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# The rotational symmetric Willmore boundary problem

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**Abstract:** In this article, we solve the rotational symmetric Willmore boundary problem with Dirichlet boundary data for any boundary conditions apart from the degenerate case when one of the sphere caps extending the boundary conditions is contained in the other. Our approach is variational, and actually our solutions are energy minimzing in the class of rotational symmetric surfaces.

**Keywords:** Willmore surfaces, surfaces of revolution, Dirichlet boundary conditions, hyperbolic space.

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## 1 Introduction

For an immersed surface $f : \Sigma \to \mathbb{R}^3$ the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |H|^2 \, \mathrm{d}\mu_g,$$

where $H$ denotes the scalar mean curvature of $f$, $g = f^* g_{euc}$ the pull-back metric and $\mu_g$ the induced area measure of $f$ on $\Sigma$.

Critical points of the Willmore functional, called Willmore surfaces or immersions, satisfy the Euler-Lagrange equation

$$\Delta_g H + |A^0|_g^2 H = 0, \tag{1.1}$$

see i.e. [KuSch02] §2, where $g = f^* g_{euc}$ is the pull-back metric and $A^0 = A - \frac{1}{2}\vec{\mathbf{H}} g$ is the tracefree second fundamental form of $f$.

In [Sch10], the second author considered the Willmore boundary problem that is to find for a given smooth embedded closed oriented one-dimensional manifold $\Gamma \subseteq \mathbb{R}^n, \neq \emptyset$, together with a smooth unit normal field $\mathbf{n} \in N\Gamma$ an immersion $f$ of a compact surface $\Sigma$ with boundary $\partial\Sigma = \Gamma$ which is Willmore on $\overset{\circ}{\Sigma}$, that is (1.1) holds for $f$, and satisfies the boundary conditions

$$f = id, \mathbf{co}_f = \mathbf{n} \quad \text{on } \partial\Sigma = \Gamma \tag{1.2}$$

where $\mathbf{co}_f$ denotes the inner conormal of $f$ at $\partial\Sigma$.

In the present article, we are seeking rotational symmetric solutions of the Willmore boundary problem. Here $\Gamma$ consists of circles with centre on the $x-$axis and in a plane orthogonal to the $x-$axis, that is

$$S_{x_0,r_0} := \{ \ (x_0, r_0 \cos\alpha, r_0 \sin\alpha) \mid \alpha \in \mathbb{R} \ \}, \tag{1.3}$$

and the rotational symmetric unit normal field $\mathbf{n}^{\beta_0}_{S_{x_0,r_0}} \in NS_{x_0,r_0}$ along $S_{x_0,r_0}$ is of the form

$$\mathbf{n}^{\beta_0}_{S_{x_0,r_0}}(x_0, r_0 \cos\alpha, r_0 \sin\alpha) = (\cos\beta_0, \sin\beta_0 \cos\alpha, \sin\beta_0 \sin\alpha) \quad \text{for } \alpha \in \mathbb{R} \tag{1.4}$$

and some fixed $\beta_0 \in \mathbb{R}$. We see $\mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$ is horizontal and constant for $\sin\beta_0 = 0$, and $\mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$ is vertical for $\cos\beta_0 = 0$, which we even allow. Actually applying an inversion at an appropriate point of the $x-$axis, which leaves the rotational symmetry with respect to the $x-$axis and the Euler-Lagrange equation (1.1) unchanged by conformal invariance of the Willmore functional, see [Ch74], we can transform the rotational symmetric Willmore boundary problem always to an equivalent problem with non-vertical normal field.

Obviously there is a unique sphere cap $cap(S_{x_0,r_0}, \mathbf{n}^{\beta_0}_{S_{x_0,r_0}})$ satisfying the boundary conditions (1.2) for $\Gamma = S_{x_0,r_0}$ and $\mathbf{n} = \mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$, which in the vertical case is part of the plane $\{x = x_0\}$ as a generalized sphere cap. As this sphere cap is a Willmore surface and even minimizes the Willmore energy under all surfaces satisfying the boundary conditions (1.2), as the round spheres are the global closed minimizers of the Willmore energy, see [Wil82], there is always a solution of the rotational symmetric boundary condition when $\Gamma$ consists of only one component. Therefore we consider in the present article boundary conditions $\Gamma$ consisting of exactly two components.

Now rotational symmetric immersions of a connected compact surface with boundary in $\mathbb{R}^3$ and outside the $x-$axis admit a regular profile curve in the upper half plane $\gamma : [0, L] \to \mathcal{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0 \ \}$ with $|\gamma'| \neq 0, L > 0$, up to reparametrization, and we get an equivalent rotationally symmetric immersion by putting $f_\gamma : [0, L] \times S^1 =: \Sigma_L \to \mathbb{R}^3$ with

$$f_\gamma(t, e^{i\alpha}) := (\gamma^1(t), \gamma^2(t) \cos\alpha, \gamma^2(t) \sin\alpha) \quad \text{for } t \in [0, L], \alpha \in \mathbb{R}. \tag{1.5}$$

Clearly $f_\gamma$ satisfies the boundary conditions (1.2) for $\Gamma = S_{x_-,r_-} + S_{x_+,r_+}$ and $\mathbf{n} = \mathbf{n}^{\beta_\pm}_{S_{x_\pm,r_\pm}}$, when

$$\begin{aligned} \gamma(0) = (x_-, r_-), \quad & \gamma'(0)/|\gamma'(0)| = (\cos\beta_-, \sin\beta_-), \\ \gamma(L) = (x_+, r_+), \quad & \gamma'(L)/|\gamma'(L)| = -(\cos\beta_+, \sin\beta_+). \end{aligned} \tag{1.6}$$

The setting in (1.6) even allows $(x_-, r_-, e^{i\beta_-}) = (x_+, r_+, e^{i\beta_+})$ .

To obtain rotationally symmetric Willmore immersions, we use an observation from [BryGr86] rsp. [HJPi92], which connects the Willmore energy with the hyperbolic elastic energy of the corresponding profile curve (see also [LaSi84a]). Denoting the curvature $\kappa_\gamma^{\mathcal{H}}$ of $\gamma$ with respect to the hyperbolic metric $y^{-2}\delta_{ij}$ on $\mathcal{H}$ , this reads

$$\frac{1}{2}\int_{\Sigma_L} |A^0_{f_\gamma}|^2 \, \mathrm{d}vol_{f_\gamma} = \frac{\pi}{2}\int_0^L |\kappa_\gamma^{\mathcal{H}}(t)|^2 \, |\gamma'(t)|/\gamma^2(t) \, \mathrm{d}t =: \frac{\pi}{2}\mathcal{F}(\gamma). \tag{1.7}$$

On the other hand by the Gauß equations and the Gauß-Bonnet theorem, see for example [Sch10] (1.1),

$$\mathcal{W}(f_\gamma) = \frac{1}{2}\int_{\Sigma_L} |A^0_{f_\gamma}|^2 \, \mathrm{d}vol_{f_\gamma} - \int_{\partial\Sigma_L} \kappa_{g_{f_\gamma}} \, \mathrm{d}vol_{f_\gamma|\partial\Sigma_L} + 2\pi\chi(\Sigma_L), \tag{1.8}$$

where $g_f = f^* g_{euc}$ is the pull-back of $f$ , $\kappa_{g_f}$ the geodesic curvature of $\partial\Sigma_L$ with respect to $g_f$ and $\chi$ is the Euler characteristic. As the geodesic curvature and the Euler characteristic remain unchanged under compactly supported perturbations of $\gamma$ , we see that $\gamma$ is a critical point of $\mathcal{F}$ if and only if the first variation of the Willmore functional at $f_\gamma$ vanishes for rotationally symmetric, compactly supported perturbations. This suffices to establish that $f_\gamma$ is a Willmore immersion, see [Ei17] §3.3, [DaDeGr08] Theorem 3.9, Step 2 and [GaGrSw91] Lemma 8.2. So in order to find rotationally symmetric Willmore immersions, we have to find regular profile curves which are critical points of $\mathcal{F}$ . Critical points of $\mathcal{F}$ , which are additionally parametrized by hyperbolic arc length, are called free elastica and were fully classified in [LaSi84b].

Dall'Acqua, Deckelnick and Grunau obtained in [DaDeGr08] graph solutions of the rotationally symmetric Willmore boundary problem when the boundary conditions in (1.6) are symmetric and horizontal, that is $r_+ = r_-$ and $\sin\beta_\pm = 0$ . This was extended in [DaFrGrSchi11] to the symmetric, non-horizontal case. The first author and Grunau obtained in [EiGr17] and [Ei17] solutions in the non-symmetric, horizontal case under the assumption that the boundary conditions in (1.6) admit a rotationally symmetric surface with Willmore energy strictly smaller than $4\pi$ . Adding half spheres at each side, this assumption is equivalent to the boundary conditions in (1.6) admitting a closed rotationally symmetric surface with Willmore energy strictly smaller than $8\pi$ . In this case, the general existence result in [Sch10] gives a smooth embedded solution for the Willmore boundary problem in $S^3$ , but the rotational symmetry is left open there. Mandel showed a symmetry breaking result in [Man18]. In contrast to this symmetry breaking result, energy minimizing solutions are symmetric in case of symmetric and horizontally pointing inward boundary data, see [EiKo17] or [Ei17] Theorem 1.5. This of course implies non-uniqueness, which has already been observed in [Ei16]. Natural boundary conditions have been considered in [BeDaFr10], [BeDaFr13], [DaDeWh13] and [DeGr09].

In this article, we obtain solutions for all rotationally symmetric Dirichlet boundary conditions, apart from one degenerate case. Actually our solutions are energy-minimizing in the class of rotationally symmetric immersions.

**Theorem 1.1** *For any* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *, if none of sphere caps* $cap(S_{x_\pm,r_\pm}, \mathbf{n}^{\beta_\pm}_{S_{x_\pm,r_\pm}})$ *is contained in the other, there exists a regular profile curve in the upper half plane* $\gamma : [0,L] \to \mathcal{H} := \{(x,y) \in \mathbb{R}^2 \mid y > 0\ \}$ *with* $|\gamma'| \neq 0, L > 0$ *, which satisfies the boundary conditions (1.6) and minimizes the Willmore energy subject to these boundary conditions, that is*

$$\mathcal{W}(f_\gamma) = \mathcal{W}(x_\pm, r_\pm, \beta_\pm) := \inf\{\mathcal{W}(f_{\tilde{\gamma}}) \mid \tilde{\gamma} \textit{ satisfies } (1.6)\ \}.$$

*In particular* $\gamma$ *is a free elastica and the corresponding rotational symmetric immersion* $f_\gamma$ *is a Willmore immersion.* □

The approach in [DaDeGr08], [DaFrGrSchi11], [EiGr17] and [Ei17] is variational. Here we will also proceed by a variational approach, but combined with some geometric measure theory, and find minimizer which may exhibit one singularity, i.e. a point touching the extended $x$-axis $\mathbb{R} \cup \{\infty\}$, see Theorem 3.7. Outside this singularity, the minimizer will be critical w.r.t. the elastic energy $\mathcal{F}$. The classification in [LaSi84b] then yields it to be part of Moebius transformed catenoids or half circles. If there is some catenoid part present in the minimizer, we obtain in Proposition 2.3 a curve with less energy, which is possible because any Moebius transformed catenoid has positive elastic energy, and get a contradiction. If both parts of the minimizer are half circles, we explicitely construct a comparison curve satisfying symmetric boundary conditions and with total Willmore energy of strictly less than $8\pi$ . This excludes singularites, and the minimizer is regular. Finally in the case when one of the sphere caps is contained in the other, we give in Proposition 4.1 an argument that there is no minimizer.

## 2 Preliminary energy bounds

Any rotational symmetric immersion outside the $x-$axis is obtained after reparametrization by a regular profile curve in the upper half plane $\gamma : [0,L] \to \mathcal{H} := \{(x,y) \in \mathbb{R}^2 \mid y > 0\ \}$ with $|\gamma'| \neq 0, L > 0$ , by putting $f_\gamma : [0,L] \times S^1 =: \Sigma_L \to \mathbb{R}^3$ with

$$f_\gamma(t, e^{i\alpha}) := (\gamma^1(t), \gamma^2(t)\cos\alpha, \gamma^2(t)\sin\alpha) \quad \text{for } t \in [0,L], \alpha \in \mathbb{R}.$$

For $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ as in Theorem 1.1, we define the infimal energy

$$\mathcal{W}(x_\pm, r_\pm, \beta_\pm) := \inf\{\mathcal{W}(f_\gamma) \mid \gamma \text{ satisfies } (1.6)\ \}. \tag{2.1}$$

We will often write $\mathcal{W}(x_\pm, r_\pm, e^{i\beta_\pm}) := \mathcal{W}(x_\pm, r_\pm, \beta_\pm)$ in slight abuse of notation. We start with upper semicontinuity when the boundary conditions are converging.

