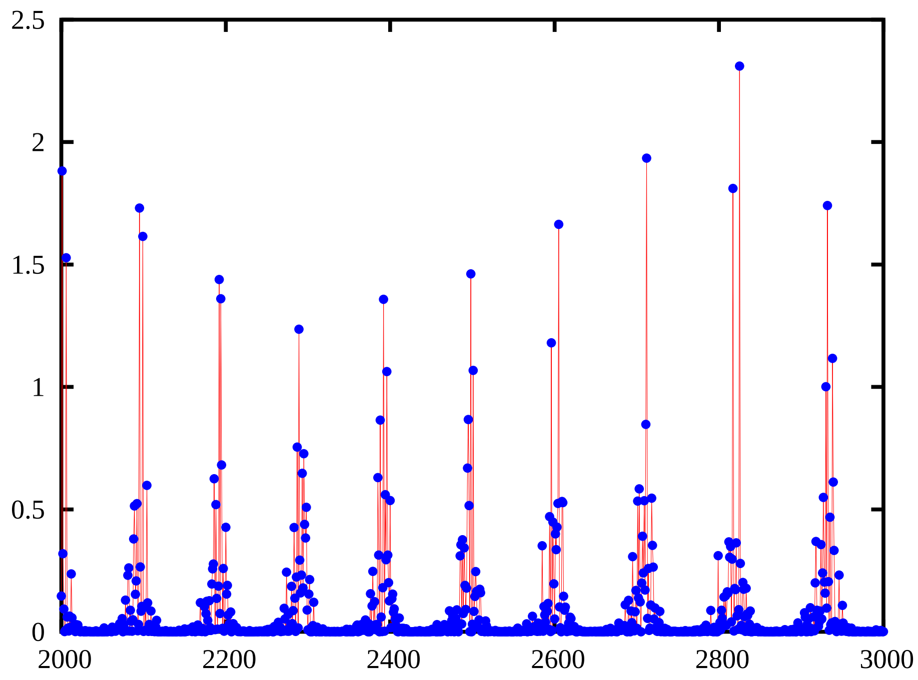
- not really a definition
- not constant in time ;-)

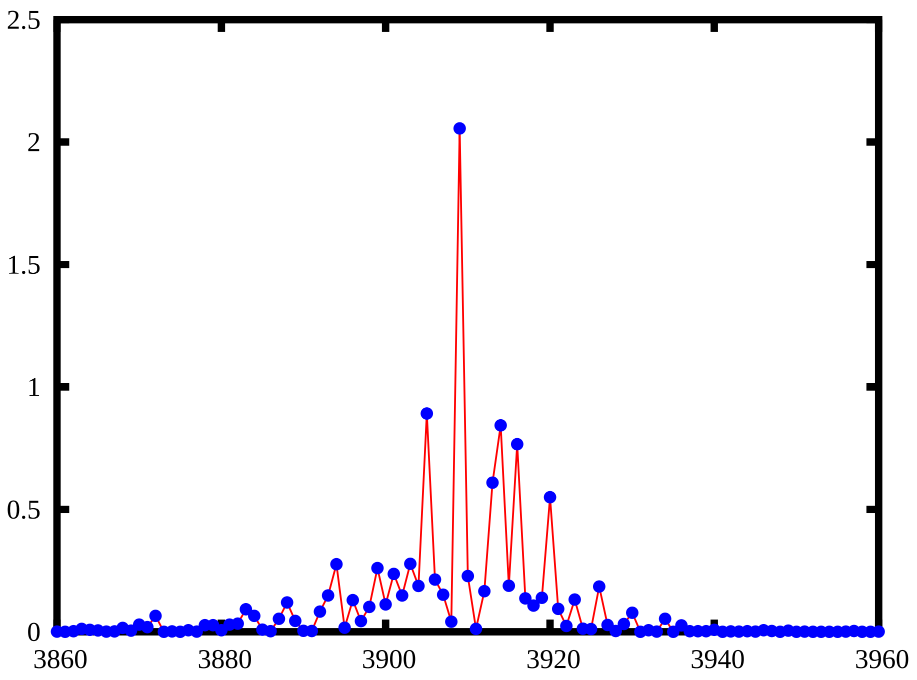
Some more details

Expectation: scars should occur at around energies $E_n^{\text{scar}} = (k_n^{\text{scar}})^2$ where

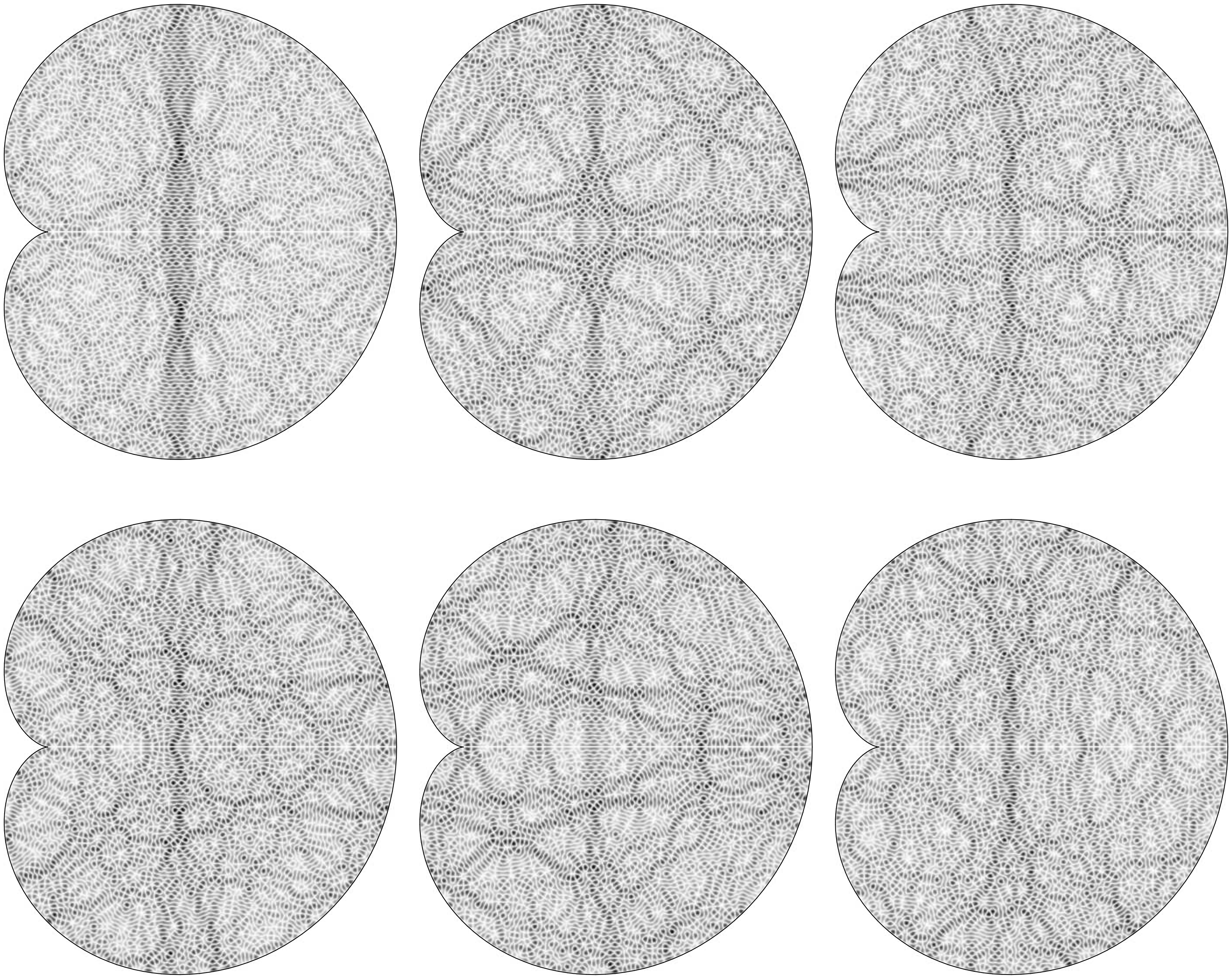
$$k_n^{\text{scar}} = \frac{2\pi}{l_\gamma}\left(n + \frac{\nu_\gamma}{4}\right) \qquad (57)$$

Plot of a scar measure: (via Poincaré Husimi function)

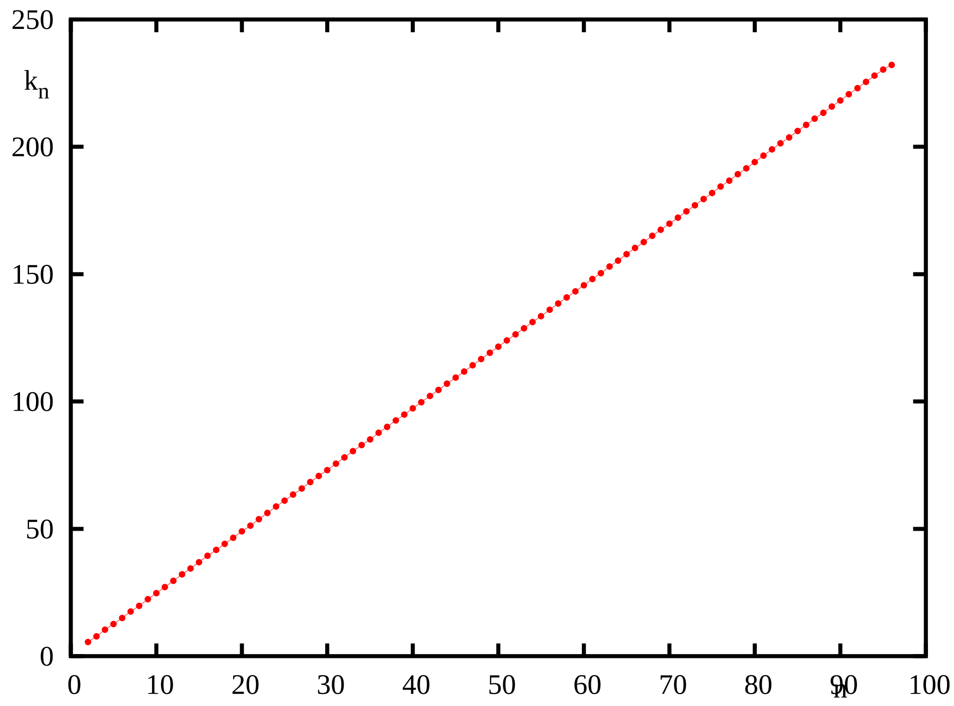




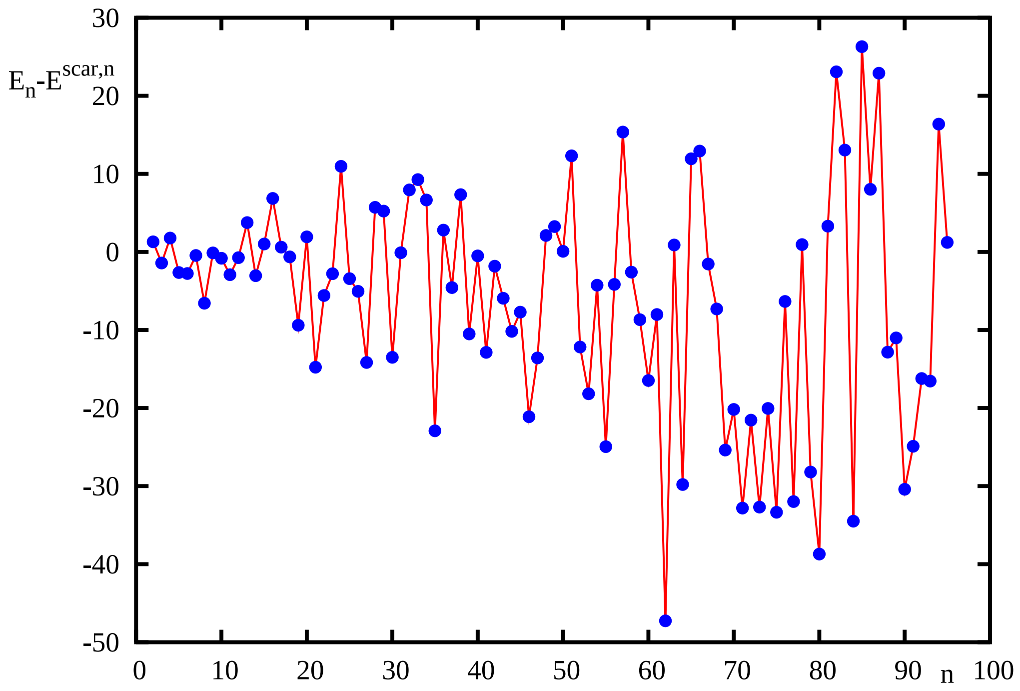
Eigenfunctions in the cluster:



Energies of scars:



difference to $k_n^{\mathbf{scar}}$



Quite strong fluctuations !!

