Zwei Würfel

$\frac{1}{12}$, $\frac{1}{6}$, $\frac{1}{2}$, 86%

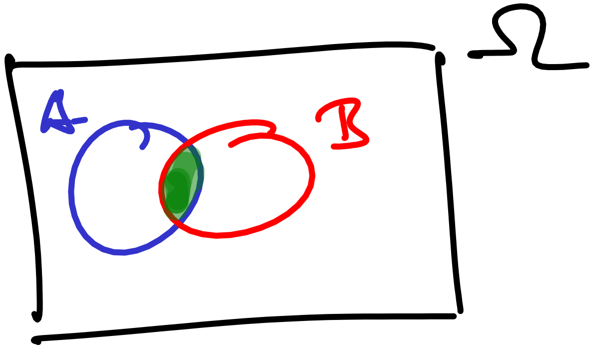
$1-\left(\frac{1}{6}\cdot\frac{5}{6}\right)=\frac{31}{36}$

$\frac{2}{6}=\frac{1}{3}$

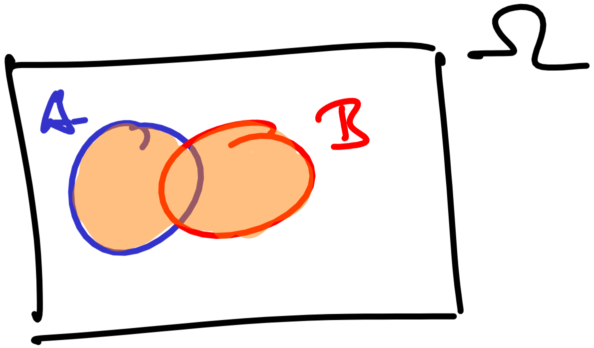
Bsp: Potenzmenge $\qquad \Omega=\{a,b,c\}$

$\mathcal{P}(\Omega)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\Omega\}$

leere Menge $\emptyset=\{\ \}$ $\qquad$ $=\{a,b,c\}$

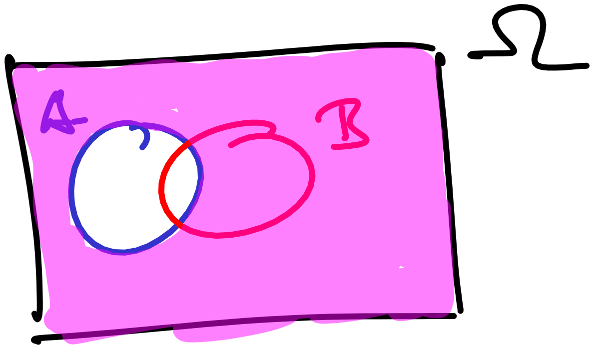


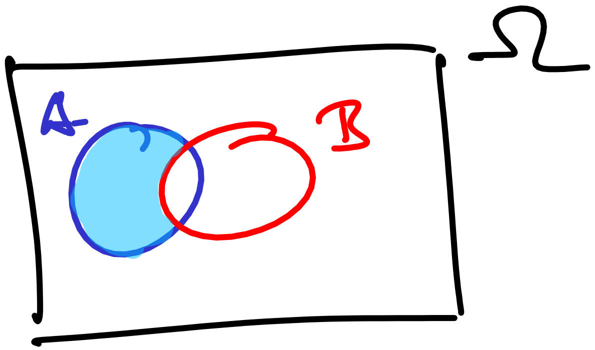
$A \cap B$



$A \cup B$

(A und B disjunkt $\Leftrightarrow$ $A \cap B = \emptyset$)





$A^c = \{\omega \in \Omega \mid \omega \notin A\}$

Komplement, Gegenereignis

$A \setminus B = \{\omega \in A \mid \omega \notin B\}$

$= A \cap B^c$

Laplacescher W'Raum

W'keit für jedes Elementarereignis gleich groß

$$\omega_1, \omega_2 \in \Omega : \quad \mathbb{P}[\omega_1] = \mathbb{P}[\omega_2]$$

Ereignis $A \subset \Omega$

$$\mathbb{P}[A] = \frac{\#A}{\#\Omega} = \frac{\text{Anzahl der Elemente von } A}{\text{Anzahl der Elemente von } \Omega}$$

$$= \frac{\text{"Anzahl günstiger Fälle"}}{\text{"Anzahl möglicher Fälle"}}$$

Würfel

$\Omega = \{1,2,3,4,5,6\}$

$A = \{1,3,5\}$, $B = \{3,4,5,6\}$

☒ Laplace-Würfel (d.h. fairer Würfel)

$P[A] = \frac{3}{6} = \frac{1}{2}$, $P[B] = \frac{4}{6} = \frac{2}{3}$

☒ "unfairer" Würfel, z.B.