**Proposition 2.1** *For any* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *and* $(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm}) \to (x_\pm, r_\pm, \beta_\pm)$ *, we have*

$$\limsup_{k\to\infty} \mathcal{W}(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm}) \leq \mathcal{W}(x_\pm, r_\pm, \beta_\pm). \tag{2.2}$$

**Proof:**
Let $\gamma : [0,1] \to \mathcal{H}, |\gamma'| \neq 0, L = 1$ , be any regular profile curve satisfying the boundary

conditions (1.6). We choose $\chi \in C_0^\infty([0,1/2[)$ with $0 \le \chi \le 1, \chi \equiv 1$ on $[0,1/4]$ , and put identifying $\mathbb{R}^2 \cong \mathbb{C}$ that

$$\eta_k(t) := \chi(t)\Big((x_{k,-} - x_-) + t(e^{i\beta_{k-}} - e^{i\beta_-})\Big) +$$

$$+\chi(1-t)\Big(((x_{k,+} - x_+) + (t-1)(e^{i\beta_{k+}} - e^{i\beta_+})\Big) \quad \text{for } 0 \le t \le 1.$$

Clearly by $(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm}) \to (x_\pm, r_\pm, \beta_\pm)$ , we have $\eta_k \to 0$ strongly for example in $C^2(0,1)$ . Also $\gamma + \eta_k$ satisfies the boundary conditions (1.6) with $(x_\pm, r_\pm, \beta_\pm)$ replaced by $(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm})$ , and it is a regular profile curve for large $k$ , hence

$$\mathcal{W}(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm}) \le \mathcal{W}(f_{\gamma+\eta_k}) \to \mathcal{W}(f_\gamma)$$

and

$$\limsup_{k\to\infty} \mathcal{W}(x_{k,\pm}, r_{k,\pm}, \beta_{k,\pm}) \le \mathcal{W}(f_\gamma).$$

Taking the infimum over all $\gamma$ , we get (2.2).

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Returning to $f_\gamma$ , its pull-back metric is given in the chart $(t, e^{i\alpha}) \mapsto (t, \alpha)$ of $\Sigma_L = [0, L] \times S^1$ by

$$(g_{\gamma,ij}) := ((f_\gamma^* g_{euc})_{ij}) = diag(|\gamma'|^2, (\gamma^2)^2), \tag{2.3}$$

hence $f_\gamma$ is conformal if and only if $|\gamma'| = \gamma^2$ , that is $\gamma : [0, L] \to \mathcal{H}$ is parametrized by arc length with respect to the hyperbolic metric $y^{-2}\delta_{ij}$ on $\mathcal{H}$ .

Next we observe that for any $S_{x_0,r_0}, \mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$ as in (1.3), (1.4), there is a unique sphere cap $cap(x_0, r_0, \beta_0)$ satisfying the boundary conditions (1.2) for $\Gamma = S_{x_0,r_0}$ and $\mathbf{n} = \mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$ given by

$$\begin{aligned} cap(x_0, r_0, \beta_0) := \{(x, y, z) \in \mathbb{R}^3 \mid \quad & |x - x_0 - r_0 \tan \beta_0|^2 + y^2 + z^2 = r_0^2/(\cos \beta_0)^2, \\ & (\cos \beta_0)\ (x - x_0) \ge 0 \ \} \end{aligned} \tag{2.4}$$

for $\cos \beta_0 \neq 0$ , that is for non-vertical normal field $\mathbf{n}^{\beta_0}_{S_{x_0,r_0}}$ . In the vertical case, we define $cap(x_0, r_0, \beta_0)$ as a generalized sphere cap being part of a plane, namely we put

$$cap(x_0, r_0, \beta_0) := \{(x_0, y, z) \mid (\sin \beta_0)\ (y^2 + z^2 - r_0^2) \ge 0\ \} \tag{2.5}$$

for $\cos \beta_0 = 0$ . The sphere caps have a unique intersection point with the extended $x-$axis $(\mathbb{R} \times \{(0,0)\}) \cup \{\infty\} =: \bar{\mathbb{R}}$ , which we call the focal point of the sphere cap, that is we put

$$focal(x_0, r_0, \beta_0) := cap(x_0, r_0, \beta_0) \cap \{(x, 0, 0) \mid x \in \mathbb{R}\ \} \tag{2.6}$$

for $\cos \beta_0 \neq 0$ and

$$\begin{aligned} focal(x_0, r_0, \beta_0) &:= (x_0, 0, 0) \quad \text{for } \sin \beta_0 = -1, \\ focal(x_0, r_0, \beta_0) &:= \infty \quad \text{for } \sin \beta_0 = 1 \end{aligned} \tag{2.7}$$

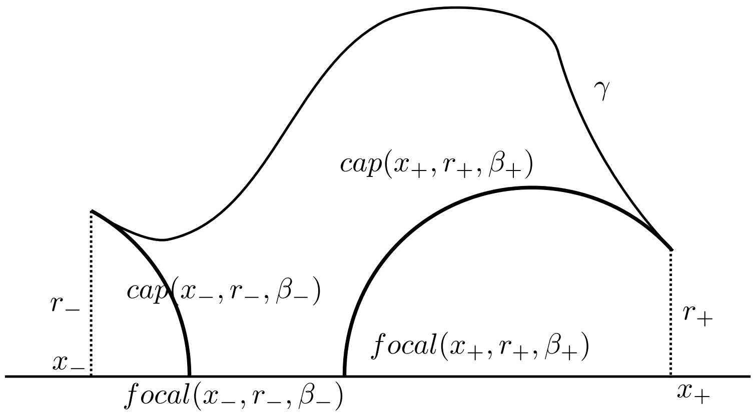


Figure 1: Definition of focal points and sphere caps.

in the vertical case when $\cos\beta_0 = 0$ , see Figure 1 below.

We define the image varifold of $f_\gamma$ by

$$f_{\gamma,\#}\mathcal{H}^2 \lfloor \Sigma_L := \#(f^{-1}(.)) \cdot \mathcal{H}^2 \lfloor f(\Sigma_L), \tag{2.8}$$

which is an integral varifold, but it does not have square integrable weak mean curvature due to the boundary. For the notions in geometric measure theory, we refer to [Sim]. Adding the inverse sphere caps, $f_\gamma$ for $\gamma$ satisfying the boundary conditions (1.6) extends to a closed rotationally symmetric $C^{1,1}-$immersion and putting

$$\hat{\mu}_\gamma := (f_{\gamma,\#}\mathcal{H}^2 \lfloor \Sigma_L) + (\mathcal{H}^2 \lfloor cap(x_-, r_-, \beta_- + \pi)) + (\mathcal{H}^2 \lfloor cap(x_+, r_+, \beta_+ + \pi)), \tag{2.9}$$

$\hat{\mu}_\gamma$ is an integral varifold with bounded weak mean curvature. Using as in [KuSch04] §A the density at infinity $\theta^2(., \infty)$ , that is

$$\theta^2(\hat{\mu}, \infty) := \lim_{\varrho \to \infty} \mu(B_\varrho(0))/(\pi\varrho^2),$$

we see that

$$\mathcal{W}(\hat{\mu}_\gamma) + 4\pi \cdot \theta^2(\hat{\mu}_\gamma, \infty) \tag{2.10}$$

adds $4\pi$ for each infinite sphere cap. Moreover using (1.8) and the pointwise invariance of $|A_f^0|^2 vol_f$ , see [Ch74], this expression remains unchanged for any inversion at a point of the $x-$axis. Likewise, we see from (1.7) and (1.8) and recalling that the profile curves of sphere caps are geodesics in the hyperbolic metric, hence have vanishing hyperbolic curvature, that

$$\frac{\pi}{2}\mathcal{F}(\gamma) = \mathcal{W}(\hat{\mu}_\gamma) + 4\pi \cdot \theta^2(\hat{\mu}_\gamma, \infty) - 4\pi. \tag{2.11}$$

To get a better energy comparison with different boundary conditions, we add the inverse sphere caps to the annulus immersions. More precisely we put

$$\begin{aligned} \mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) &:= \mathcal{W}(x_\pm, r_\pm, \beta_\pm) + \\ &+ \mathcal{W}\Big(cap(x_-, r_-, \beta_- + \pi)\Big) + 4\pi \cdot \theta^2(\mathcal{H}^2 \lfloor cap(x_-, r_-, \beta_- + \pi), \infty) + \\ &+ \mathcal{W}\Big(cap(x_+, r_+, \beta_+ + \pi)\Big) + 4\pi \cdot \theta^2(\mathcal{H}^2 \lfloor cap(x_+, r_+, \beta_+ + \pi), \infty), \end{aligned} \tag{2.12}$$

where $\mathcal{W}(x_\pm, r_\pm, \beta_\pm)$ is given in (2.1) and $\theta^2(., \infty)$ is the density at infinity, see (2.10), which adds $4\pi$ for each infinite inverse sphere cap and gives invariance of $\mathcal{W}_{closed}$ for any inversion at a point of the $x-$axis.

Our main observation for the following upper bound for the energy is that two disjoint spheres whose focal points are much closer than the radii can be joined by a catenoid to get a rotational symmetric surface whose Willmore energy is smaller than $8\pi$ , which is the Willmore energy of the two spheres.

**Proposition 2.2** *For* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *as in Theorem 1.1, we have*

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) < 12\pi. \tag{2.13}$$

*If the focal point of the corresponding sphere caps coincide, that is*

$$focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+), \tag{2.14}$$

*then*

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) \leq 8\pi. \tag{2.15}$$

**Proof:**
After an inversion at an appropriate point of the $x-$axis, which leaves $\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm)$ unchanged, we may assume that both boundary conditions $(x_\pm, r_\pm, \beta_\pm)$ are non-vertical. Then both focal points $focal(x_\pm, r_\pm, \beta_\pm) =: \hat{x}_\pm e_1$ are real. First we consider the case that the focal points do not coincide

$$focal(x_-, r_-, \beta_-) \neq focal(x_+, r_+, \beta_+).$$

Then the sphere

$$S := \{(x, y, z) \mid x - (\hat{x}_+ + \hat{x}_-)/2|^2 + y^2 + z^2 = |\hat{x}_+ - \hat{x}_-|^2/4 > 0 \ \}$$

touches the sphere caps $cap(x_\pm, r_\pm, \beta_\pm)$ at their distinct focal points $focal(x_\pm, r_\pm, \beta_\pm)$ . Therefore the spheres

$$S_\pm := cap(x_\pm, r_\pm, \beta_\pm) \cup cap(x_\pm, r_\pm, \beta_\pm + \pi)$$

lie in the closure of the interior of $S$ or in the closure of the exterior of $S$ . If both $S_\pm$ lie in the closure of the interior of $S$ , we stretch $S =: S_0$ to

$$S_\delta := \{(x, y, z) \mid x - (\hat{x}_+ + \hat{x}_-)/2|^2 + y^2 + z^2 = |\hat{x}_+ - \hat{x}_-|^2/4 + \delta > 0 \ \},$$

and if both $S_\pm$ lie in the closure of the exterior of $S$ , we shrink $S =: S_0$ to

$$S_\delta := \{(x, y, z) \mid x - (\hat{x}_+ + \hat{x}_-)/2|^2 + y^2 + z^2 = |\hat{x}_+ - \hat{x}_-|^2/4 - \delta > 0 \ \}.$$

If one of the spheres $S_\pm$ lies in the closure of the interior of $S$ and the other lies in the closure of the exterior of $S$ , then both $S_\pm$ lie either on the left or the right side of their focal points $focal(x_\pm, r_\pm, \beta_\pm)$ . Moreover none of the spheres $S_\pm$ coincide with $S$ in particular $S_\pm$ are stricly contained in the interior or the exterior of $S$ apart from its focal points. In this case we translate $S =: S_0$ to

$$S_\delta := S \pm \delta e_1$$

accordingly. In any of these cases, $S_\delta$ is disjoint to the spheres $S_\pm$ for $\delta > 0$ , but comes arbitrarily close to the focal points $focal(x_\pm, r_\pm, \beta_\pm)$ for $\delta \searrow 0$ . There are inversions at points $x_\delta^\pm e_1 \to x_0^\pm e_1$ of the $x-$axis which transform $S_\delta$ and one of the spheres $S_\pm$ to spheres of same radius bounded from below and above, still touching for $\delta = 0$ and coming arbitrarily close for $\delta \to 0$ . For these spheres sufficiently close, there is by Proposition A.1 a catenoid touching the two spheres at an arbitrarly small opening angle, hence touching the sphere caps, and we get a rotational symmetric surface with less Willmore energy than the two spheres and arbitrarily small sphere caps removed, thereby keeping the boundary conditions (1.6) for $(x_\pm, r_\pm, \beta_\pm)$ . As the spheres have energy $12\pi$ , the constructed surface has energy less than $12\pi$ , which gives (2.13).