(mean spacing: $\frac{4\pi}{\mathcal{A}} = \frac{4\pi}{3\pi/4} = \frac{16}{3} = 5.333\ldots$)

Remark: For surfaces of constant negative curvature:

no scars observed, see

- Aurich, Steiner '95
- Auslaender, Fishman '98

Possible types of scars: (simplified)

- "super-strong scarring":
  quantum limit is a δ function on a periodic orbit
- strong scarring:
  quantum limit is a δ function on a periodic orbit
  $+$ Liouville measure
- "soft scarring":
  quantum limit is the Liouville measure

Some results on this:

- For any Anosov map on the torus: weight for scars (on a finite union of periodic orbit): $< (\sqrt{5} - 1)/2$

  ([F. Bonechi, S. De Biévre 2003])

- Explicit construction of a sequence of states for the cat map for which the quantum limit is the sum of

  $1/2$ Lebesgue $+ 1/2\ \delta$ on any periodic orbit.

  ([F. Faure, S. Nonnenmacher, S. De Bièvre 2003])

- weight for scars (on a finite or countable union of p.o.): $< 1/2$,

  ([F. Faure, S. Nonnenmacher 2003])

Quantum unique ergodicity:

- proven for ergodic linear parabolic maps on $\mathbb{T}^2$

  ([Marklof, Rudnick 2000])
- for certain cat maps: QUE for joint eigenstates with Hecke operators, ([Rudnick, Kurlberg 2000])

  (not all eigenstates are of this type)
- for sequences of joint eigenstates of the Laplacian and Hecke operators on arithmetic surfaces ([Lindenstrauss 2003])

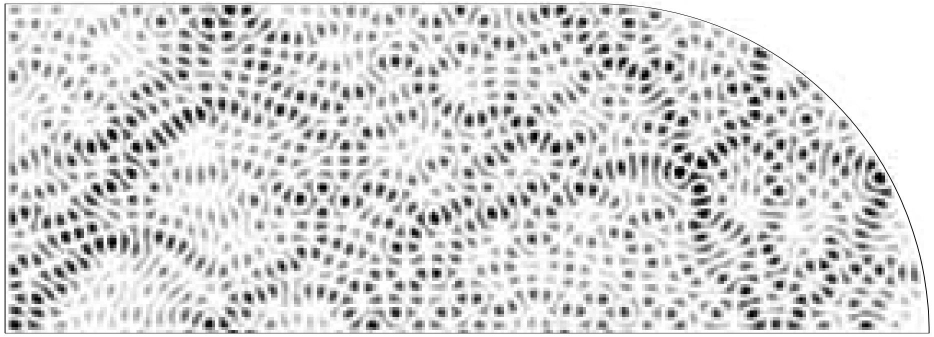
  (all eigenstates are conjectured to be of this type)

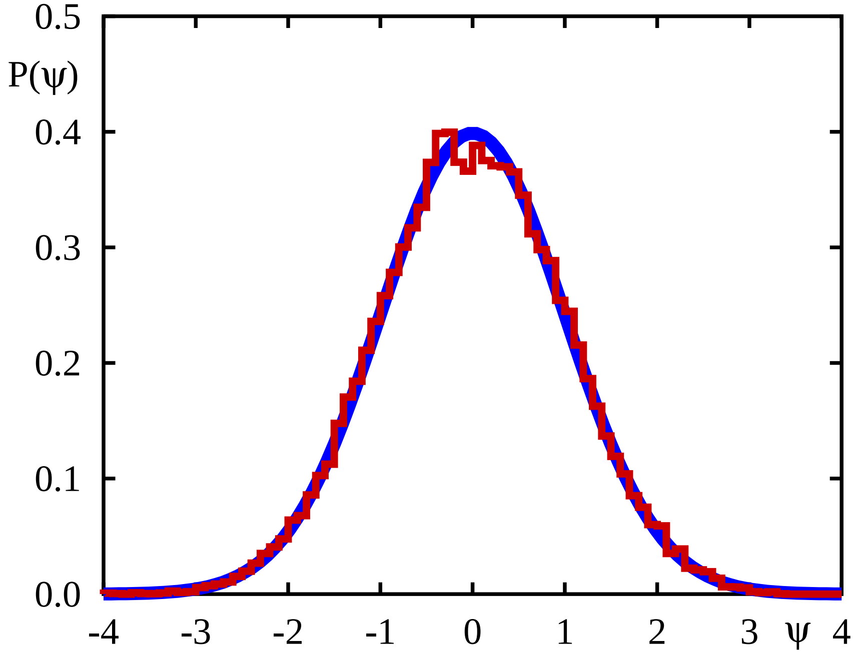
Other extreme:

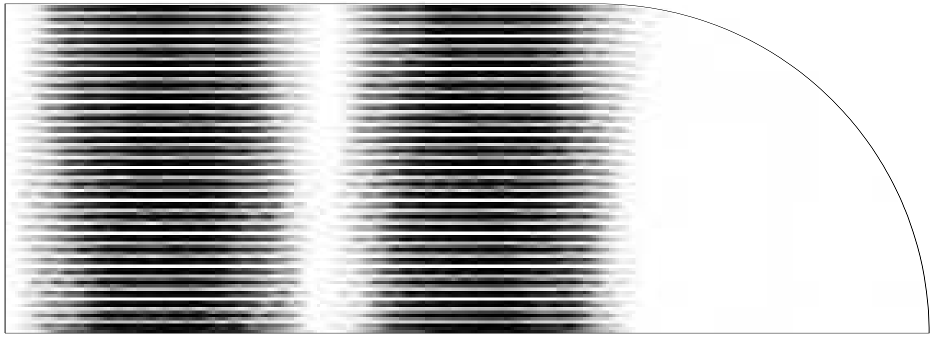
- class of ergodic piecewise affine transformation on $\mathbb{T}^2$:
  all classical invariant measures appear as quantum limits.

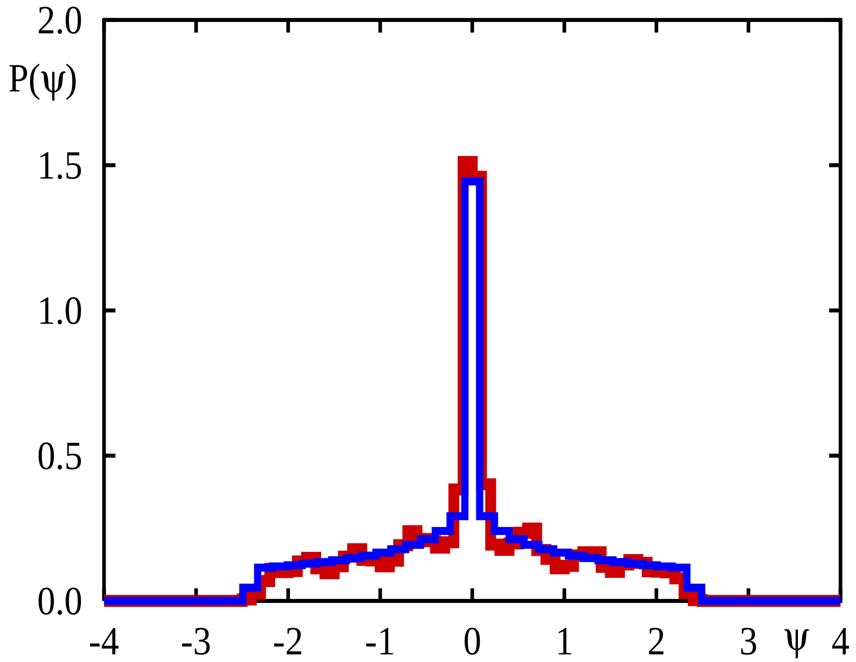
  ([C. Chang, T. Krüger, R. Schubert, S. Troubetzkoy])