$P[\{1\}] = \frac{1}{12}$, $P[\{6\}] = \frac{3}{12} = \frac{1}{4}$

$P[\{j\}] = \frac{1}{6}$, $j = 2,3,4,5$

erfüllt auch (1) & (2) aber nicht Laplacesch

$P[A] = \frac{1}{12} + \frac{1}{6} + \frac{1}{6} = \frac{5}{12}$

$P[B] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$

Beweise für Folgerungen

(i) $$\mathbb{P}[A \cup A^c] = \mathbb{P}[\Omega] \underset{(1)}{=} 1$$

$$\underset{(2)}{=} \mathbb{P}[A] + \mathbb{P}[A^c]$$

$$\Rightarrow \mathbb{P}[A^c] = 1 - \mathbb{P}[A]$$

(ii) $$\mathbb{P}[\emptyset] \underset{(i)}{=} 1 - \mathbb{P}[\emptyset^c] = 1 - \mathbb{P}[\Omega] \underset{(1)}{=} 0$$

(iii) $$\mathbb{P}[A \cup B] = \mathbb{P}[A \cup (B \setminus A)]$$

$$\underset{(2)}{=} \mathbb{P}[A] + \underbrace{\mathbb{P}[B \setminus A] + \mathbb{P}[A \cap B]} - \mathbb{P}[A \cap B]$$

$$= \mathbb{P}[A] + \mathbb{P}[\underbrace{(B \setminus A) \cup (A \cap B)}_{=B}] - \mathbb{P}[A \cap B]$$

Bsp: Würfel

$A = \{1,3,5\}, \quad B = \{3,4,5,6\}$

$$P[A \cup B] = P[\{1,3,4,5,6\}]$$

$$= \frac{5}{6} \quad \text{für Laplace-W.}$$

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$= P[\{1,3,5\}] + P[\{3,4,5,6\}] + P[\{3,5\}]$$

$$= \frac{3}{6} + \frac{4}{6} - \frac{2}{6} = \frac{5}{6} \quad \text{(Laplace-W.)}$$

Bsp: Bedingte W'heiten

Würfel, A, B wie bisher

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[\{3,5\}]}{\mathbb{P}[\{3,4,5,6\}]}$$

$$= \frac{2/6}{4/6} = \frac{2}{4} = \frac{1}{2} \quad \text{(Laplace-W.)}$$

$$\mathbb{P}[B|A] = \frac{\mathbb{P}[A\cap B]}{\mathbb{P}[A]} = \frac{\mathbb{P}[\{3,5\}]}{\mathbb{P}[\{1,3,5\}]}$$

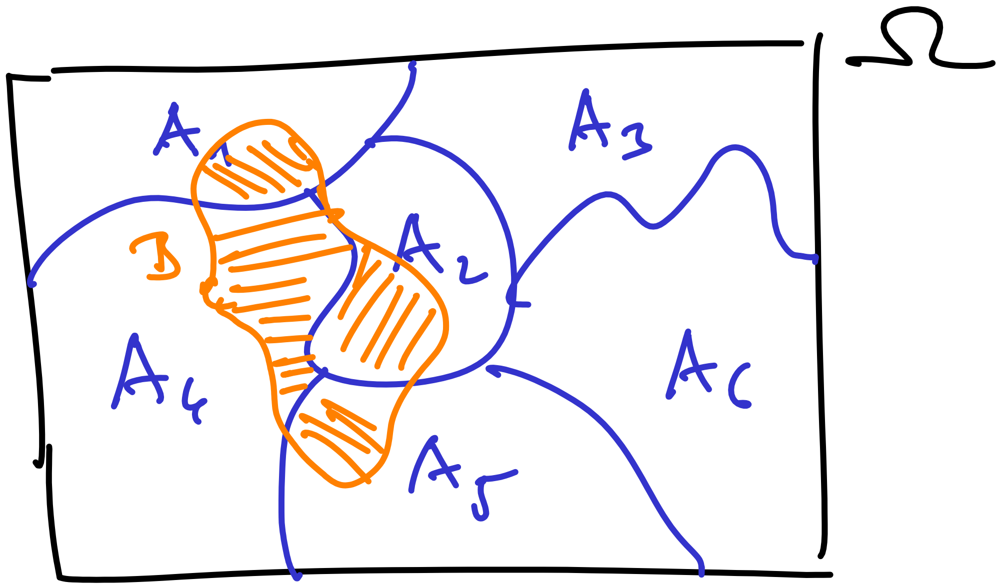
$$= \frac{2/6}{3/6} = \frac{2}{3} \quad \text{(Laplace-W.)}$$