If the focal points of the corresponding sphere caps coincide, that is (2.14), we may assume after an inversion at an appropriate point of the $x-$axis to simplify the notation that

$$e_1 = focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+),$$
$$S_- = \partial B_1(0), e_1 \in S_+ \subseteq \mathbb{R}^3 - B_1(0),$$

that is $S_+$ lies outside $S_-$ or coincides with $S_-$ . We further assume that $S_+$ is not a plane. Identifying $\mathbb{R}^2 \cong \mathbb{C}$ , we see, as $(e^{i\beta_-}, 0)$ is tangential at $(x_-, r_-, 0)$ and directing to $focal(x_-, r_-, \beta_-) = (1, 0, 0)$ along the sphere $S_-$ , that

$$(x_-, r_-) = e^{i(\beta_- + \pi/2)}$$

with $|\beta_-| < \pi/2$ , as $r_- > 0$ . Choosing $-\pi/2 < \beta_0 < \beta_-$ , we see that $e^{i(\beta_0+\pi/2)} \in cap(x_-, r_-, \beta_-)$ , the corresponding sphere caps have the same focal point

$$focal(e^{i(\beta_0+\pi/2)}, \beta_0) = focal(x_-, r_-, \beta_-)$$

and the boundary conditions $(x_-, r_-, \beta_-), (e^{i(\beta_0+\pi/2)}, \beta_0 + \pi)$ can be joined along the sphere $S_-$ , in particular

$$\mathcal{W}_{closed}((x_-, r_-, \beta_-), (e^{i(\beta_0+\pi/2)}, \beta_0 + \pi)) = 4\pi. \tag{2.16}$$

Now the sphere cap $cap(e^{i(\beta_0+\pi/2)}, \beta_0 - \varepsilon)$ for small $\varepsilon > 0$ stays inside $S_-$ , hence does not intersect $S_+$ , but its focal point approaches

$$focal(e^{i(\beta_0+\pi/2)}, \beta_0 - \varepsilon) \to focal(e^{i(\beta_0+\pi/2)}, \beta_0) =$$
$$= focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+).$$

Inserting a small catenoid after appropriate inversion as above, we see for small $\varepsilon > 0$ that

$$\mathcal{W}_{closed}((e^{i(\beta_0+\pi/2)}, \beta_0 - \varepsilon), (x_+, r_+, \beta_+)) < 8\pi$$

and subtracting the additional sphere caps

$$\mathcal{W}((e^{i(\beta_0+\pi/2)}, \beta_0 - \varepsilon), (x_+, r_+, \beta_+)) <$$
$$< 8\pi - \mathcal{W}(cap(e^{i(\beta_0+\pi/2)}, \pi + \beta_0 - \varepsilon)) - \mathcal{W}(cap(x_+, r_+, \pi + \beta_+)).$$

As obviously by successively joining the boundary conditions

$$\mathcal{W}(x_\pm, r_\pm, \beta_\pm) \leq$$

$$\leq \mathcal{W}((x_-, r_-, \beta_-), (e^{i(\beta_0+\pi/2)}, \pi + \beta_0 - \varepsilon)) + \mathcal{W}((e^{i(\beta_0+\pi/2)}, \beta_0 - \varepsilon), (x_+, r_+, \beta_+)),$$

we obtain letting $\varepsilon \to 0$ and using Proposition 2.1 for the first term that

$$\mathcal{W}(x_\pm, r_\pm, \beta_\pm) \leq \mathcal{W}((x_-, r_-, \beta_-), (e^{i(\beta_0+\pi/2)}, \pi + \beta_0)) + 8\pi +$$

$$-\mathcal{W}(cap(e^{i(\beta_0+\pi/2)}, \pi + \beta_0)) - \mathcal{W}(cap(x_+, r_+, \pi + \beta_+)).$$

Now adding the additional sphere caps gives

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) \leq \mathcal{W}_{closed}((x_-, r_-, \beta_-), (e^{i(\beta_0+\pi/2)}, \pi + \beta_0)) + 8\pi +$$

$$-\mathcal{W}(cap(e^{i(\beta_0+\pi/2)}, \pi + \beta_0) - \mathcal{W}(cap(e^{i(\beta_0+\pi/2)}, \beta_0)).$$

As $cap(e^{i(\beta_0+\pi/2)}, \pi + \beta_0) \cup cap(e^{i(\beta_0+\pi/2)}, \beta_0)$ give a full sphere, we have

$$\mathcal{W}(cap(e^{i(\beta_0+\pi/2)}, \pi + \beta_0) + \mathcal{W}(cap(x_+, r_+, \beta_0)) = 4\pi$$

and obtain (2.15) by observing (2.16).

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In our limit procedures, we will obtain the regular profile curves of round spheres or vertical planes, but also the regular profile curves of Möbius transforms of catenoids, see Propositon 3.7. For the catenoids, we will need the following energy estimate.

**Proposition 2.3** *Let* $\gamma_i : [0, \infty[ \to \mathcal{H}, i = 1, 2$ *, be the regular profile curves of parts of a round sphere, of a vertical plane or of a Möbius transform of a catenoid with*

$$\lim_{t\to\infty} \gamma_i(t) \in \mathbb{R}e_1 \cup \{\infty\} = \bar{\mathbb{R}}. \tag{2.17}$$

*Then for any* $t_0 > 0$ *there exists* $t_1, t_2 \geq t_0$ *with*

$$\mathcal{W}_{closed}(\gamma_i(t_i), \gamma_i'(t_i)/|\gamma_i'(t_i)|) \leq 8\pi. \tag{2.18}$$

**Proof:**
If both $\gamma_\pm$ are the regular profile curves of parts of a round sphere or of a vertical plane, then (2.17) means that the focal points of the sphere caps coincide, and (2.18) follows from (2.15). To simplify the notation, we may assume after an inversion at a point of the $x-$axis or a horizontal translation that $0 = \lim_{t\to\infty} \gamma_i(t)$ . Then the inversions $\tilde{\gamma}_i := \gamma_i/|\gamma_i|^2$ are the regular profile curves of parts of a vertical plane or of a horizontal translation and homothetie of the standard catenoid.

As we may assume that the catenoid appears, we examine $\tilde{\gamma}_{cat}(t) := (M + \alpha t, \alpha \cosh t)$ with $M \in \mathbb{R}, \alpha > 0$ and its inversion $\gamma_{cat} := \tilde{\gamma}_{cat}/|\tilde{\gamma}_{cat}|^2$ . We want to write $\gamma_{cat}$ for large $t$ as a graph over the positive $y-$axis

$$(\varphi(y(t)), y(t)) = \gamma_{cat}(t) = \tilde{\gamma}_{cat}(t)/|\tilde{\gamma}_{cat}|^2(t) = \frac{(M + \alpha t, \alpha \cosh t)}{(M + \alpha t)^2 + \alpha^2 \cosh^2 t} \quad \text{for large } t \tag{2.19}$$

and for some smooth $\varphi$ . Clearly $y$ is smoothly defined. Abbreviating the reciprocal of the denominator by $A(t) := ((M + \alpha t)^2 + \alpha^2 \cosh^2 t)^{-1}$ , we get differentiating

$$y'(t) = (\alpha \cosh t \cdot A(t))' = \alpha \sinh t \cdot A(t) + \alpha \cosh t \cdot A'(t), \tag{2.20}$$

$$A'(t) = -2A(t)^2(\alpha(M+\alpha t) + \alpha^2 \sinh t \cosh t).$$

In the following, we abbreviate for $a, b > 0$ defined for large $t$ that

$$a(t) \approx b(t) :\Longleftrightarrow \qquad \lim_{t\to\infty} a(t)/b(t) = 1,$$
$$a(t) \sim b(t) :\Longleftrightarrow \quad 0 < \liminf_{t\to\infty} a(t)/b(t) \le \limsup_{t\to\infty} a(t)/b(t) < \infty$$

and likewise for $a, b \neq 0$ defined for small positive $y$ and $y \searrow 0$.

Then

$$A'(t)/A(t) = -2\frac{\alpha(M+\alpha t) + \alpha^2 \sinh t \cosh t}{(M+\alpha t)^2 + \alpha^2 \cosh^2 t} \approx -2, \tag{2.21}$$

and

$$\frac{y'(t)}{A(t)\cosh t} = \alpha \tanh t + \alpha A'(t)/A(t) \approx -\alpha < 0,$$

in particular $y'(t) < 0$ for large $t$, and $t \mapsto y(t)$ has a smooth inverse for small $y$, as $y(t) \to 0$, since $\gamma_{cat}(t) \to 0$ for $t \to \infty$. Therefore there exists a smooth $\varphi$ defined for small $y$ satisfying (2.19). Then

$$y(t)e^t/2 \approx y(t)\cosh t = \frac{\alpha \cosh^2 t}{(M+\alpha t)^2 + \alpha^2 \cosh^2 t} \approx \alpha^{-1} > 0 \tag{2.22}$$

$$\frac{\log(1/y(t))}{t} = 1 - t^{-1}\log(e^t y(t)) \approx 1,$$

$$\varphi(y) = \frac{M+\alpha t}{(M+\alpha t)^2 + \alpha^2 \cosh^2 t} \approx \alpha^{-1} t/\cosh^2 t \approx \alpha y^2 \log(1/y). \tag{2.23}$$

Again differentiating, we get

$$\varphi'(y(t))y'(t) = \varphi(y(t))' = ((M+\alpha t)\cdot A(t))' = \alpha A(t) + (M+\alpha t)\cdot A'(t),$$

hence by (2.20) and (2.21) that

$$\varphi'(y) = \frac{\alpha A(t) + (M+\alpha t)\cdot A'(t)}{\alpha \sinh t \cdot A(t) + \alpha \cosh t \cdot A'(t)} = \frac{\alpha + (M+\alpha t)\cdot A'(t)/A(t)}{\alpha \sinh t + \alpha \cosh t \cdot A'(t)/A(t)} \approx$$

$$\approx \frac{-2\alpha t}{-\alpha \cosh t} \approx 2\alpha y \log(1/y), \tag{2.24}$$

in particular $\varphi'(y) > 0$ for small $y > 0$.

Now for the boundary data of $\gamma_{cat}$ or likewise of *graph* $\varphi$, we consider the sphere caps

$$cap(\gamma_{cat}(t), \gamma'_{cat}(t)/|\gamma'_{cat}(t)|) = cap\Big(\varphi(y), y, \frac{(-\varphi'(y), -1)}{\sqrt{1+\varphi'(y)^2}}\Big) =: cap(y)$$

for $y = y(t)$ for large $t$ or likewise small $y > 0$. Now $(1, -\varphi'(y))/\sqrt{1+\varphi'(y)^2}$ is normal to the sphere cap $cap(y)$ at $(\varphi(y), y) \in cap(y)$, hence the center of the sphere cap $cap(y)$ is $\xi(y)e_1 \in \mathbb{R}$ with

$$\xi(y) := \varphi(y) + y/\varphi'(y) \tag{2.25}$$

and its radius is given by

$$\varrho(y) := \sqrt{1+\varphi'(y)^2} \cdot y/\varphi'(y). \tag{2.26}$$

As the direction of the boundary condition $(-\varphi'(y),-1)/\sqrt{1+\varphi'(y)^2}$ is pointing to the left by (2.24), the focal point of the sphere cap $cap(y)$ is given by

$$focal(y) := focal\Big(\varphi(y), y, \frac{(-\varphi'(y),-1)}{\sqrt{1+\varphi'(y)^2}}\Big) = (\xi(y) - \varrho(y))e_1. \tag{2.27}$$

Clearly

$$\xi(y), \varrho(y) \to 0 \quad \text{for } y \searrow 0$$

by (2.23), (2.24), (2.25) and (2.26). Even more

$$\frac{\xi(y)-\varrho(y)}{\varrho(y)} = \frac{\varphi(y) + y/\varphi'(y) - \sqrt{1+\varphi'(y)^2}\cdot y/\varphi'(y)}{\sqrt{1+\varphi'(y)^2}\cdot y/\varphi'(y)} =$$

$$= \frac{\varphi'(y)\varphi(y)/y + 1 - \sqrt{1+\varphi'(y)^2}}{\sqrt{1+\varphi'(y)^2}} \to 0 \quad \text{for } y \searrow 0 \tag{2.28}$$

by (2.23), (2.24), (2.25) and (2.26).