Amplitude distribution revisited



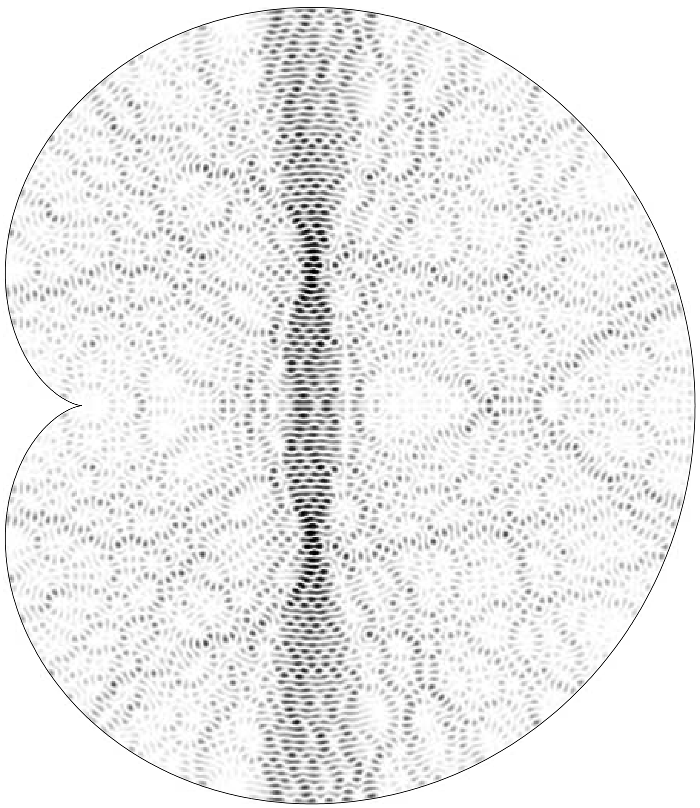


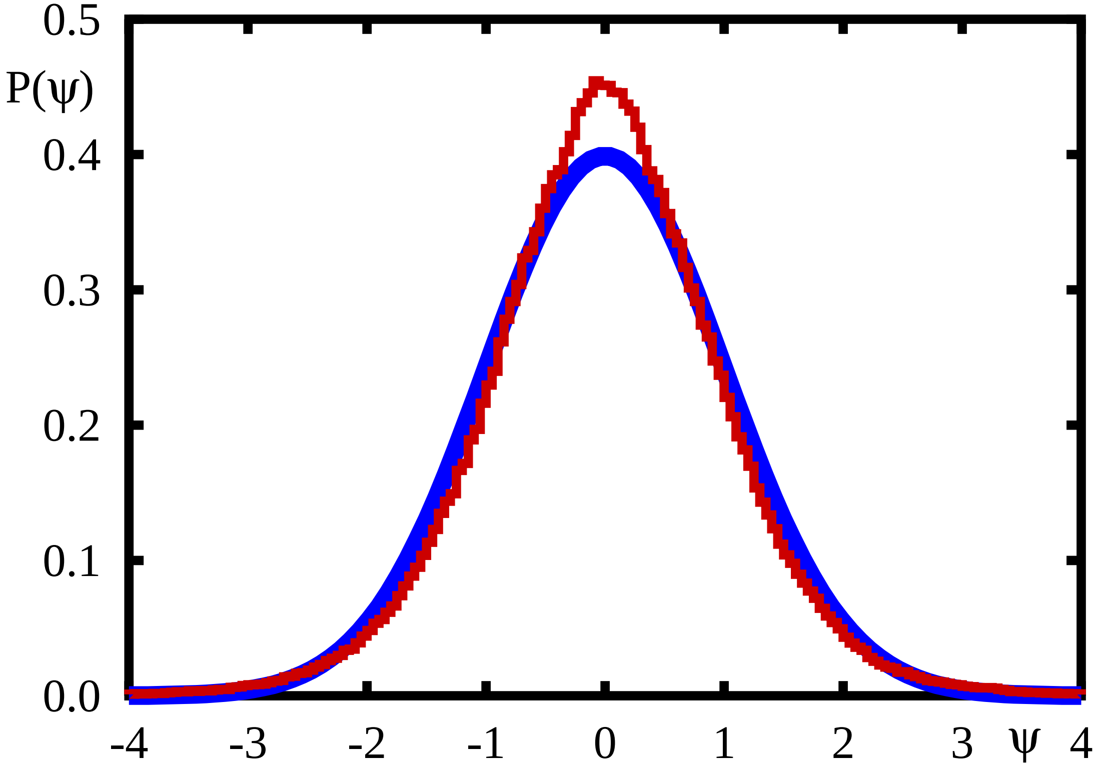




Analytical expression available:

Amplitude distribution for scars





BBM and scars: Modification of the random wave conjecture

Eigenfunctions of classically chaotic systems behave like random waves, but in general only for a subsequence of density one.

Theorem (Szegö limit theorem)
For classical pseudodifferential operators one has

$$\lim_{E\to\infty} \frac{1}{N(E)} \sum_{E_n\le E} \langle \psi_n, A\psi_n\rangle = \overline{\sigma(A)} \tag{58}$$

I.e. quantum mechanical mean values approach the classical mean.

Theorem (Egorov, special case)
Under certain assumptions

$$\sigma\left(U_t^* A U_t\right) = \sigma(A) \circ \phi^t \tag{59}$$

I.e.: time evolution for finite times and quantization commute in the semiclassical limit.

Consider now (following [R. Schubert, 2001])

$$S_2(E,A) := \frac{1}{N(E)} \sum_{n: E_n \le E} \left| \langle \psi_n, A\psi_n \rangle - \overline{\sigma(A)} \right|^2 . \tag{60}$$

Define

$$\overline{A_T} := \frac{1}{T} \int_0^T U_t^* [A - \overline{\sigma(A)}] U_t \, \mathrm{d}t \tag{61}$$

for which (as $U_t|\psi_n\rangle = \exp(-\mathrm{i}tE_n)|\psi_n\rangle$)

$$\langle \psi_n, \overline{A_T}\psi_n \rangle = \langle \psi_n, A\psi_n \rangle - \overline{\sigma(A)} . \tag{62}$$

Thus

$$\left| \langle \psi_n, A\psi_n \rangle - \overline{\sigma(A)} \right|^2 = \left| \langle \psi_n, \overline{A_T}\psi_n \rangle \right|^2 \tag{63}$$

$$\le ||\overline{A_T}\psi_n|| = \langle \psi_n, \overline{A_T}^* \overline{A_T}\psi_n \rangle \tag{64}$$

V Sketch: proof of the QET

With

$$S_2(E,A) \leq \frac{1}{N(E)} \sum_{n:E_n \leq E} \langle \psi_n, \overline{A_T}^* \overline{A_T} \psi_n \rangle \tag{66}$$

and the Szegö limit theorem we then obtain

$$\lim_{E\to\infty} S_2(E,A) \leq \frac{1}{\mathrm{vol}(\Sigma_1)} \int_{\Sigma_1} \sigma(\overline{A_T}^* \overline{A_T})\, \mathrm{d}\mu \tag{67}$$

$$= \int_{\Sigma_1} \sigma(\overline{A_T})^* \sigma(\overline{A_T})\, \mathrm{d}\mu \tag{68}$$

Applying the Egorov theorem we have

$$\sigma(\overline{A_T})(p,q) = \frac{1}{T} \int_0^T \sigma(A) \circ \phi^t(p,q)\, \mathrm{d}t - \overline{\sigma(A)} \ . \tag{69}$$

Thus

$$\lim_{E\to\infty} S_2(E,A) = \lim_{E\to\infty} \frac{1}{N(E)} \sum_{n:E_n\leq E} \left| \langle \psi_n, A\psi_n\rangle - \overline{\sigma(A)} \right|^2 \tag{71}$$

$$= 0 \ . \tag{72}$$

This is the mean value of a sequence of positive numbers.

Thus there exists a subsequence $\{n_j\}$ of density one, such that

$$\lim_{n_j\to\infty} \langle \psi_{n_j}, A\psi_{n_j}\rangle = \overline{\sigma(A)} \tag{73}$$

(see e.g. Walters)

Moreover this holds for all A: diagonal argument, Zelditch '87.

V QET – Summary

- **Eigenfunctions in strongly chaotic systems:**
  - Random wave model
  - Quantum ergodicity theorem: For ergodic systems: almost all eigenfunctions become equidistributed
    Possible exceptional eigenfunctions:
    bouncing ball modes, scars, ...
  - QET $\Longrightarrow$ semiclassical eigenfunction hypothesis
    (for ergodic systems, restricted to subsequence of density one)
- Not discussed
  - Rate of quantum ergodicity $S_1(E, A) = aE^{-1/4}$ (?)
  - influence of non-quantum ergodic subsequences on the rate
  - Gaussian (?) flucutations of $\langle \psi_n, A\psi_n \rangle$.

Further topics

- Autocorrelation function and rate of quantum ergodicity
- Poincaré Husimi representation and quantum ergodicity
- Time evolution in chaotic systems

([AB, R. Schubert 2002])

Local Autocorrelation function

$$C^{\mathrm{loc}}(\boldsymbol{q}, \delta\boldsymbol{x}) := \psi^*(\boldsymbol{q} - \delta\boldsymbol{x}/2)\psi(\boldsymbol{q} + \delta\boldsymbol{x}/2) \ . \tag{74}$$

In terms of the Wigner function

$$W_n(\boldsymbol{p}, \boldsymbol{q}) := \frac{1}{(2\pi)^2} \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{q}'} \psi_n^*(\boldsymbol{q} - \boldsymbol{q}'/2)\psi_n(\boldsymbol{q} + \boldsymbol{q}'/2) \, \mathrm{d}q' \ , \tag{75}$$

one has [Berry '77]

$$C_n^{\mathrm{loc}}(\boldsymbol{q}, \delta\boldsymbol{x}) = \int W_n(\boldsymbol{p}, \boldsymbol{q}) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\delta x} \, \mathrm{d}p \ . \tag{76}$$

VI Autocorrelation function and rate of qerg

For ergodic systems the quantum ergodicity theorem implies

$$W_{n_j}(\boldsymbol{p},\boldsymbol{q}) \to \frac{\delta(H(\boldsymbol{p},\boldsymbol{q}) - E_{n_j})}{\text{vol}(\Sigma_{E_{n_j}})} , \tag{77}$$

One gets for chaotic billiards in two dimensions [Berry '77]

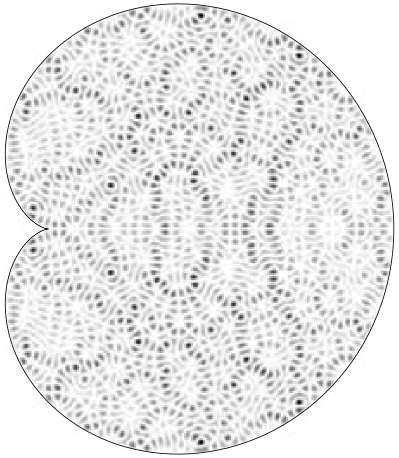
$$C^{\text{loc}}(\boldsymbol{q},\delta\boldsymbol{x}) \to \frac{1}{\text{vol}(\Omega)} J_0(\sqrt{E}|\delta\boldsymbol{x}|) , \tag{78}$$

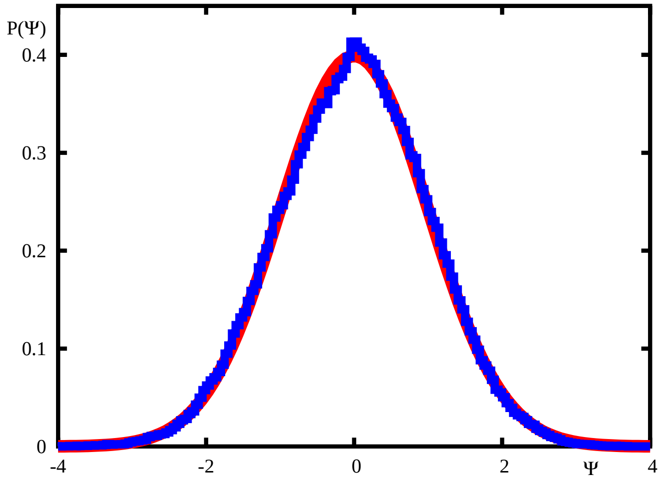
weakly as a function of $\boldsymbol{q}$ in the limit $E \to \infty$.