weiteres Ereignis: $C = \{6\}$ Laplace

$\mathbb{P}[A] = \frac{1}{2}$, $\mathbb{P}[C] = \frac{1}{6}$

$$\mathbb{P}[A\cap C] = \mathbb{P}[\{\}] = 0 \neq \mathbb{P}[A]\cdot\mathbb{P}[C] = \frac{1}{2}\cdot\frac{1}{6}$$

zum Satz von Bayes



Beweis:

$$P[A_j|B] = \frac{P[A_j \cap B]}{P[B]} \cdot \frac{P[A_j]}{P[A_j]}$$

$$= \frac{P[B|A_j] \cdot P[A_j]}{P[B]}$$ ← auch oft gewählt

$$= \frac{P[B|A_j] \cdot P[A_j]}{P[B \cap \Omega]} = \frac{P[B|A_j] \cdot P[A_j]}{P[B \cap (\bigcup_{k=1}^{n} A_k)]}$$

$$B \cap \left(\dot{\bigcup}_{k=1}^{n} A_k \right) = \dot{\bigcup}_{k=1}^{n} (B \cap A_k)$$

$$= \frac{P[B|A_j] \cdot P[A_j]}{P\left[\dot{\bigcup}_{k=1}^{n} \underbrace{(B \cap A_k)}_{\text{disjunkt, da } A_k \text{ disjunkt sind}}\right]} = \frac{P[B|A_j] \cdot P[A_j]}{\sum_{k=1}^{n} P[B \cap A_k]}$$

$$= \frac{P[B|A_j] \cdot P[A_j]}{\sum_{k=1}^{n} P[B|A_k] \cdot P[A_k]}$$

Def. bed. W'keit

Bsp: diagnostische Test

$$\mathbb{P}[A_1 | B] = ?$$

$$\mathbb{P}[A_1] = 0{,}01 \implies \mathbb{P}[A_2] = 0{,}99$$

da $A_2 = A_1^c$

$$\mathbb{P}[B | A_1] = 0{,}98$$

$$\mathbb{P}[B^c | A_2] = 0{,}95 \implies \mathbb{P}[B | A_2] = 0{,}05$$

$$\mathbb{P}[A_1 | B] = \frac{\mathbb{P}[B|A_1] \cdot \mathbb{P}[A_1]}{\mathbb{P}[B|A_1] \cdot \mathbb{P}[A_1] + \mathbb{P}[B|A_2] \cdot \mathbb{P}[A_2]}$$

$$= \frac{0{,}98 \cdot 0{,}01}{0{,}98 \cdot 0{,}01 + 0{,}05 \cdot 0{,}99} = \frac{98}{98 + 495} \approx \frac{1}{6}$$

Würfel vom Anfang

W = weißer Würfel zeigt 6

M = mind. ein Würfel zeigt 6

$$P[W|M] = ?$$

$$P[W] = \frac{1}{6} \Rightarrow P[W^c] = \frac{5}{6}$$

$$P[M|W] = 1$$

$$P[M|W^c] = \frac{1}{6}$$

$$P[W|M] = \frac{P[M|W] \cdot P[W]}{P[M|W] \cdot P[W] + P[M|W^c] \cdot P[W^c]}$$

$$= \frac{1 \cdot \frac{1}{6}}{1 \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{5}{6}} = \frac{6}{6+5} = \frac{6}{11}$$