The inverted catenoid can also lie on the left side of the $y-$axis, and this is obtained by considering $t \to -\infty$ .

Now we distinguish three cases. First we consider that $\gamma_1 = \gamma_{cat}$ parametrizes an inverted catenoid, and $\gamma_2$ parametrizes a part of a round sphere or of a vertical plane, hence after an inversion that $\gamma_2$ parametrizes a part of the $y-$axis which ends in the origin.

Next we move a circle with center $\xi e_1$ on the $x-$axis and small radius $r > 0$ for large $\xi \gg 0$ to the left until it touches the inverted catenoid parametrized by $\gamma_{cat}$ say in $(x_r, y_r) \in \gamma_{cat}(\mathbb{R}) \cup \{0\}$ . Then obviously, since the center of this touching ball lies on the $x-$axis, we have $0 \leq y_r \leq r$ is small. Now since $\gamma_{cat}$ coincides for large $t$ with *graph* $\varphi$ and $\varphi$ is not quadratically decaying by (2.23), the ball cannot touch the inverted catenoid from the right at the origin, hence $y_r > 0$ and this ball touches *graph* $\varphi$ in $(x_r, y_r) = (\varphi(y_r), y_r)$ and completely lies in the half plane $[x > 0]$ . Then the part of this ball left of $x = x_r$ is the sphere cap $cap(y_r)$ , in particular its focal point $focal(y_r)$ lies in $[x > 0]$ , that is $\xi(y_r) - \varrho(y_r) > 0$ . As $r$ can be chosen arbitrarily small, we conclude

$$\exists y_j \searrow 0 : \xi(y_j) - \varrho(y_j) > 0. \tag{2.29}$$

Rescaling the radius $r = \varrho(y_r)$ to $1$ , the distance of the ball to the $y-$axis is the distance of the rescaled focal point to the origin that is

$$0 < d(r) := \frac{\xi(y_r) - \varrho(y_r)}{\varrho(y_r)} \to 0 \quad \text{for } r \to 0 \tag{2.30}$$

by (2.28), as $y_r \leq r \to 0$ .

After an inversion at an appropriate point of the $x-$axis, the ball and the $y-$axis are mapped on balls of same radius bounded from below and above, at distance $\tilde{d}(r)$ which is up to a bounded factor $d(r)$ . Then by Proposition A.1 for small $r > 0$ , these balls can be touched from above by a catenoid rotationally symmetric with respect to the $x-$axis and symmetric with respect to the balls, and this catenoid seperates on each ball a sphere cap of opening angle $0 < \beta(\tilde{d}(r)) \to 0$ for $r \to 0$ . On the other hand, the opening angle $\omega(y)$ of the sphere cap $cap(y)$ can be calculated as in (A.4) by the slope of the boundary condition as

$$\tan(\omega(y)/2) = \varphi'(y) \in ]0, \infty[, \tag{2.31}$$

this time it is not the reciprocal of the slope, as we consider the graph of $\varphi$ over the $y-$axis. Therefore when $r$ decreases to $0$ , the opening angle $\omega(y_r)$ of the sphere cap $cap(y_r)$ also decreases to $0$ . In order for the catenoid to touch the sphere caps, and hence gives a connection between the boundary data of the inverted sphere caps, we have to ensure that $\beta(\tilde{d}(r))$ is smaller than both opening angles of the inverted sphere cap $cap(y_r)$ and of the inverted part of the vertical line for small $r$ .

The opening angle $\tilde{\omega}(y_r)$ of the inverted sphere cap is up to a bounded factor $\omega(y_r)$ , whereas the opening angle of the inverted $y-$axis is given by the inversion of the line segment of the $y-$axis starting independend of $r$ at $\gamma_2(t_0)$ , ending in the origin and being stretched by $r^{-1} \to \infty$ , and therefore this opening angle is positively bounded from below for $r \to 0$ . Therefore in order for the catenoid to yield a connection between the boundary data of the inverted sphere caps, it is sufficient to establish

$$\lim_{r \to 0} \frac{\beta(\tilde{d}(r))}{\omega(y_r)} = 0. \tag{2.32}$$

First we know from Proposition A.1 (A.1) and (2.30) that

$$\liminf_{r \to 0} \frac{d(r)}{\tan^2(\beta(\tilde{d}(r))/2)} \geq \lim_{\tilde{d} \searrow 0} \frac{c_0 \tilde{d}}{\tan^2(\beta(\tilde{d})/2)} = \infty \tag{2.33}$$

for some given $c_0 > 0$ independent of $r$ . Next we continue from (2.28) and improve and using $\sqrt{1+x} \geq 1 + x/2 - x^2/4$ to

$$\frac{\xi(y) - \varrho(y)}{\varrho(y)} \leq \frac{\varphi'(y)\varphi(y)/y - \varphi'(y)^2/2 + \varphi'(y)^4/4}{\sqrt{1 + \varphi'(y)^2}},$$

hence

$$\limsup_{y \searrow 0} \frac{\xi(y) - \varrho(y)}{\varrho(y)} \varphi'(y)^{-2} \leq \limsup_{y \searrow 0} \Big( \varphi(y)/(y\varphi'(y)) - 1/2 + \varphi'(y)^2/4 \Big) \leq 0, \tag{2.34}$$

as

$$y\varphi'(y) \approx 2\alpha y^2 \log(1/y) \approx 2\varphi(y)$$

by (2.23) and (2.24). Then (2.30) and (2.31) yield

$$\limsup_{r \to 0} \frac{d(r)}{\tan^2(\omega(y_r)/2)} \leq \limsup_{y \searrow 0} \frac{\xi(y) - \varrho(y)}{\varrho(y)} \varphi'(y)^{-2} \leq 0,$$

hence with (2.33) and since $\beta(\tilde{d}(r)), \omega(y_r) \to 0$ for $r \to 0$ that

$$\lim_{r \to 0} \frac{\beta(\tilde{d}(r))}{\omega(y_r)} = \lim_{r \to 0} \frac{\tan(\beta(\tilde{d}(r))/2)}{\tan(\omega(y_r)/2)} = 0,$$

which is (2.32).

Therefore a catenoid touches the inverted sphere caps from above for small $r > 0$, hence for $y_r = y(t_1)$ with $t_1 > t_0$ for small $r > 0$ that

$$\mathcal{W}_{closed}(\gamma_1(t_1), \gamma_1'(t_1)/|\gamma_1'(t_1)| \ ; \ \gamma_2(t_0), \gamma_2'(t_0)/|\gamma_2'(t_0)|) < 8\pi,$$

which gives (2.18), and the proposition is proved in the case exactly one $\gamma_i$ parametrizes an inverted catenoid.

In the second case, we consider that both $\gamma_i$ parametrize inverted catenoids which both lie locally right to the $y-$axis. Then $\gamma_i$ parametrize for large $t$ the graphs of smooth function $\varphi_i$ with $M_i \in \mathbb{R}, \alpha_i > 0$ as above in the definition of $\gamma_{cat}$. Indexing the sphere caps $cap_i$ corresponding to $\varphi_i$ their centers $\xi_i$, their radii $\varrho_i$, their focal points $focal_i$ and the functions $y_i$, we start with (2.29) choosing $y_0 > 0$ with $y_0 = y_1(\tilde{t})$ for some $\tilde{t} > t_0$ and

$$\xi_1(y_0) - \varrho_1(y_0) > 0.$$

Then by continuity, (2.28) and (2.29), there exists $y_2 > 0$ with $y_2 = y_2(t_2)$ for some $t_2 > t_0$ and

$$0 < \xi_2(y_2) - \varrho_2(y_2) < \xi_1(y_0) - \varrho_1(y_0).$$

and again by continuity and (2.28) there exists $y_1 > 0$ with $y_1 = y_2(t_1)$ for some $t_1 > \tilde{t} > t_0$ and

$$\xi_1(y_1) - \varrho_1(y_1) = \xi_2(y_2) - \varrho_1(y_2) > 0.$$

This means

$$focal(\gamma_1(t_1), \gamma_1'(t_1)/|\gamma_1'(t_1)|) = focal_1(y_1) =$$
$$= focal_2(y_2) = focal(\gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|),$$

hence by Proposition 2.2 (2.15) that

$$\mathcal{W}_{closed}(\gamma_i(t_i), \gamma_i'(t_i)/|\gamma_i'(t_i)|) \leq 8\pi,$$

which is (2.18) and proves the second case.

Finally, we consider the case when both $\gamma_i$ parametrize inverted catenoids which lie locally opposite the $y-$axis, say $\gamma_1$ on the right and $\gamma_2$ on the left. Then $\gamma_i$ parametrize for large $t$ the graphs of smooth function $\varphi_1, -\varphi_2 > 0$ with $M_i \in \mathbb{R}, \alpha_i > 0$ as above in the definition of $\gamma_{cat}$.

It will be important that the focal points lie on the same side of the $y-$axis as the inverted cateniod, that is

$$\xi_i(y) - \varrho_i(y) > 0 \quad \text{for small } y. \tag{2.35}$$

Actually, we will not prove (2.35), but instead we give an easy argument which will serve for our purpose. If $\gamma_1 = \gamma_{cat}$ does not satisfy (2.35), there is $\tilde{y}_j \searrow 0$ with $\xi_1(\tilde{y}_j) - \varrho_1(\tilde{y}_j) \leq 0$, hence by continuity and (2.29), there is $\hat{y}_j \searrow 0$ with $\xi_1(\hat{y}_j) - \varrho_1(\hat{y}_j) = 0$ or likewise $focal_1(\hat{y}_j) = 0$. Then there is $t_1 > t_0$ with $focal_1(y(t_1)) = 0$, and we replace $\gamma_1$ by $\gamma_1^*$, which coincides with $\gamma_1$ on $[0, t_1]$, but parametrizes the sphere cap $cap_1(y(t_1))$ on $[t_1, \infty[$. Now from the first case with only one inverted catenoid, there exist $t_1^*, t_2 > t_1 > t_0$ with

$$\mathcal{W}_{closed}(\gamma_1^*(t_1^*), (\gamma_1^*)'(t_1^*)/|(\gamma_1^*)'(t_1^*)| \ ; \ \gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|) \leq 8\pi.$$

Since $\gamma_1^*$ parametrizes a sphere cap on $[t_1, \infty[\ni t_1^*$ , we obviously have

$$\mathcal{W}_{closed}(\gamma_1^*(t_1), (\gamma_1^*)'(t_1)/|(\gamma_1^*)'(t_1)|\ ;\ \gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|) \leq$$

$$\leq \mathcal{W}_{closed}(\gamma_1^*(t_1^*), (\gamma_1^*)'(t_1^*)/|(\gamma_1^*)'(t_1^*)|\ ;\ \gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|),$$

hence

$$\mathcal{W}_{closed}(\gamma_1(t_1), (\gamma_1)'(t_1)/|(\gamma_1)'(t_1)|\ ;\ \gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|) =$$

$$= \mathcal{W}_{closed}(\gamma_1^*(t_1), (\gamma_1^*)'(t_1)/|(\gamma_1^*)'(t_1)|\ ;\ \gamma_2(t_2), \gamma_2'(t_2)/|\gamma_2'(t_2)|) \leq 8\pi,$$

which is (2.18), and we may assume (2.35).