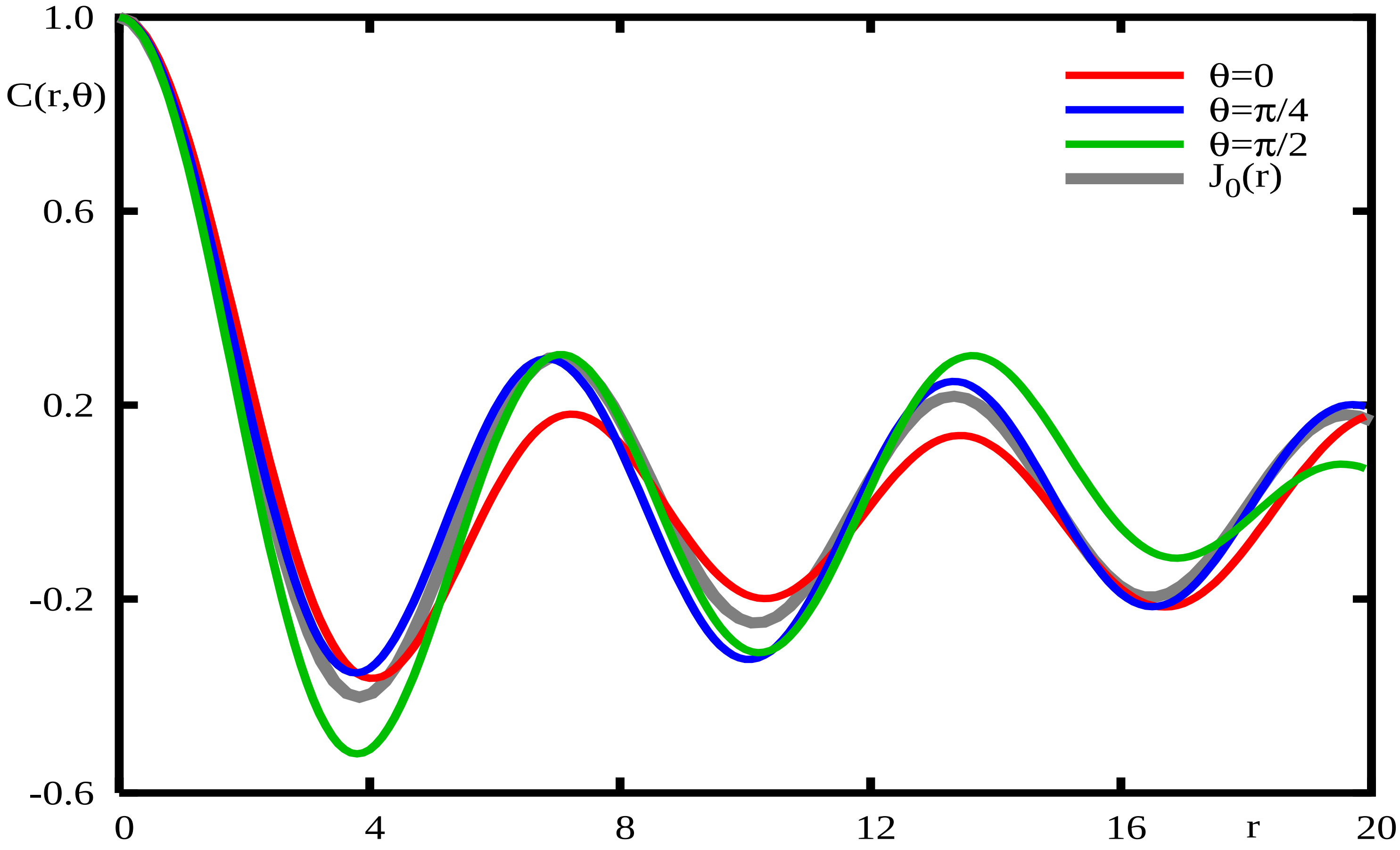
Numerical tests (using a local average of $C^{\text{loc}}(\boldsymbol{q},\delta\boldsymbol{x})$):

Agreement is not too good — quite strong fluctuations

Question: Can one understand/describe these results ?




P(Ψ)
0.4
0.3
0.2
0.1
0
-4
-2
0
2
Ψ
4


1.0
C(r,θ)
0.6
0.2
-0.2
-0.6
0
4
8
12
16
r
20
θ=0
θ=π/4
θ=π/2
J0(r)

Correlation length expansion

$$C_\rho(\boldsymbol{q}, \delta \boldsymbol{x}) = \iint \rho(\boldsymbol{q} - \boldsymbol{q}) W(\boldsymbol{p}, \boldsymbol{q}) \mathrm{e}^{\mathrm{i} p \delta x / \sqrt{E}} \, \mathrm{d}p \mathrm{d}q \quad . \tag{79}$$

For $\rho = 1$ one gets

$$C(\delta \boldsymbol{x}) = J_0(|\delta \boldsymbol{x}|) + 2\pi \sum_{l=1}^{\infty} (-1)^l \left[a_{2l} \cos(2l\theta) + b_{2l} \sin(2l\theta) \right] J_{2l}(|\delta \boldsymbol{x}|) + O(E^{-1/\cdots}$$

where the coefficients a_{2l} and b_{2l} are the Fourier coefficients

$$a_{2l} = \frac{1}{\pi} \int_0^{2\pi} \mathcal{I}(\varphi) \cos(2l\varphi) \, \mathrm{d}\varphi \qquad b_{2l} = \frac{1}{\pi} \int_0^{2\pi} \mathcal{I}(\varphi) \sin(2l\varphi) \, \mathrm{d}\varphi \, , \tag{80}$$

of the *radially integrated momentum density* [Życzkowski '92; AB, Schubert '99]

$$\mathcal{I}(\varphi) := \int_0^{\infty} |\hat{\psi}(r \mathbf{e}(\varphi))|^2 r \, \mathrm{d}r \; . \tag{81}$$

Relation to the rate of quantum ergodicity

$$C(\delta \boldsymbol{x}) = J_0(|\delta \boldsymbol{x}|) + 2\pi \sum_{l=1}^{\infty} (-1)^l \left[a_{2l} \cos(2l\theta) + b_{2l} \sin(2l\theta) \right] J_{2l}(|\delta \boldsymbol{x}|) + O(E^{-1/2}) \ ,$$

If the classical system is ergodic and ψ_{n_j} is a quantum ergodic sequence of eigenfunctions, then for $j \to \infty$

$$\langle \psi_{n_j}, \hat{A}_{2l}(\boldsymbol{q}) \psi_{n_j} \rangle \sim \overline{a_{2l}} = \delta_{l0} \tag{82}$$

$$\langle \psi_{n_j}, \hat{B}_{2l}(\boldsymbol{q}) \psi_{n_j} \rangle \sim \overline{b_{2l}} = 0 \ . \tag{83}$$

Thus for $E \to \infty$ we recover $C(r, \theta) = J_0(r)$.

Deviations are determined by the *rate of quantum ergodicity*.

Removing the angular dependence

As

$$\frac{1}{2\pi}\int_0^{2\pi} C(r,\theta)\,\mathrm{d}\theta = J_0(r) + O(E^{-1/2}) \ , \tag{84}$$

we consider the second moment,

$$\sigma^2(r) := \frac{1}{2\pi}\int_0^{2\pi} [C(r,\theta) - J_0(r)]^2\,\mathrm{d}\theta \ . \tag{85}$$

Inserting the expansion of the autocorrelation function $C(\delta\boldsymbol{x})$

$$\sigma^2(r) = 2\pi^2 \sum_{l=1}^{\infty} (a_{2l}^2 + b_{2l}^2)[J_{2l}(r)]^2 (1 + O(E^{-1/2})) \ . \tag{86}$$

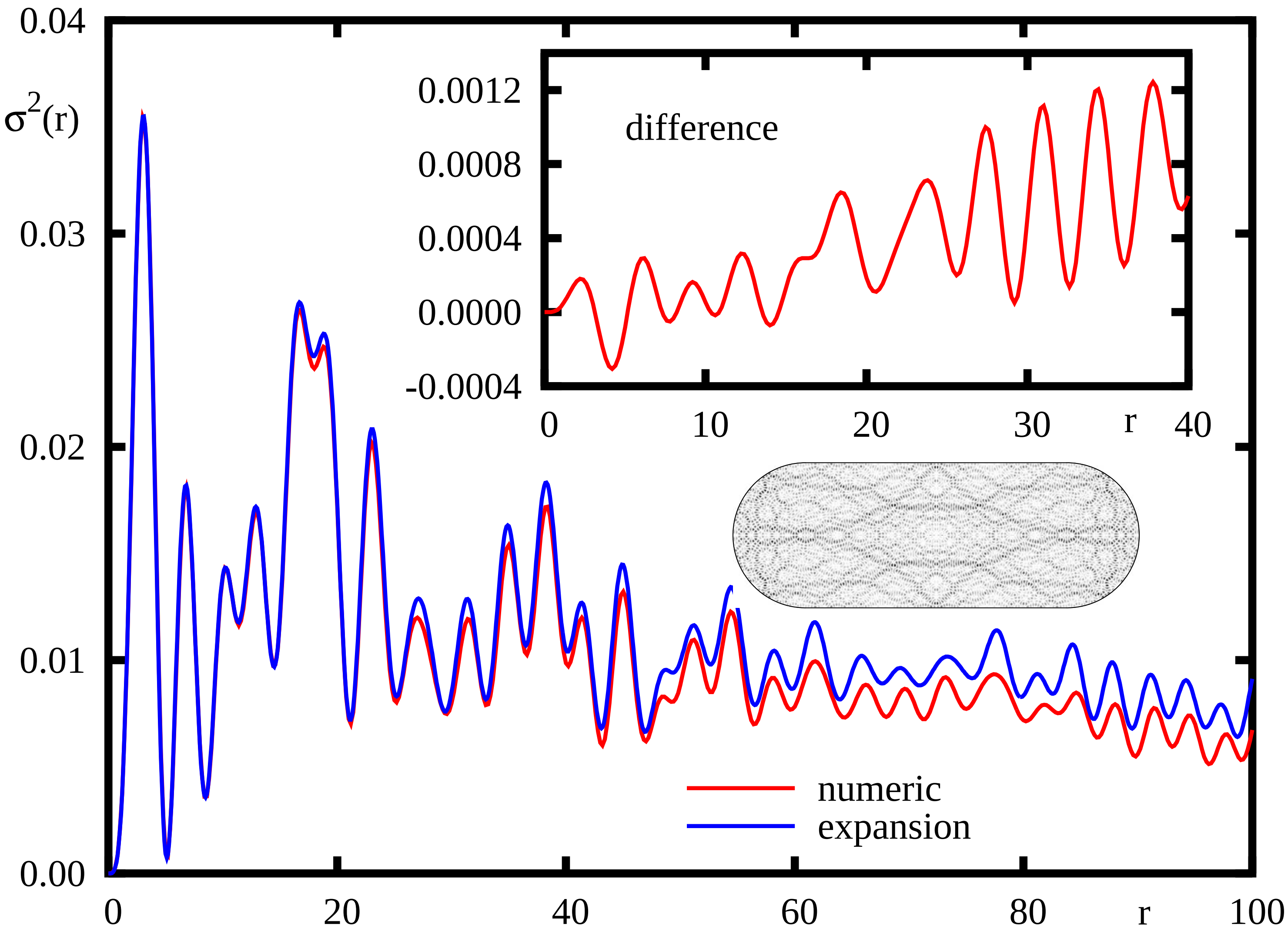
VI Autocorrelation function and rate of qerg

Second moment $\sigma^2(r)$

$$\sigma^2(r) := \frac{1}{2\pi} \int_0^{2\pi} [C(r,\theta) - J_0(r)]^2 \, d\theta \; . \qquad (87)$$

Expansion gives

$$\sigma^2(r) = 2\pi^2 \sum_{l=1}^{\infty} (a_{2l}^2 + b_{2l}^2)[J_{2l}(r)]^2 \, (1 + O(E^{-1/2})) \; . \qquad (88)$$



According to [Eckhardt et. al. '95] we expect in the mean (under suitable conditions on the system)

$$\frac{1}{N(E)} \sum_{E_n \le E} \left[\langle \psi_{n_j}, \hat{A}\psi_{n_j} \rangle - \overline{A} \right]^2 \sim \frac{4\sigma_{\mathrm{cl}}^2(A)}{\mathrm{vol}(\Omega)} \frac{1}{\sqrt{E}} \tag{89}$$

for any pseudodifferential operator $\hat{A}$ of order zero with symbol A.

Here $\overline{A}$ denotes the mean value of A, and $\sigma_{\mathrm{cl}}(A)/\sqrt{T}$ is the variance of the fluctuations of

$$\frac{1}{T} \int_T A(\boldsymbol{p}(t), \boldsymbol{q}(t))\, \mathrm{d}t \tag{90}$$

around $\overline{A}$.