This time we choose $y > 0$ small with $y = y_1(\tilde{t}_1) = y_2(\tilde{t}_2)$ for some $\tilde{t}_1, \tilde{t}_2 > t_0$ and get from (2.35) that the sphere caps $(-1)^{i-1} cap_i(y_i)$ lie completely right respectively left to the $y-$axis. Moreover by (2.24) and (2.26), the radii of these sphere caps are related by

$$\varrho_1(y) = \sqrt{1 + \varphi_1'(y)^2} \cdot y/\varphi_1'(y) \approx 1/(2\alpha_1 \log y) \sim 1/(2\alpha_2 \log y) \approx \varrho_2(y) \qquad (2.36)$$

up to a bounded factor. Stretching by the reciprocal of $\varrho(y) \in \{\varrho_i(y)\}$ , the sphere cap get radii bounded from below and above, and their distance is

$$0 < d(y) := \frac{\xi_1(y) - \varrho_1(y)}{\varrho(y)} + \frac{\xi_2(y) - \varrho_2(y)}{\varrho(y)} \to 0 \quad \text{for } y \searrow 0 \qquad (2.37)$$

by (2.28), (2.35) and (2.36).

As in the first case after an inversion at an appropriate point of the $x-$axis, the sphere caps are mapped on sphere caps of radius $1$ and at distance $\tilde{d}(y)$ which is up to a bounded factor $d(y)$ . Then by Proposition A.1 for small $y > 0$ , these balls corresponding to the sphere caps can be touched from above by a catenoid rotationally symmetric with respect to the $x-$axis and symmetric with respect to the balls, and this catenoid seperates on each ball a sphere cap of opening angle $0 < \beta(\tilde{d}(y)) \to 0$ for $y \to 0$ . On the other hand, the opening angle $\omega_i(y)$ of the sphere caps $cap_i(y)$ can be calculated as in (2.31) by the slope of the boundary condition as

$$\tan(\omega_i(y)/2) = \varphi_i'(y) \in ]0, \infty[, \qquad (2.38)$$

in particular by (2.24) that $\omega_i(y) \to 0$ for $y \to 0$ and

$$\omega_1(y) \sim \omega_2(y). \qquad (2.39)$$

The opening angles $\tilde{\omega}_i(y)$ of the inverted sphere caps are up to a bounded factor $\omega_i(y)$ , and we therefore estimate $\omega_i(y)$ . If $\xi_1(y) - \varrho_1(y) \geq \xi_2(y) - \varrho_2(y)$ , we get for the distance by (2.36) and (2.37) that

$$\tilde{d}(y) \leq C' d(y) \leq C \frac{\xi_1(y) - \varrho_1(y)}{\varrho_1(y)}$$

for some $C, C' < \infty$ independent of $y$ , hence in any case by (2.34) and (2.38) that

$$\limsup_{y \searrow 0} \frac{\tilde{d}(y)}{\sup_i \tan^2(\omega_i(y)/2)} \leq C \limsup_{y \searrow 0} \sup_i \left( \frac{\xi_i(y) - \varrho_i(y)}{\varrho_i(y)} \varphi_i'(y)^{-2} \right) \leq 0.$$

As

$$\lim_{y\searrow 0}\frac{\tilde d(y)}{\tan^2(\beta(\tilde d(y))/2)}=\lim_{\tilde d\searrow 0}\frac{\tilde d}{\tan^2(\beta(\tilde d)/2)}=\infty$$

by Proposition A.1 (A.1) and $\tilde d(y),\beta(\tilde d(y)),\omega_i(y)\to 0$ for $y\searrow 0$ , we get

$$\lim_{y\searrow 0}\frac{\beta(\tilde d(y))}{\sup_i\omega_i(y)}=\lim_{y\searrow 0}\frac{\tan(\beta(\tilde d(y))/2)}{\sup_i\tan(\omega_i(y)/2)}=0,$$

hence with (2.39) that

$$\lim_{y\searrow 0}\frac{\beta(\tilde d(y))}{\inf_i\tilde\omega_i(y)}=0.$$

Therefore $\tilde\omega_i(y)<\beta(\tilde d(y))$ for small $y>0$ and a catenoid touches the inverted sphere caps from above, hence for $y=y_1(t_1)=y_2(t_2)$ with $t_1,t_2>t_0$ for small $y>0$ that

$$\mathcal{W}_{closed}(\gamma_1(t_1),\gamma_1'(t_1)/|\gamma_1'(t_1)|\ ;\ \gamma_2(t_2),\gamma_2'(t_2)/|\gamma_2'(t_2)|)<8\pi,$$

which gives (2.18) also in the third case, and the proposition is fully proved.

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## 3 Direct method

We are seeking the solutions of the Willmore boundary problem in Theorem 1.1 by the direct method of considering a minimizing sequence for the Willmore energy of rotationally symmetric immersions satisfying the boundary conditions (1.6). We repeat the necessary definitions and theorems of [CoVe13] here, on which we build our minimizing scheme. We start with the concept of generalized generators, which allows its corresponding surface of revolution to touch the $x$-axis.

**Definition 3.1 (see Def. 3 in [CoVe13])** *We say that $\gamma:[0,1]\to\mathbb{R}^2$ is a generalized generator, if $\gamma$ is Lipschitz continuous, $|\gamma'|\equiv\ell(\gamma)$ and $\gamma_2(t)>0$ for $\mathcal{L}^1$-almost every $t\in(0,1)$, $\gamma''\in L^1_{Loc}(\{\gamma_2>0\};\mathbb{R}^2)$ exists in a weak sense and there exists a $C>0$, such that*

$$\int_0^1(k_1^\gamma)^2+(k_2^\gamma)^2\,d\mu_\gamma<C. \tag{3.1}$$

*Here $\kappa_{1/2}^\gamma$ are the principal curvatures of $f_\gamma$, $\ell(\gamma)$ denotes the length of the curve $\gamma$ w.r.t. to the euclidean metric and $\mu_\gamma$ is the corresponding measure of the surface of revolution, i.e.*

$$\mu_\gamma:=2\pi\gamma^2|\gamma'|\mathcal{L}^1\lfloor[0,1]. \tag{3.2}$$

□

Please note, that in [CoVe13] the revolution of the curve was performed around the $y$-axis, hence the coordinates have been changed here to reflect this. We have the following regularity.

**Lemma 3.2 (see Lemma 3 in [CoVe13])** *Let $\gamma$ be a generalized generator as in Definition 3.1. Then for any subinterval $[a,b] \subset [0,1] \cap \{\gamma_2 > 0\}$ we have*

$$\gamma \in W^{2,2}((a,b),\mathbb{R}^2) \text{ and } \gamma' \text{ has a unique extension to } C^0([a,b],\mathbb{R}^2).$$

□

Next we have the behaviour of the tangent on the $x$-axis.

**Lemma 3.3 (see Lemma 4 in [CoVe13])** *Let $\gamma$ be a generalized generator as in Definition 3.1. Let $a,b \in [0,1]$ be such that $\gamma_2(a) = \gamma_2(b) = 0$, $\gamma_2(t) > 0$ for all $t \in (a,b)$. Then the limits of $\gamma'$ as $t \to a^+$ rsp. $t \to b^-$ exist and furthermore*

$$\lim_{t\to a^+} \gamma_1'(t) = \lim_{t\to b^-} \gamma_1'(t) = 0.$$

*Also either*

$$\lim_{t\to a^+} \gamma_2'(t) = \ell(\gamma), \ \lim_{t\to b^-} \gamma_2'(t) = -\ell(\gamma)$$

*or*

$$\lim_{t\to a^+} \gamma_2'(t) = -\ell(\gamma), \ \lim_{t\to b^-} \gamma_2'(t) = \ell(\gamma)$$

*holds.* □

Next we define the convergence for a sequence of generators suitable for our variational endeavor.

**Definition 3.4 (see Def. 4 and Def. 6 in [CoVe13])** *Let $\gamma_n$ be a sequence of generalized generators and $\gamma$ be a generalized generator as in Definition 3.1 as well. We say $\gamma_n$ convergences weakly as generators to $\gamma$, if and only if the following holds*

$$\gamma_n \to \gamma \text{ uniformely in } C^0([0,1],\mathbb{R}^2), \tag{3.3}$$

$$\gamma_n' \to \gamma' \text{ strongly in } L^2((0,1),\mathbb{R}^2), \tag{3.4}$$

$$\mu_{\gamma_n} \to \mu_\gamma \text{ weakly as measures} \tag{3.5}$$

$$\sup_{n\in\mathcal{N}} \int_{[0,1]} |\gamma_n''|^2 \, d\mu_{\gamma_n} < \infty, \tag{3.6}$$

$$\text{for all } \varphi \in C_c^\infty(\mathbb{R},\mathbb{R}^2) \text{ we have } \lim_{n\to\infty} \int \gamma_n'' \cdot \varphi \, d\mu_{\gamma_n} = \int \gamma'' \cdot \varphi \, d\mu_\gamma. \tag{3.7}$$

□

Please note, that (3.7) together with (3.5) and (3.6) is also called convergence of the measure function pair $(\gamma_n'', \mu_{\gamma_n})$ to $(\gamma'', \mu_\gamma)$, see e.g. [Hu86, §4]. Now we turn our attention to lower-semicontinuity.

**Proposition 3.5 (see Prop. 1 in [CoVe13])** *Let $\gamma_n$ be a sequence of generalized generators converging in the sense of Definition 3.4 to a generalized generator $\gamma$. Then*

$$\liminf_{n\to\infty} \mathcal{W}(f_{\gamma_n}) \geq \mathcal{W}(f_\gamma).$$

□

Since we only deal with the Willmore energy, Proposition 3.5 can be directly proven by e.g. [Sim93, 39.4] and density of smooth functions in $L^2$ (see e.g. [Ei19] (4.11) to (4.14) for a proof in this way). In [CoVe13] this result is more involved, because it also covers the Canham-Helfrich energy, which is in general not lower semicontinuous (see e.g. [GB93] p. 550, Remark (ii)). At last we cite the necessary compactness result.

**Proposition 3.6 (see Def. 1, Eq. 49, Eq. 50 and Prop. 2 in [CoVe13])** *Let* $\gamma_n : [0,1] \to \mathbb{R} \times [0,\infty)$ *with* $\gamma_n \in C^1((0,1),\mathbb{R}^2) \cap W^{2,2}_{loc}((0,1),\mathbb{R}^2)$ *with* $\gamma_{n,2}(0) = \gamma_{n,2}(1) = 0$, $|\gamma_n'(t)| = \ell(\gamma_n)$ *and* $\gamma_{n,2}(t) > 0$ *for all* $t \in (0,1)$. *Furthermore let*

$$\sup_{n\in\mathcal{N}} \mathcal{W}(f_{\gamma_n}) < \infty \tag{3.8}$$

*and*

$$\sup_{n\in\mathcal{N}} \mu_{\gamma_n}(\mathbb{R}) < \infty. \tag{3.9}$$

*Then either there exists a subsequence (after relabeling and possibly after a translation)* $\gamma_n$ *which converges to a generalized generator* $\gamma$ *in the sense of Definition 3.4 or there exists a point* $(z,0) \in \mathbb{R}^2$ *such that* $\gamma_n$ *converges strongly in* $W^{1,2}((0,1),\mathbb{R}^2)$ *to that point.* □

Please note, that (3.9) is a uniform bound on the area of the surfaces $f_{\gamma_n}$.

Next we combine the a priori estimates in Proposition 2.2 and the variational setup of [CoVe13] to prove the following proposition.

**Proposition 3.7** *Let*

$$M := \left\{ \begin{array}{c} \gamma : [0,1] \to \mathbb{R} \times [0,\infty) \text{ a generalized generator} \\ \text{satisfying the boundary conditions (1.6)} \\ \#\{t \in [0,1] : \ \gamma_2(t) = 0\} \leq 1 \end{array} \right\}.$$

*Then there exists a* $\gamma_{min} : [0,1] \to \mathbb{R} \times [0,\infty) \cup \{\infty\}$ *satisfying (1.6) with*

$$\mathcal{W}_{closed}(f_{\gamma_{min}}) = \inf_{\gamma\in M} \mathcal{W}(f_\gamma). \tag{3.10}$$

*Furthermore there is at most one* $t \in (0,1)$ *with* $\gamma_{min}(t) \in \mathbb{R} \times \{0\} \cup \{\infty\}$. *Outside of* $t$ *the curve* $\gamma$ *is regular. If there exists such a* $t \in (0,1)$, *then outside of* $t$, $f_{\gamma_{min}}$ *is either part of a half sphere or a Moebius transformed catenoid. If on the other hand no such* $t$ *occurs, then* $\gamma \in M$ *is a smooth critical point for* $\mathcal{W}(f_.)$ *rsp.* $\mathcal{F}$. □

We start with a lemma, which will enable us to estimate the number of possible singularities. A similar estimate is [CoVe13, Lemma 6], but ours gives a slightly better constant, which we will actually need later.

**Lemma 3.8** *Let* $\gamma$ *be a generalized generator as in Definition 3.1. Let* $a, b \in [0,1]$ *be such that* $\gamma_2(a) = \gamma_2(b) = 0$, $\gamma_2(t) > 0$ *for all* $t \in (a,b)$. *Then the Willmore energy of* $f_{\gamma|_{[a,b]}}$ *satisfies*

$$\mathcal{W}(f_{\gamma|_{[a,b]}}) \geq 4\pi.$$

**Proof:**
Let $a < t_a < t_b < b$. Then we use for a regular curve $c \in W^{2,2}((0,L),\mathcal{H})$ that

$$\frac{2}{\pi}\mathcal{W}(f_c) = \mathcal{F}(c) - 4\left[\frac{c_2'}{\sqrt{(c_1')^2+(c_2')^2}}\right]_0^L, \tag{3.11}$$

see e.g. [Ei14, Thm 3.10] for an elementary proof, on $[t_a, t_b]$ and obtain

$$\mathcal{W}(f_{\gamma|_{[a,b]}}) \geq \mathcal{W}(f_{\gamma|_{[t_a,t_b]}}) = \frac{\pi}{2}\left(\mathcal{F}(\gamma|_{[t_a,t_b]}) - 4\left[\frac{\gamma_2'}{\sqrt{(\gamma_1')^2+(\gamma_2')^2}}\right]_{t_a}^{t_b}\right)$$
$$\geq -2\pi\left(\frac{\gamma_2'(t_b)}{\sqrt{(\gamma_1')^2(t_b)+(\gamma_2'(t_b))^2}} - \frac{\gamma_2'(t_a)}{\sqrt{(\gamma_1'(t_a))^2+(\gamma_2'(t_a))^2}}\right).$$

Since $\gamma_2(t) > 0$ in $(a,b)$, Lemma 3.3 yields

$$\lim_{t_a\downarrow a}\gamma_2'(t_a) = \ell(\gamma),\ \lim_{t_b\uparrow b}\gamma_2'(t_b) = -\ell(\gamma).$$

By letting $t_a \downarrow a$ and $t_b \uparrow b$ in the above estimate, Lemma 3.3 yields

$$\mathcal{W}(S(\gamma|_{[a,b]})) \geq -2\pi\left(-\frac{\ell(\gamma)}{|\ell(\gamma)|} - \frac{\ell(\gamma)}{|\ell(\gamma)|}\right) = 4\pi.$$

///

Let us now prove Proposition 3.7.

**Proof of Proposition 3.7:**
Let $\gamma_n \in M$ (see (3.7)) be a minimizing sequence w.r.t. to $\mathcal{W}$. After applying a suitable Moebius transformation, we add two sphere caps to the surface of revolution belonging to $\gamma_n$, such that we obtain a sequence of closed, bounded surfaces, see Figure 2 and (2.9). Since the Willmore energy is invariant w.r.t. conformal changes, the $\gamma_n$ are still a minimizing sequence (for possibly new boundary values, but neverthelesss we obtain the result by applying the inverted version of the first Moebius transformation at the end).

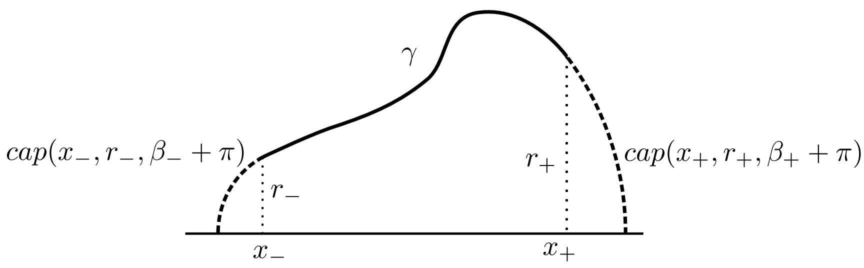


Figure 2: Adding Spheres to minimizing sequence to obtain closed surfaces.

After reparametrization this yields a new sequence of generalized generators, which we call $c_n$. These satisfy

$$\begin{gathered} c_{n,2}(0) = c_{n,2}(1) = 0, \\ \exists 0 < t_-^n < t_+^n < 1 \text{ such that }, \\ c_n(t_-^n) = (x_-, r_-), c_n'(t_-^n)/|c_n'(t_-^n)| = (\cos\beta_-, \sin\beta_-), \\ c_n(t_+^n) = (x_+, r_+), c_n'(t_+^n)/|c_n'(t_+^n)| = -(\cos\beta_+, \sin\beta_+), \\ c_n|_{[t_-^n, t_+^n]} \text{ after reparametrization is the same curve as } \gamma_n, \\ c_n|_{[0,t_-^n]} \text{ and } c_n|_{[t_+^n,1]} \text{ represent constant spheres independent of } n. \end{gathered}$$

In a first step we additionally assume

$$\sup_n \operatorname{diam}(f_{\gamma_n}) < \infty. \tag{3.12}$$

By [Sim93, Lemma 1] and the bound on the Willmore energy we have

$$\sup_n \operatorname{area}(f_{\gamma_n}) < \infty. \tag{3.13}$$

After possibly extracting a subsequence we can distinguish two cases:

1. For all $n \in \mathbb{N}$ there exists exactly one $t_n \in (t_-^n, t_+^n)$ such that $c_{n,2}(t_n) = 0$.
2. For all $n \in \mathbb{N}$ we have $c_{n,2}(t) > 0$ for all $t \in (0, 1)$.

In the second case Proposition 3.6 is already applicable. In the first case we subdivide $c_n$ at $t_n$ in two generalized generators. Both of these then satisfy the assumptions of Theorem 3.6. Hence we can extract another subsequence such that we get two limits and convergence in the sense of Definition 3.4. The boundary values rule out the need of a translation and the convergence to one point. The two limiting generalized generators can then by (3.3) be joined again.
In both cases we obtain a subsequence and a generalized generator $c : [0, 1] \to [0, \infty)$, such that $c_n \to c$ in the sense of Definition 3.4. The added half circles, the interior regularity Lemma 3.3 and the Sobolev embedding yield, that $c$ satisfies the boundary values, i.e. there exists $0 < t_- < t_+ < 1$ such that

$$\begin{gathered} c(t_-) = (x_-, r_-), c'(t_-)/|c'(t_-)| = (\cos\beta_-, \sin\beta_-), \\ c(t_+) = (x_+, r_+), c'(t_+)/|c'(t_+)| = -(\cos\beta_+, \sin\beta_+). \end{gathered}$$

The lower semicontinuity result Proposition 3.5 together with (2.13) yield

$$\mathcal{W}(f_c) \leq \liminf_{n\to\infty} \mathcal{W}(f_{c_n}) = \liminf_{n\to\infty} \mathcal{W}_{closed}(f_{\gamma_n}) < 12\pi.$$

If $c$ would now have at least two points $0 < t_1 < t_2 < 1$ such that $c_2(t_1) = c_2(t_2) = 0$, then we can apply Proposition 3.7 on three distinct subintervals and would obtain

$$\mathcal{W}(f_c) \geq 12\pi.$$

This is a contradiction to the strict inequality in the above calculation. Hence $c$ has at most one point in $(t_-, t_+)$ in which it touches the $x$-axis. Furthemore this yields after a reparametrization

$$\gamma := c|_{[t_-,t_+]} \in M.$$

Proposition 3.5 yields $\gamma$ to be minimizing w.r.t. the Willmore energy in this class. Outside of a possible singularity on the $x$-axis, $\gamma$ is then by Lemma 3.2 and [GaGrSw91, Lemma 8.2] critical w.r.t. to the elastic energy and the Willmore energy. Then the arguments of [EiGr17, §5] apply outside this singularity and $\gamma$ is smooth and an elastica in all $t \in [0,1]$ with $\gamma_2(t) > 0$. If we do not have a singularity, the theorem is proven. Hence let us assume we find exactly one $t \in ]0,1[$ with $\gamma_2(t) = 0$. Since after reparametrization $\gamma$ is an elastica outside of $t$ and has finite elastic energy by (3.11) and Proposition 2.2, the classification in [LaSi84b, table 2.7 c)] yields $\gamma$ outside of $t$ to be either part of a circle with center on the $x$-axis or part of a Moebius transformed graph of $x \mapsto \cosh(x)$. The corresponding surface of revolution to the latter curve is a catenoid. This can be seen by the discussion in [LaSi84b], in which elastica have been divided into wavelike, orbitlike, asymptotic geodesic and constant curvature type (cf. [LaSi84b, table 2.7 c)]). Orbitlike elastica never reach the $x$-axis, see e.g. [LaSi84b, Figure 2] or [Ei16, 5.4]. Wavelike elastica need infinitely many periods in their curvature to reach the $x$-axis (see e.g. [LaSi84b, Figure 1 c) or Prop. 5.1] or [Ei16, Lemma 7.8]). Then by e.g. [LaSi84b, table 2.7 c)] or [EiKo17, Lemma 3.11] their elastic energy is infinite. In the case of constant elastic curvature the case of $\pm\sqrt{2}$ (see [LaSi84b, table 2.7 c)]) refers to the Clifford torus and therefore never reaches the $x$-axis. The other constant curvature elastica are geodesics and therefore half circles or straight lines. The case asymptotic geodesic corresponds to the catenoid. Therefore the case for bounded diameter is closed.

In the remaining case we assume after possibly extracting a subsequence

$$\operatorname{diam}(f_{c_n}) \to \infty. \tag{3.14}$$

Here we use the same idea as in [EiGr17, Lemma 4.3] (which is encapsulated in [EiGr17, Figure 5]). By (3.14) we find $t_n \in (t_-^n, t_+^n)$ such that

$$|c_n(t_n)| \to \infty.$$

Now let $D = B_1(0)$ be the hyperbolic disc equipped with the Poincaré metric, see e.g. [Ra, Eq. 4.5.4]. Furthermore let $Q : \mathcal{H} \to D$ be the Cayley transformation, which is an isometry, see e.g. [Ra, Eq. 4.6.1]. Now let $R_\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation around zero counterclockwise with angle $\varphi$. Then $R_\varphi|_D \to D$ is an isometry as well, see e.g. [Ra, Thm. 4.5.2]. We define (cf. [EiGr17, Figure 5, proof of Lemma 4.3])

$$\Phi_\varphi := Q^{-1} \circ R_\varphi \circ Q : \mathcal{H} \to \mathcal{H}. \tag{3.15}$$

$\Phi_\varphi$ is then a one-parameter family of isometries, such that $\Phi_0 = id$. Furthermore for $\varphi \in (0, 2\pi)$ we have

$$\Phi_\varphi(\infty) \in \mathbb{R} \times \{0\},$$

i.e. has a finite value. Additionally we find exactly one $x_\varphi \in \mathbb{R}$, such that

$$\Phi_\varphi((x_\varphi, 0)) = \infty.$$

By possibly extracting another subsequence, we can assume that the possible singularity of $c_n$ (i.e. the point apart from $c_n(0)$ and $c_n(1)$ meeting the $x$-axis) either does not exist, is bounded in $n$ or converges to $\pm\infty$. In either of these cases we can choose $\varphi \neq 0$ in such a way, that for all $n \in \mathbb{N}$ we have, that $c_n$ never comes close to $x_\varphi$.
Since the 'boundary' spheres $c_n([0, t^n_-])$ and $c_n([t^n_+, 1])$ are independent of $n$, we obtain $\Phi_\varphi(c_n([0, t^n_-]))$ and $\Phi_\varphi(c_n([t^n_+, 1])$ are independent of $n$ as well. Furthermore they are again part of circles and therefore define new Dirichlet boundary values. For these boundary values $\Phi_\varphi(c_n)$ is by conformal invariance again a minimizing sequence w.r.t. the Willmore and elastic energy. Furthermore the diameter is now bounded. Hence we can argue as in the other case and obtain after possibly extracting a subsequence a limit $c_\varphi$ of $\Phi_\varphi(c_n)$ w.r.t. to the convergence in Definition 3.4. Again arguing as in the other case, we only have one singularity in the inner part of $c_\varphi$ and it is critical w.r.t. to the elastic and Willmore energy outside this singularity. Therefore it is smooth there as well. Furthermore by (3.2) we have

$$\lim_{n\to\infty} \Phi_\varphi(c_n(t_n)) = (x'_\varphi, 0), \tag{3.16}$$

for an $x'_\varphi \in \mathbb{R}$ (which satisfies $\Phi_\varphi(\infty) = (x'_\varphi, 0)$). Hence $(x'_\varphi, 0)$ has to be this singularity. Also $c_\varphi$ has to consist of parts of half circles and asymptotic geodesic curves. Then $\Phi_\varphi^{-1}(c_\varphi)$ satisfies the original boundary values, has a point at infinity and has minimal Willmore energy.