Considering the mean of this function over all eigenfunctions up to energy E, and combining the previous

$$\frac{1}{N(E)} \sum_{E_n \le E} \left[\langle \psi_{n_j}, \hat{A} \psi_{n_j} \rangle - \overline{A} \right]^2 \sim \frac{4\sigma_{\text{cl}}^2(A)}{\text{vol}(\Omega)} \frac{1}{\sqrt{E}} \tag{91}$$

and

$$\sigma^2(r) = 2\pi^2 \sum_{l=1}^{\infty} (a_{2l}^2 + b_{2l}^2)[J_{2l}(r)]^2 \, (1 + O(E^{-1/2})) \; . \tag{92}$$

we get

$$\overline{\sigma}^2(E, r) := \frac{1}{N(E)} \sum_{E_n \le E} \sigma_n^2(r) \tag{93}$$

$$\sim \frac{8\pi^2}{\text{vol}(\Omega)} \sum_{l=1}^{\infty} \left[\sigma_{\text{cl}}(A_{2l})^2 + \sigma_{\text{cl}}(B_{2l})^2 \right] \, [J_{2l}(r)]^2 \frac{1}{\sqrt{E}} \; .$$

- Origin of fluctuations around $J_0(r)$:
  deviations from quantum ergodicity at finite energies
- Thus: **Autocorrelation function allows to study the**

  rate of quantum ergodicity!

Remarks on $\overline{\sigma}^2(E, r)$:

– Efficient quantity to measure the dependence of the rate of quantum ergodicity on different length scales.

– For larger $r \equiv |\delta \boldsymbol{x}|$, one needs to incorporate higher terms in l which corresponds to expectation values of faster oscillating observables.

Question: Poincaré representation of eigenstates?

AB, S. Fürstberger, R. Schubert: *Poincaré Husimi representation of eigenstates in quantum billiards* (2003)

Natural starting point: **normal derivative** of the eigenfunction

$$u_n(s) := \langle \hat{\boldsymbol{n}}(s), \nabla\psi_n(\boldsymbol{x}(s))\rangle \ , \tag{94}$$

Coherent states on the billiard boundary $\partial\Omega$

$$c^b_{(q,p),k}(s) := \left(\frac{k}{\pi\sigma}\right)^{1/4} \sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}k[p(s-q+mL)+\frac{\mathrm{i}}{2\sigma}(s-q+mL)^2]} \ , \tag{95}$$

where $(q,p) \in \partial\Omega \times \mathbb{R}$.

Husimi function: $$h_n(q,p) = \left|\langle c^b_{(q,p),k}, u_n\rangle\right|^2 \tag{96}$$

Husimi function on the Poincaré section $\mathcal{P}$:

$$h_n(q,p) = \frac{1}{2\pi k_n} \left| \int_{\partial\Omega} \overline{c}^{\,b}_{(q,p),k_n}(s)\; u_n(s)\, \mathrm{d}s \right|^2 . \tag{97}$$

([Crespi, Perez, Chang '93; Tualle,Voros '95])

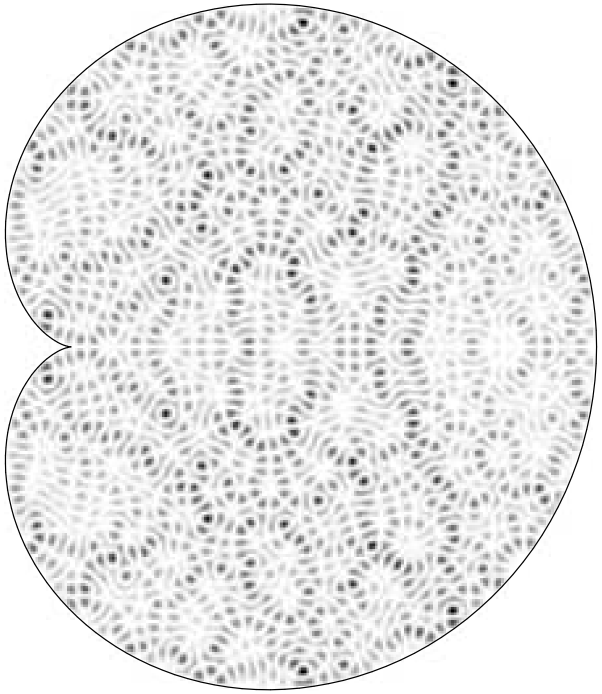
Alternative Poincaré Husimi representation:

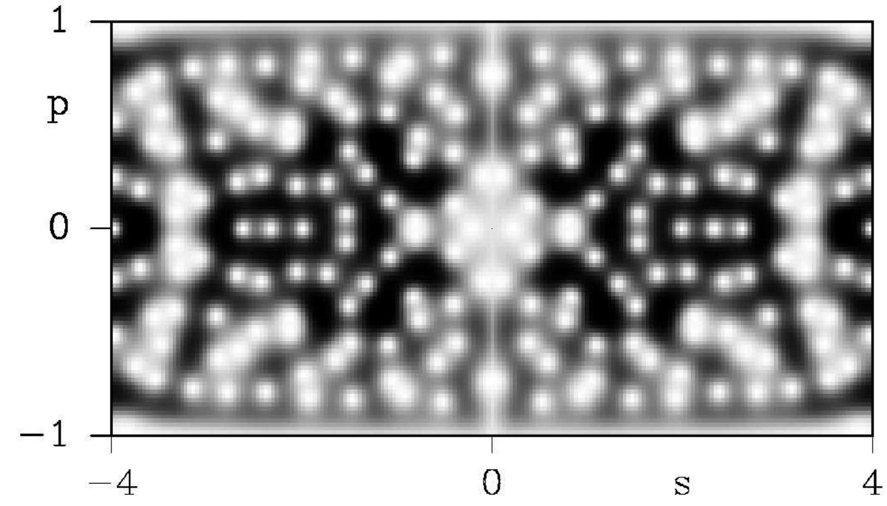
$$\widetilde{h}_n(q,p) = \frac{1}{2k_n^2} \frac{\left| \int_{\partial\Omega} \overline{c}^{\,b}_{(q,p),k_n}(s)\; u_n(s)\; \langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle\, \mathrm{d}s \right|^2}{\int_{\partial\Omega} \overline{c}^{\,b}_{(q,p),k_n}(s) c^{b}_{(q,p),k_n}(s)\; \langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle\, \mathrm{d}s} \tag{98}$$

([Simonotti,Vergini,Saraceno '97])

VII Poincaré Husimi functions – examples

1277:





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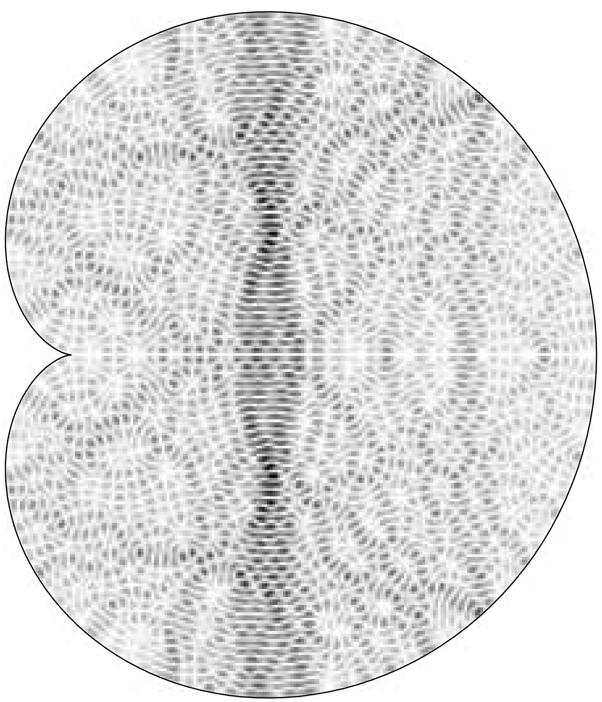
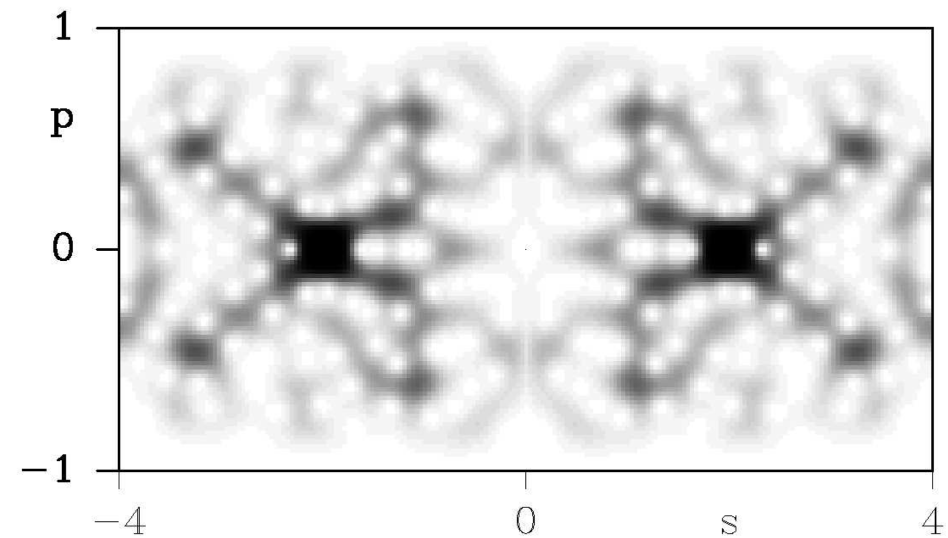




Mean behaviour:

For Husimi functions in phase space:

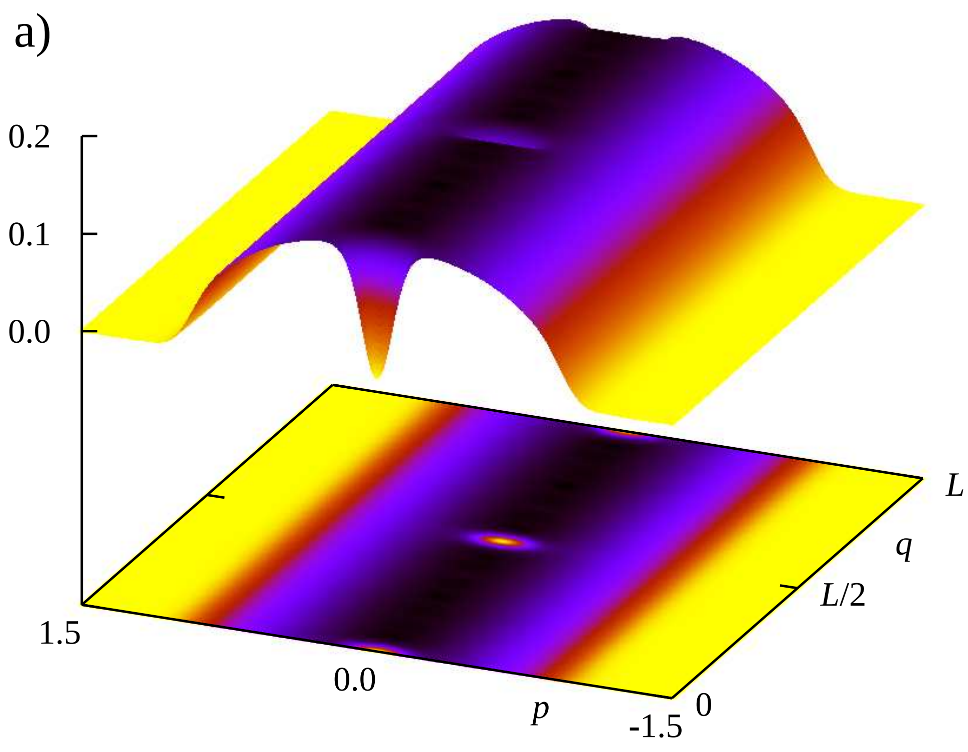
$$\lim_{k\to\infty} \frac{1}{N(k)} \sum_{k_n \le k} H_n^B(\boldsymbol{p}, \boldsymbol{q}) = \frac{1}{\pi \operatorname{vol}(\Omega)} \chi_\Omega(\boldsymbol{q}) \delta(1 - |\boldsymbol{p}|^2) \ .$$

And on the boundary?