///

We are ready to prove Theorem 1.1 in two special cases.

**Proposition 3.9** *For any* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *with*

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) < 8\pi \tag{3.17}$$

*or*

$$focal(x_-, r_-, \beta_-) \neq focal(x_+, r_+, \beta_+), \tag{3.18}$$

*there exists a regular profile curve in the upper half plane* $\gamma : [0, L] \to \mathcal{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0 \}$ *with* $|\gamma'| \neq 0, L > 0$ *which minimizes the Willmore energy subject to these boundary conditions, that is*

$$\mathcal{W}(f_\gamma) = \mathcal{W}(x_\pm, r_\pm, \beta_\pm), \tag{3.19}$$

*in particular* $\gamma$ *is a free elastica satisfying the boundary conditions (1.6), and the corresponding rotational symmetric immersion* $f_\gamma$ *is a Willmore immersion.*

**Proof:**
The result in case (3.17) follows directly by Proposition 3.7 with Lemma 3.8. Hence we assume the focal points to be different, i.e. (3.18).

Let $\gamma$ be the minimizer found in Theorem 3.7. We proceed by contradiction and assume we have a singularity, i.e. $\gamma$ consists of two $\gamma_\pm$ either satisfying one part of (1.6), being Moebius transformed catenoids or half circles, being parametrized by hyperbolic arclength and such that

$$\lim_{t\to\infty} \gamma_+(t) = \lim_{t\to\infty} \gamma_-(t) \in \mathbb{R}e_1 \cup \{\infty\}.$$

By a Moebius transformation of the upper half plane (see e.g. the second part of the proof of Theorem 3.7) we can rule out the $\infty$ case for the singularity. Since the focal points do not align and $\gamma$ is continuous, one of the $\gamma_\pm$ has to be a Moebius transformed catenoid. By Proposition 3.9 we find $t_\pm$, such that

$$W_{closed}(\gamma_\pm(t_\pm), \gamma_\pm'/|\gamma_\pm'(t_\pm)|) \leq 8\pi.$$

Since at least one of them is a Moebius transformed catenoid, their elastic energy is positive, i.e. we find a $c_\gamma > 0$, such that w.l.o.g.

$$\mathcal{F}(\gamma_+|_{[t_+,\infty)}) = c_\gamma > 0.$$

Then (3.11) together with Lemma 3.3 as in Lemma 3.8 (remember, half circle have zero elastic energy)

$$\mathcal{W}_{closed}(f_{\gamma_\pm|_{[t_\pm,\infty)}}) = \frac{\pi}{2}\mathcal{F}(\gamma_\pm|_{[t_\pm,\infty)}) + 4\pi.$$

Hence

$$\mathcal{W}_{closed}(f_{\gamma_+|_{[t_+,\infty)} \oplus \gamma_-|_{[t_-,\infty)}}) = 8\pi + C_\gamma$$

for some $C_\gamma > 0$. Again by Proposition 3.9 we obtain a smooth regular curve $c : [0,1] \to \mathcal{H}$ satisfying the following boundary conditions

$$\begin{aligned} c(0) &= \gamma_+(t_+), \quad c'(0)/|c'(0)| = \gamma_+'(t_+)/|\gamma_+'(t_+)|, \\ c(1) &= \gamma_-(t_-), \quad c'(1)/|c'(1)| = \gamma_-'(t_-)/|\gamma_-'(t_-)|, \end{aligned}$$

and

$$\mathcal{W}_{closed}(f_c) < 8\pi + \frac{C_\gamma}{2}.$$

Since the boundary conditions are the same and the added sphere caps do have the same Willmore energy, we obtain

$$\mathcal{W}(f_{\gamma_+|_{[t_+,\infty)} \oplus \gamma_-|_{[t_-,\infty)}}) > \mathcal{W}(f_c).$$

By inserting $c$ into $\gamma$ in place of $\gamma_+|_{[t_+,\infty)} \oplus \gamma_-|_{[t_-,\infty)}$, we obtain $\gamma_{comp} \in M$ (cf. (3.7)), i.e. a $C^{1,1}$-regular curve satisfying the original boundary conditions. By construction we also have

$$\mathcal{W}_{closed}(f_\gamma) > \mathcal{W}_{closed}(f_{\gamma_{com}}),$$

which is a contradiction to (3.10) and $\gamma$ being a minimizer. Hence there is no singularity in $\gamma$ and it is therefore a smooth, critical point of the Willmore energy satisfying the desired boundary data.

///

For boundary conditions with intersecting inverse sphere caps, the closed $C^{1,1}-$immersion obtained in (2.9) by adding the inverse sphere caps obviously contains a double point, and we get from the Li-Yau inequality in [LY82] that $\mathcal{W}_{closed} \geq 8\pi$, in particular such boundary conditions are not covered by (3.17). On the other hand boundary conditions which do not satisfy (3.18), but none of the sphere caps is contained in the other, satisfy (3.17), as we will see in the next section.

# 4 The case of coinciding focal points

For establishing Theorem 1.1 by Proposition 3.9 only the case of coinciding focal points of the sphere caps of the boundary conditions is left open. First we clarify that there is no minimizing solution when one of the sphere caps of the boundary conditions is contained in the other.

**Proposition 4.1** *For* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *as in Theorem 1.1, such that one of the corresponding sphere caps* $cap(x_\pm, r_\pm, \beta_\pm)$ *is contained in the other, there exists no regular profile curve in the upper half plane* $\gamma : [0, L] \to \mathcal{H} := \{(x, y) \in \mathbb{R}^2 \mid y > 0\ \}$ *with* $|\gamma'| \neq 0, L > 0$ *, which satisfies the boundary conditions (1.6) and minimizes the Willmore energy subject to these boundary conditions.*

**Proof:**
As one of the sphere caps is contained in the other, the focal points of the boundary conditions and the inverse boundary conditions coincide, that is

$$\hat{x} := focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+) \neq$$
$$\neq \hat{y} := focal(x_-, r_-, \beta_- + \pi) = focal(x_+, r_+, \beta_+ + \pi).$$

We assume that both are finite.

Now if on contrary such a minimizing regular profile curve $\gamma$ exist, then the closed rotationally symmetric $C^{1,1}-$immersed surface $\Sigma := cap(x_-, r_-, \beta_- + \pi) \oplus f_\gamma \oplus cap(x_+, r_+, \beta_+ + \pi)$ or more precisely the corresponding varifold $\hat{\mu}_\gamma$ in (2.9) satisfies with Proposition 2.2 (2.15) that

$$\mathcal{W}(\hat{\mu}_\gamma) = \mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) \leq 8\pi.$$

Inverting at the inverse focal point $\hat{y} \neq \hat{x}$ , both sphere caps are getting infinite, hence (2.10) gives

$$\mathcal{W}(I(\hat{\mu}_\gamma)) = \mathcal{W}(\hat{\mu}_\gamma) - 8\pi = 0.$$

This says that the inversion $I(\hat{\mu}_\gamma)$ is minimal, in particular it is smooth and analytic, see [Mo58], hence so is $\hat{\mu}_\gamma$ . But $\hat{\mu}_\gamma$ contains the sphere caps $cap(x_\pm, r_\pm, \beta_\pm + \pi)$ , hence by analyticity coincide with the sphere caps. This is not possible, as the regular profile curve $\gamma$ is compactly contained in the hyperbolic plane $\mathcal{H}$ , hence does not come close to the focal point $\hat{x}$ being an element of the $x-$axis.

///

The remaining case of coinciding focal points in Theorem 1.1 will follow by Proposition 3.9 under assumption (3.17) when we improve the estimate (2.15) of Proposition 2.2.

**Proposition 4.2** *For* $x_\pm \in \mathbb{R}, r_\pm > 0, \beta_\pm \in \mathbb{R}$ *as in Theorem 1.1 with coinciding focal points of the corresponding sphere caps, that is*

$$focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+), \tag{4.1}$$

*but none of the sphere caps is contained in the other, we have the strict inequality*

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) < 8\pi \tag{4.2}$$

**Proof:**
As the focal points of the sphere caps coincide, but none of the sphere caps is contained in the other, the sphere caps are disjoint apart from the common focal point, and as in the proof of Proposition 2.2, we may assume after an inversion at an appropriate point of the $x-$axis, that the sphere caps have the same radius, say 1, and lie opposite each other at the common focal point, which we may further assume to be the origin.

Then after possibly interchanging, the boundary conditions are of the form

$$(x_\pm, r_\pm, \beta_\pm) = (\pm 1 + e^{i(\beta_\pm \mp \pi/2)}, \beta_\pm) \quad \text{with } |\beta_+|, |\beta_- - \pi| < \pi/2.$$

Clearly by tracing along the sphere caps, $\mathcal{W}_{closed}$ is monotonically non-decreasing when the boundary points approach the origin, more precisely for $-\pi/2 < \beta_-^2 < \beta_-^1 < \pi/2, \pi/2 < \beta_+^1 < \beta_+^2 < 3\pi/2$ , we have

$$\mathcal{W}_{closed}(\pm 1 + e^{i(\beta_\pm^1 \mp \pi/2)}, \beta_\pm^1) \leq \mathcal{W}_{closed}(\pm 1 + e^{i(\beta_\pm^2 \mp \pi/2)}, \beta_\pm^2),$$

and actually the monotonicity is strict by an analyticity argument as in Proposition 4.1. Therefore it suffices to prove for symmetric boundary data that

$$\mathcal{W}_{closed}(\pm x_0, r_0, \pm(\beta_0 + \pi)) < 8\pi \quad \text{for} \begin{cases} (x_0, r_0) \in \partial B_1(1) \cap [Im > 0], \\ \beta_0 = \arctan r_0/x_0, \end{cases} \tag{4.3}$$

and further it suffices to prove (4.3) for a sequence $(x_0, r_0) \to 0$ or likewise for $\beta_0 \nearrow \pi/2$ .

//

We will prove (4.3) in several steps by giving a comparison regular profile curve $\gamma$ in the upper half plane which satisfies the boundary conditions (1.6). We choose $\gamma$ as a wavelike free elastica parametrized by hyperbolic length in the hyperbolic plane which at $s = 0$ has its maximal hyperbolic curvature at $s = 0$ and starts on the $y-$axis with horizontal tangent pointing to the right, that is

$$\gamma(0) \in i\mathbb{R} \cap \mathcal{H} \quad \text{and} \quad \gamma'(0) = \gamma_2(0) e_1. \tag{4.4}$$

We follow [LaSi84b] to determine the hyperbolic curvature of $\gamma$ and put

$$x_p(\varphi) := \int_0^\varphi \frac{\mathrm{d}\zeta}{\sqrt{1 - p^2 \sin^2 \zeta}} \quad \text{for } 0 \leq \varphi \leq \pi/2, 0 \leq p \leq 1 \tag{4.5}$$

and

$$cn(x, p) := \cos \varphi \quad \text{for } 0 \leq x = x_p(\varphi) \leq x_p(\pi/2), x < \infty, 0 \leq p \leq 1, \tag{4.6}$$

see for example [AbSt] section 16.1 for the Jacobi elliptic functionen $cn$ and the Jacobi amplitude $x_p$ .