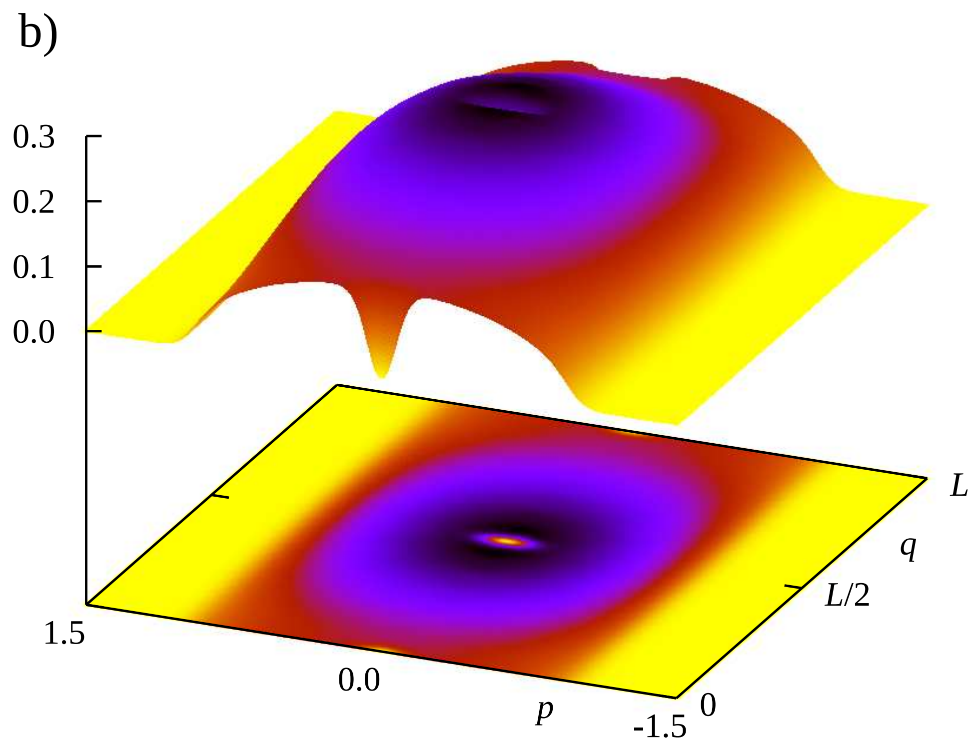
$$\mathcal{H}_k(q, p) := \frac{1}{N(k)} \sum_{k_n \le k} h_n(q, p) \to \textbf{?}$$

Plot of $\mathcal{H}_k(q,p) := \frac{1}{N(k)} \sum_{k_n \le k} h_n(q,p)$

Variant 1



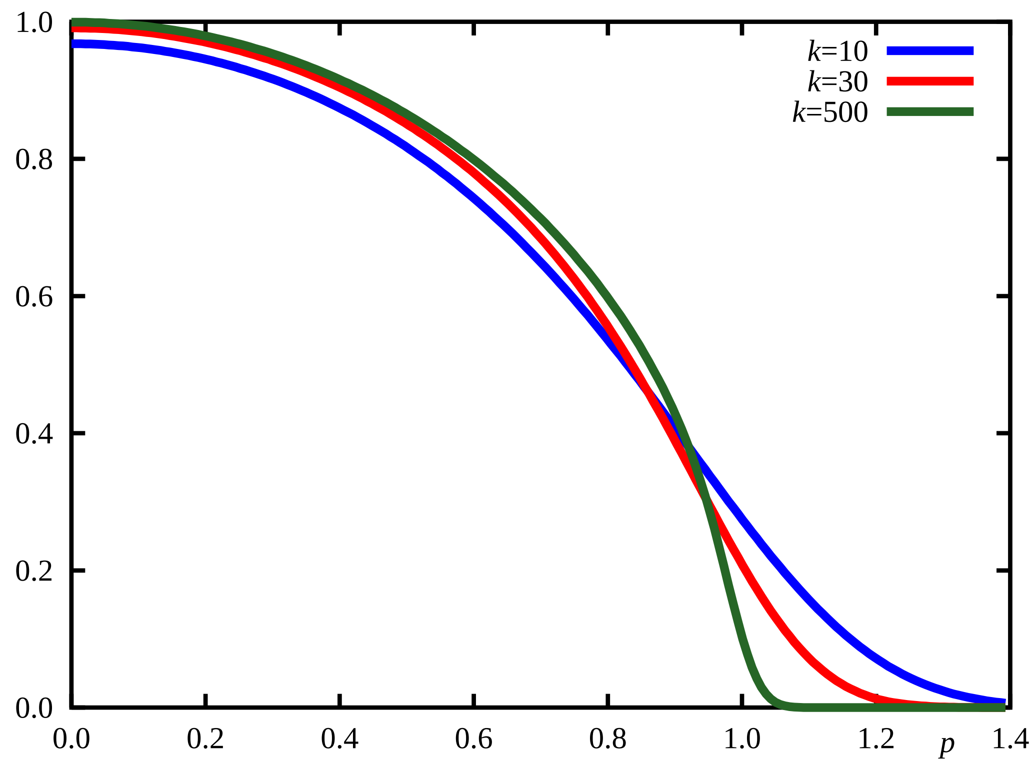
Variant 2



Analytically we show

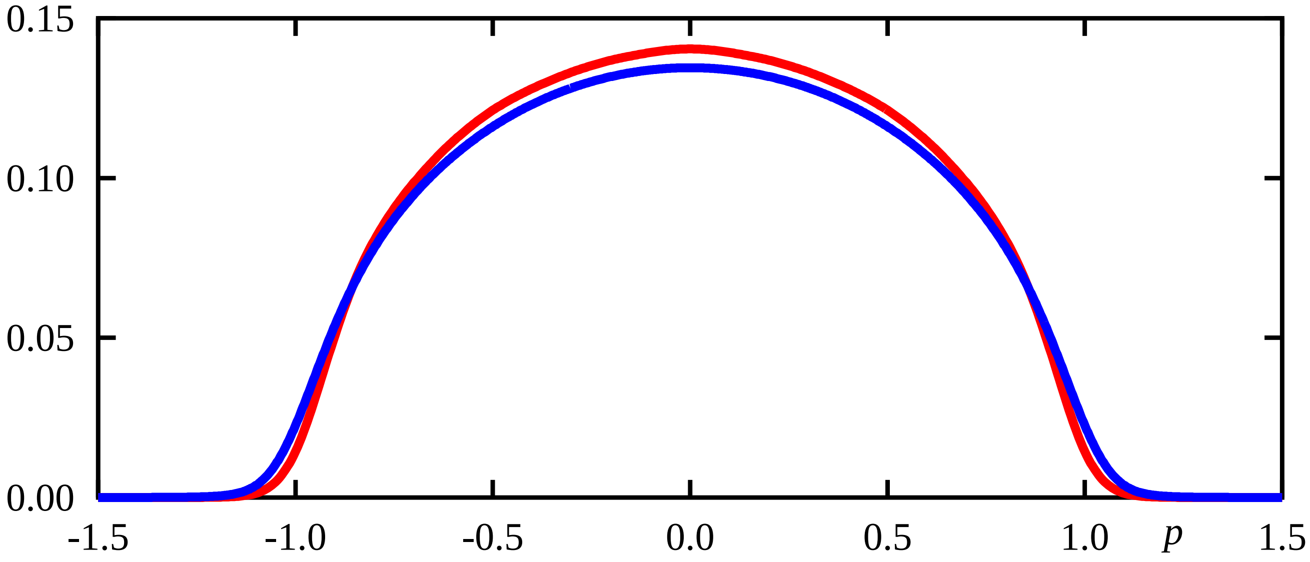
$$\mathcal{H}_k(q,p) \equiv \frac{1}{N(k)} \sum_{k_n \le k} h_n(q,p) = \frac{2}{A\pi} \sqrt{1-p^2} + O(k^{-1/2}) \ ,$$

Uniform asymptotics



Numerical comparison of the mean behaviour

Section of $\mathcal{H}_k(q, p)$ at $q = 3.0$

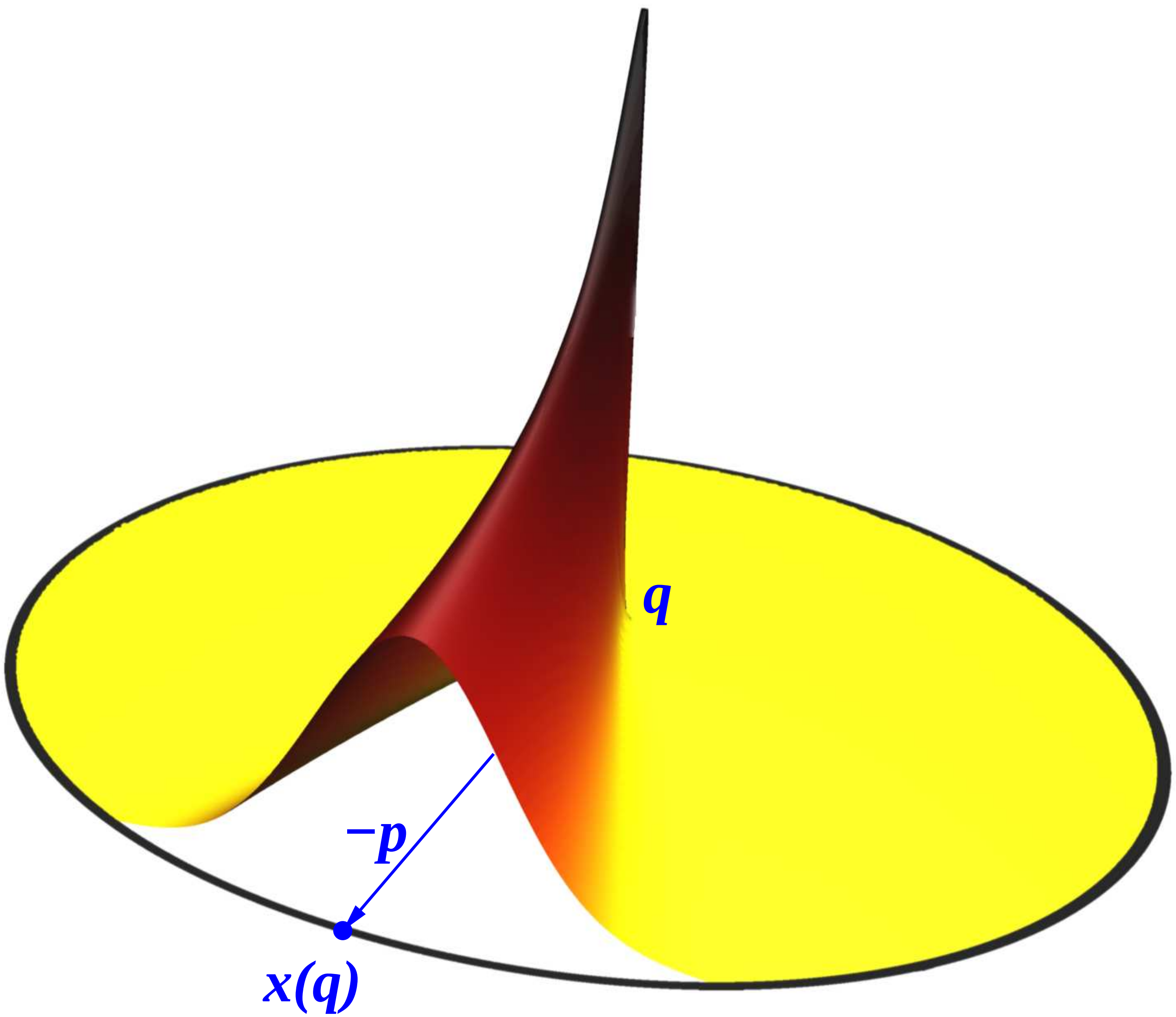


red: numerical result

blue: uniform semiclassics

Question: Does the **ad-hoc** definition of the Poincaré Husimi functions make sense ?

Approach: project coherent state in phase space onto boundary



We show

$$H_n(\boldsymbol{p},\boldsymbol{q}) = \delta_{k_n}(1-|\boldsymbol{p}|)\,\frac{1}{4}\,\frac{h_n(q,p)}{\sqrt{1-p^2}}\,(1+O(k_n^{-1/2}))\ , \qquad (99)$$

with

$$\delta_{k_n}(1-|\boldsymbol{p}|) := \left(\frac{k_n}{\pi}\right)^{1/2} \mathrm{e}^{-k_n(1-|\boldsymbol{p}|)^2}\ . \qquad (100)$$

Consequence

$$\langle\psi_n, \boldsymbol{A}\psi_n\rangle_\Omega = \int_{-1}^{1}\int_{\partial\Omega} \frac{h_n(q,p)}{4\sqrt{1-p^2}}\,\langle a\rangle(q,p)\,l(q,p)\,\mathrm{d}q\,\mathrm{d}p + O(k_n^{-1/2})\ , \qquad (101)$$

where $l(q,p)$ is the length of the orbit segment.

Thus: **physical interpretation of the Poincaré Husimi functions!**

VII Quantum ergodicity for Poincaré Husimi functions

For ergodic systems the quantum ergodicity theorem implies

- almost all Husimi functions $H_n(\boldsymbol{p}, \boldsymbol{q})$ tend weakly to $\frac{1}{2\pi \operatorname{vol}(\Omega)}$.

The relation

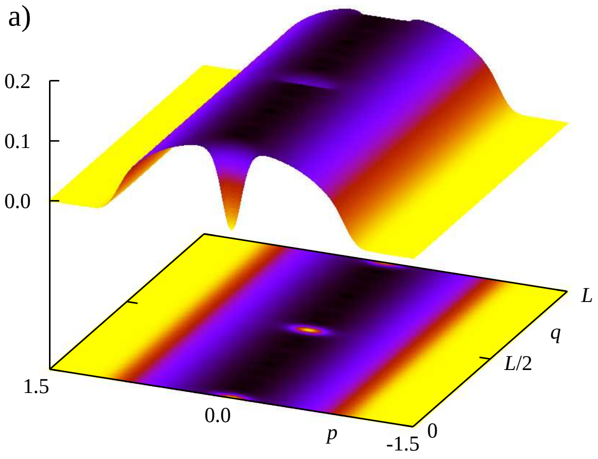
$$\langle \psi_n, \boldsymbol{A}\psi_n \rangle_\Omega = \int_{-1}^{1} \int_{\partial\Omega} \frac{h_n(q,p)}{4\sqrt{1-p^2}} \langle a \rangle(q,p) I(q,p) \,\mathrm{d}q \,\mathrm{d}p + O(k_n^{-1/2}) \ , \qquad (102)$$

then implies that almost all Poincaré Husimi functions

$$h_n(q,p) \to \frac{2}{\pi \operatorname{vol}(\Omega)} \sqrt{1-p^2} \qquad (103)$$

in the semiclassical limit (in the weak sense).

I.e.: **Quantum ergodicity theorem for the Poincaré Husimi functions**



- Mean behaviour of Poincaré Husimi functions:

  $$\sim \sqrt{1-p^2}$$

- Relation between
  - Husimi functions in phase space and
  - Poincaré Husimi functions.

Consequences:

- physical interpretation and justification of the previous ad-hoc definitions
- quantum ergodicity theorem for the Poincaré Husimi functions

Numerical experiment: Start with coherent state

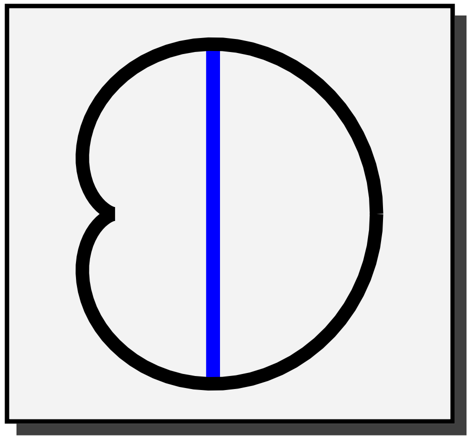
$$\mathrm{Coh}_{(\boldsymbol{p},\boldsymbol{q}),k}(\boldsymbol{x}) := \left(\frac{k}{\pi}\right)^{1/2} \mathrm{e}^{\mathrm{i}k[\langle \boldsymbol{p},\boldsymbol{x}-\boldsymbol{q}\rangle + \frac{\mathrm{i}}{2}\langle \boldsymbol{x}-\boldsymbol{q},(\boldsymbol{x}-\boldsymbol{q})\rangle]} , \qquad (104)$$

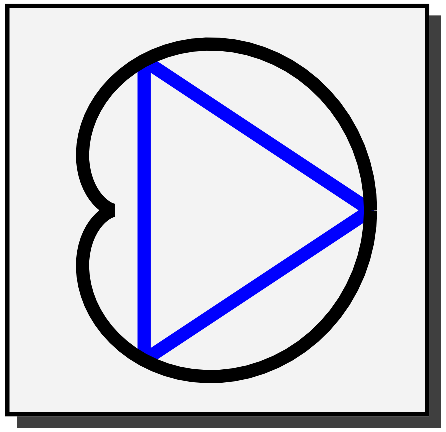
where $(\boldsymbol{p}, \boldsymbol{q}) \in \mathbb{R}^2 \times \mathbb{R}^2$ denotes the point in phase space around which the coherent state is localized.

START

Observation: follows classical trajectory for some time

Two more examples:





What happens for large times?

START

Conjecture:
Random wave description is possible!

Conjecture:
For chaotic systems the time evolution of an initially localized wavepacket leads to a random state in the limit of large times.

One consequence:

- Gaussian distribution for the components (real and imaginary) of $\psi(\boldsymbol{q}, t)$

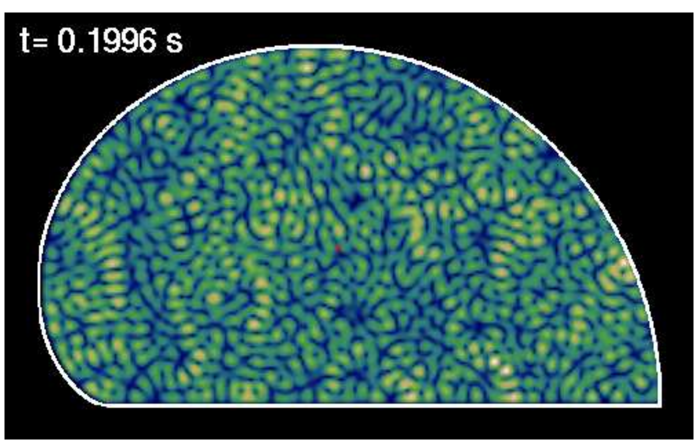
Or: $P(|\psi|^2) = \exp(-\psi)$

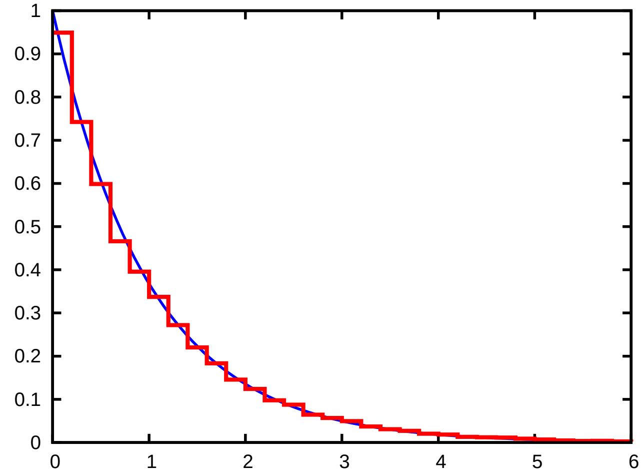
Consider amplitude distribution...

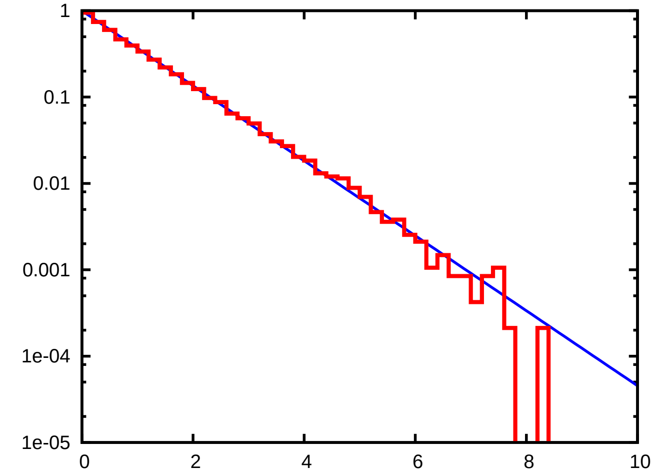
Amplitude distribution...

START

To summarize:

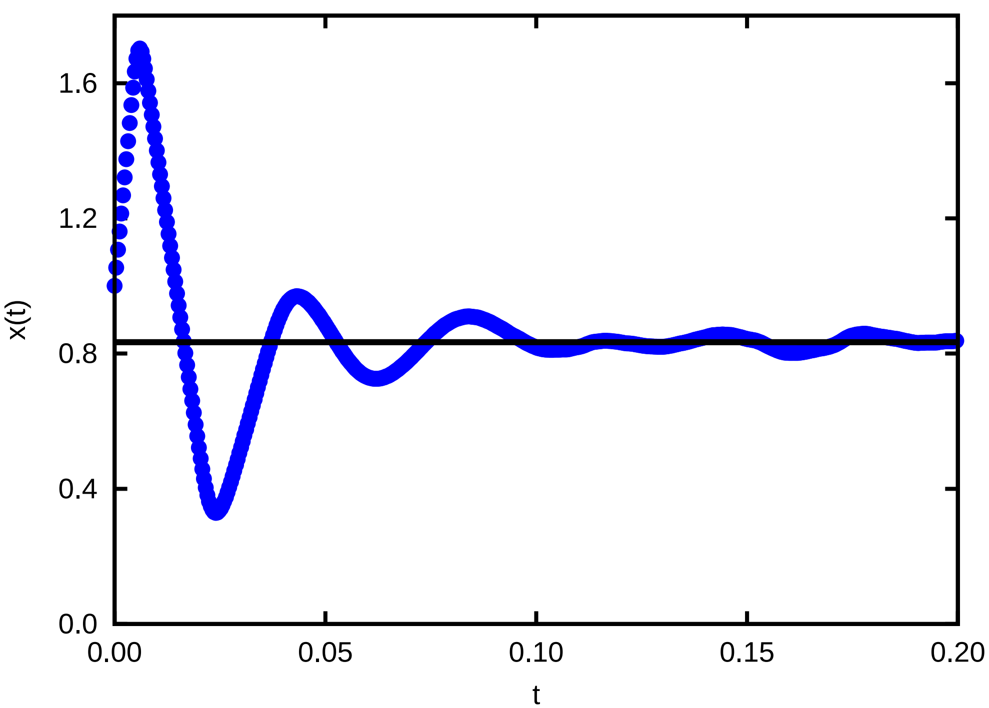




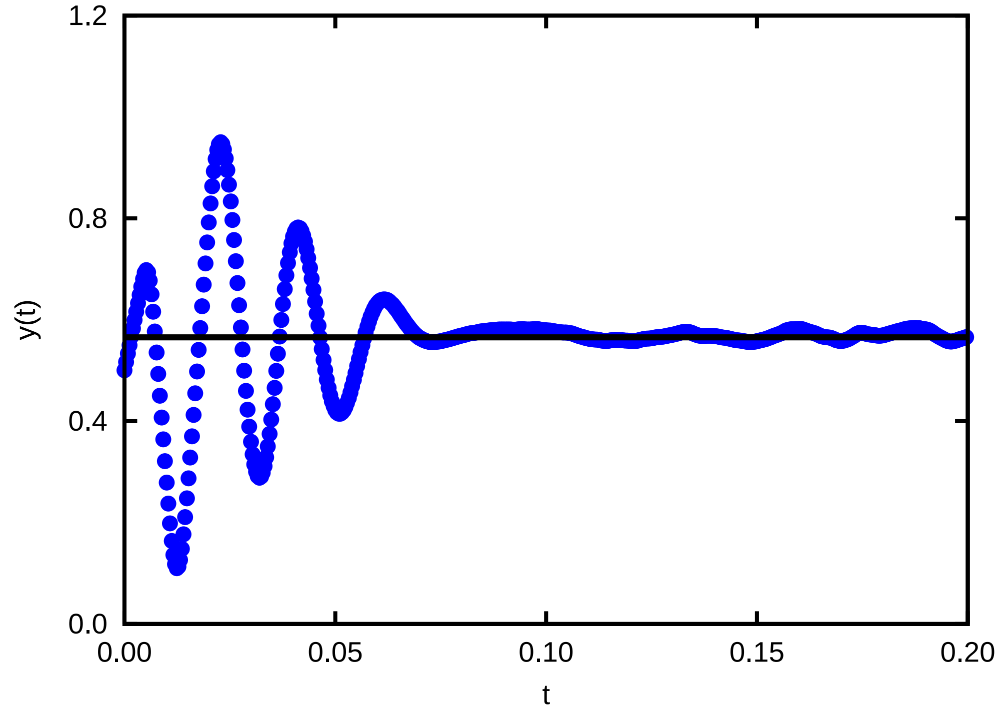


Behaviour of expectation values

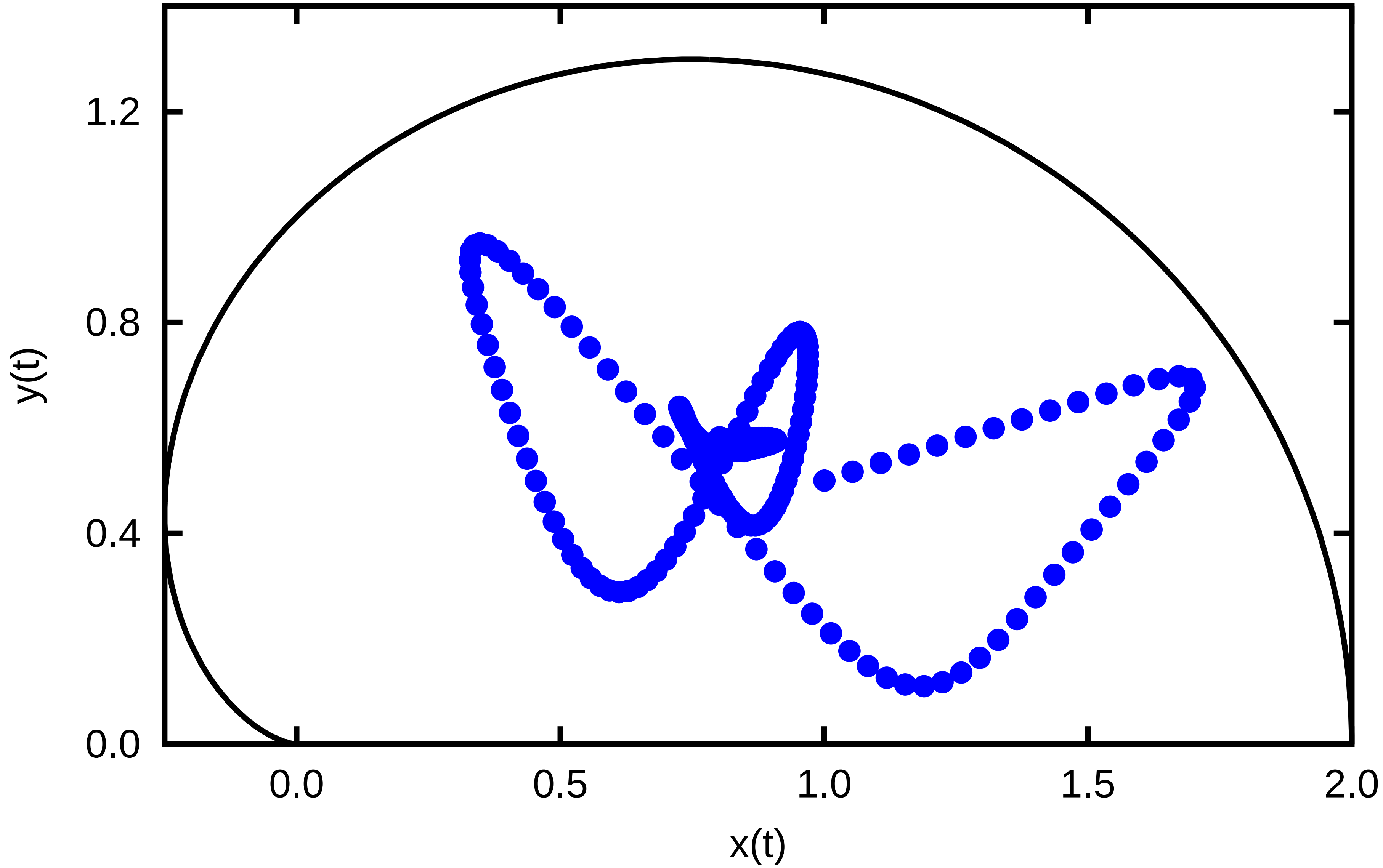
x–position



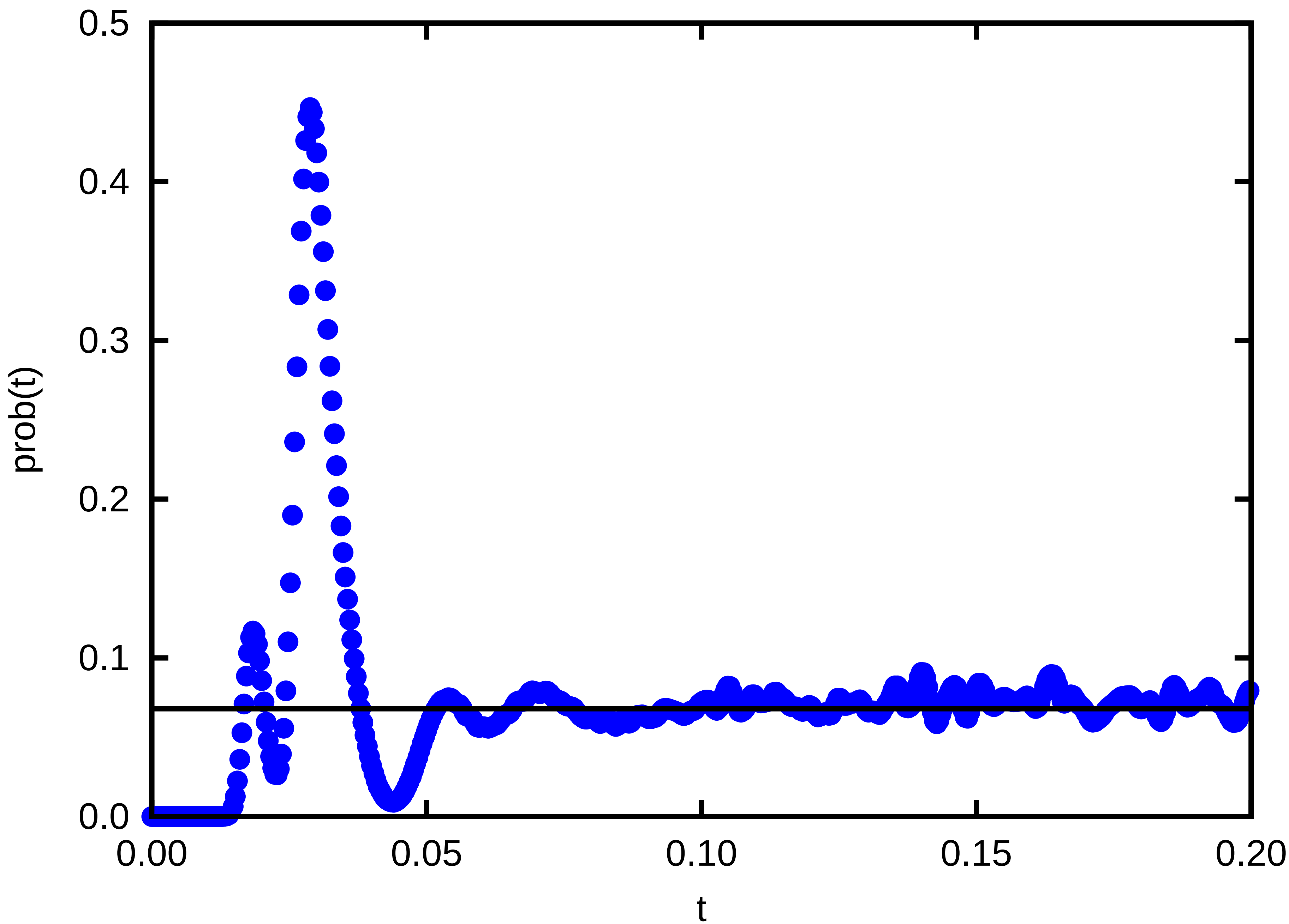
y–position



Plot of $(x(t), y(t))$



Probability to find the particle in some region D



Topics

- Classical billiards
- Quantum billiards
- Statistics of wave functions
- Quantum ergodicity
- More recent results:
  - Autocorrelation function and rate of quantum ergodicity
  - Poincaré Husimi representation and quantum ergodicity
  - Time evolution in chaotic systems

X References

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... and the references therein ;-)