The curvature function $\kappa$ for the wavelike free elastica $\gamma$ parametrized by hyperbolic arc-length $s$ is given up to an additive constant in the arc-length $s$ according to [LaSi84b] Table 2.7 (c) by

$$\begin{gathered} 2 < \kappa_0 < \infty, \\ 1/2 < p^2 = \frac{\kappa_0^2}{2(\kappa_0^2 - 2)} < 1, \\ r^2 = \tfrac{1}{2}(\kappa_0^2 - 2) > 1, \end{gathered} \tag{4.7}$$

and

$$\kappa(s) = \kappa_p(s) := \kappa_0 \cdot cn(rs,p) \quad \text{for } s \in \mathbb{R}. \tag{4.8}$$

We see that for $s=0$ the curvature $\kappa_p$ attains its maximum $\kappa_0 = \kappa_p(0)$ , as required. Clearly $cn(.,p)$ is periodic for $0 \leq p < 1$ with period $4x_p(\pi/2)$ , and therefore $\kappa_p$ is periodic and the quarter of its period is given by

$$s_0 := \frac{1}{r} x_p(\pi/2). \tag{4.9}$$

As for cos , we have from (4.6) the following relations

$$\begin{aligned} \kappa_p(s+4s_0) = \kappa_p(s), \quad \kappa_p(s+2s_0) = -\kappa_p(s), \\ \kappa_p(s_0 - s) = -\kappa_p(s_0+s), \quad \kappa_p(-s) = \kappa_p(s), \end{aligned} \tag{4.10}$$

and the points of vanishing curvature are given by

$$\kappa_p(s) = 0 \Longleftrightarrow s \in s_0 + 2s_0\mathbb{Z}, \tag{4.11}$$

and all these points are inflection points of $\gamma$ , that is the curvature changes its sign there. By the Frenet equations, the horizontal start on the $y-$axis at $s=0$ in (4.4) and as $z \mapsto -\bar{z}$ is an orientations-reversing isometry of $\mathcal{H}$ , we see

$$\gamma(-s) = -\overline{\gamma(s)}. \tag{4.12}$$

Next by [LaSi84b] Figure 1, we know that $\gamma$ as a wavelike free elastica oscillates along an axial geodesic which it crosses perpendicularly at each inflection point, see also [Ei16] Lemma 6.1, [Ei17] Lemma 5.1 or [DaSch24] Proposition 3.1. As $\gamma$ has no selfintersections, see [Ei17] Lemma 5.18, [EiGr17] Lemma 6.13 or [DaSch24] Proposition 3.1, we see from (4.11) that no inflection point lies on the $y-$axis, hence by the symmetry of $\gamma$ in (4.12), we see that this geodesic is a half circle with center at the origin. After a homothety, we may assume that this geodesic is $\partial B_1 \cap \mathcal{H}$ , and we have from (4.11) that

$$\left.\begin{aligned} \gamma((2n+1)s_0) \in \partial B_1(0) \cap \mathcal{H}, \\ \gamma'((2n+1)s_0)/\gamma_2((2n+1)s_0)) \in \{\pm\gamma((2n+1)s_0)\}, \end{aligned}\right\} \quad \forall n \in \mathbb{Z}, \tag{4.13}$$

as $\gamma$ is parametrized by hyperbolic arc-length.

The hyperbolic isometry

$$\Phi(z) := \frac{1+z}{1-z} \quad \text{for } z \in \mathbb{C} \cup \{\infty\},$$

satisfies $\Phi(-1) = 0, \Phi(i) = (1+i)(1-i) = i, \Phi(1) = \infty$ , hence sends the geodesic $\partial B_1(0) \cap \mathcal{H}$ on the $y-$axis$\cap\mathcal{H}$ , and the image $\tilde{\gamma} := \Phi(\gamma)$ is a wavelike free elastica which intersects the $y-$axis at its inflections points at $s_0 + 2s_0\mathbb{Z}$ . Fixing $\lambda := \tilde{\gamma}(s_0)/\tilde{\gamma}(-s_0) \neq 1$ , as $\tilde{\gamma}$ has no selfintersections, we see as above by the Frenet equations that $\tilde{\gamma}$ is self-similar, more precisely

$$\tilde{\gamma}(s+2s_0) = -\lambda\overline{\tilde{\gamma}(s)} \quad \forall s. \tag{4.14}$$

Moreover by [DaSch24] Proposition 3.1 (3.4), the map

$$s \mapsto |\tilde{\gamma}(s)| = |\Phi(\gamma(s))| \text{ is strictly monotone on } \mathbb{R}. \tag{4.15}$$

As for $z \in \mathcal{H}$ by elementary calculations

$$\Phi(z) = \frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{|1-z|^2} = \frac{1+z-\bar{z}-|z|^2}{|1-z|^2} = \frac{1-|z|^2+2iIm(z)}{|1-z|^2}$$

and

$$|\Phi(z)|^2 = \frac{(1-|z|^2)^2+4Im(z)^2}{|1-z|^4} =$$

$$= \frac{1-2|z|^2+|z|^4+4Im(z)^2}{|1-z|^4} = \frac{1+2|z|^2+|z|^4-4Re(z)^2}{|1-z|^4} =$$

$$= \frac{(1+|z|^2)^2}{|1-z|^4} - \frac{4Re(z)^2}{|1-z|^4} = \left(\frac{1+|z|^2}{1-2Re(z)+|z|^2}\right)^2 - \frac{4Re(z)^2}{|1-z|^4} =$$

$$= \left(1+\frac{2Re(z)}{|1-z|^2}\right)^2 - \frac{4Re(z)^2}{|1-z|^4} = 1+\frac{4Re(z)}{|1-z|^2}, \tag{4.16}$$

we conclude from (4.4) that the monotonicity in (4.15) is increasing, that is

$$s \mapsto |\tilde{\gamma}(s)| = |\Phi(\gamma(s))| \text{ is strictly increasing on } \mathbb{R}, \tag{4.17}$$

in particular $\lambda = |\tilde{\gamma}(s_0)|/|\tilde{\gamma}(-s_0)| > 1$ and

$$\pm\gamma_1(s) > 0 \quad \text{for } \pm s > 0. \tag{4.18}$$

As $\Phi$ is a hyperbolic isometry, we get denoting the hyperbolic distance by $d_{\mathcal{H}}$ using the symmetry (4.12), $\tilde{\gamma}(\pm s_0) \in i\mathbb{R}$ and the definition of $\lambda = \tilde{\gamma}(s_0)/\tilde{\gamma}(-s_0)$ that the half of the hyperbolic distance of two consecutive inflection points is given by

$$\mu := d_{\mathcal{H}}(i, \gamma(\pm s_0)) = d_{\mathcal{H}}(i, \tilde{\gamma}(\pm s_0)) = \frac{1}{2} d_{\mathcal{H}}(\tilde{\gamma}(s_0), \tilde{\gamma}(-s_0)) = \frac{1}{2}\log\lambda. \tag{4.19}$$

In particular $|\log|\tilde{\gamma}(s_0)|| = |\log|\tilde{\gamma}(-s_0)||$ , and we get by the self-similarity in (4.14) that

$$\tilde{\gamma}((2n+1)s_0) = \lambda^{n+(1/2)} i \quad \forall n \in \mathbb{Z}, \tag{4.20}$$

hence by the monotonicity in (4.17) that

$$1 < \lambda \leq |\Phi(\gamma(s))|^2 \leq \lambda^3 \quad \text{for } s_0 \leq s \leq 3s_0. \tag{4.21}$$

We see that $\gamma|[-s,s]$ satisfies the symmetric boundary data in (4.3) after applying a homothety, when

$$f(s) := focal(\gamma(s), -\gamma'(s)/|\gamma(s)|) = 0 \tag{4.22}$$

and the smooth angle $\beta$ of the tangent of $\gamma$ defined by

$$e^{i\beta} = \gamma'/|\gamma'| = \gamma'/\gamma_2, \tag{4.23}$$

as $\gamma$ is parametrized by hyperbolic arc-length, satisfies $\beta(s) \in \beta_0 + 2\pi\mathbb{Z}$ . Therefore (4.3) is implied by the following Proposition.

**Proposition 4.3** *For any* $\beta_0 < \pi/2$ *, there exists* $\kappa_0 > 2, s > 0$ *satisfying (4.22) with* $\beta_0 < \beta(s) < \pi/2$ *for appropriate choice in (4.23) and*

$$\mathcal{F}(\gamma|[0,s]) < 4. \tag{4.24}$$

□

**Proof that Proposition 4.3 $\Longrightarrow$ Proposition 4.2**:
It remains to clarify the energy condition. For $\hat{\mu}_{\gamma|[-s,s]}$ as in (2.9), we see by (2.11) that

$$\mathcal{W}_{closed}(\pm x, r, \pm(\beta+\pi)) \leq \mathcal{W}(\hat{\mu}_\gamma) = \pi\mathcal{F}(\gamma|[0,s]) + 4\pi$$

hence (4.24) implies (4.3).

///

Now the focal point of a point $(x,y) \in \mathcal{H}$ and direction $-e^{i\beta}$ being not vertical or equivalently $\cos\beta \neq 0$ is given using the equation for the sphere cap $cap(x,y,\beta)$ in (2.4) by

$$focal(x,y,\beta+\pi) = x - y\Big(\frac{1}{\cos\beta} - \tan\beta\Big) = x - y\frac{\cos\beta}{1+\sin\beta}, \tag{4.25}$$

where the last expression remains valid for $\sin\beta \neq -1$ , that is $-e^{i\beta}$ is not pointing vertically upwards. Clearly $focal(x,y,\beta+\pi) = \infty$ for $\sin\beta = -1$ . For the special case when $(x,y) \in \partial B_1(0), e^{i\beta} = (x,y)$ points in the same direction as $(x,y)$ , we obviously have $\sin\beta = y > 0$ and get

$$focal(x,y,\beta+\pi) = x - y\frac{x}{1+y} = \frac{x}{1+\sqrt{1-x^2}}. \tag{4.26}$$

For any smooth path $\xi$ parametrized by hyperbolic arc-length in the hyperbolic plane, the hyperbolic mean curvature can be calculated by elementary differential geometry, see [GuSp11] (1.17) or [DaSch24] (A.1) third identity, in terms of the derivative of the smooth angle $\beta_\xi$ of the tangent, that is $e^{i\beta_\xi} = \xi'/|\xi'| = \xi'/\xi_2$ , by

$$\kappa_\xi = \beta'_\xi + \xi'_1/\xi_2 = \beta'_\xi + \cos\beta_\xi. \tag{4.27}$$

This enables to calculate the derivative of the focal point $d_\xi$ defined as in (4.22) for $\xi$ .

**Lemma 4.4** *For any smooth path* $\xi$ *parametrized by hyperbolic length in the hyperbolic plane with hyperbolic curvature* $\kappa_\xi$ *, focal point* $d_\xi$ *as in (4.22) and*

$$\xi'_2 > -\xi_2,$$

*the derivative of the focal point is given by*

$$f'_\xi = \frac{\xi_2^2}{\xi_2 + \xi'_2}\kappa_\xi.$$

**Proof:**
Omitting the subscript $\xi$ to simplify the notation, we get differentiating the last expression in (4.25) and using (4.27) and $e^{i\beta} = \xi'/\xi_2$ , hence $\sin\beta = \xi_2'/\xi_2 \neq -1$ by assumption, that

$$f' = \xi_1' - \xi_2' \frac{\cos\beta}{1+\sin\beta} - \xi_2\beta' \frac{-\sin\beta(1+\sin\beta) - \cos^2\beta}{(1+\sin\beta)^2} =$$

$$= \xi_2\Big(\cos\beta - \frac{\sin\beta\cos\beta}{1+\sin\beta} + \beta' \frac{1+\sin\beta}{(1+\sin\beta)^2}\Big) =$$

$$= \frac{\xi_2}{1+\sin\beta}(\cos\beta(1+\sin\beta) - \sin\beta\cos\beta + \beta') =$$

$$= \frac{\xi_2}{1+\sin\beta}(\cos\beta + \beta') = \frac{\xi_2^2}{\xi_2 + \xi_2'}\kappa,$$

which is the assertion.

///

In euclidian space, we know that positive respectively negative euclidean curvature of a path means that the tangent turns counter-clockwise respectively clockwise. This is no longer true for the hyperbolic curvature, as one can see for half circle perpendicular to the $x-$axis, which are hyperbolic geodesics. It remains true, when the tangent is pointing to the left or right depending on the sign of the hyperbolic curvature. The precise statement is given here.

**Lemma 4.5** *Let* $\xi : [s_1, s_2] \to \mathcal{H}$ *be a regular smooth path with non-positive hyperbolic curvature* $\kappa_\xi \leq 0$ *on* $[s_1, s_2]$ *. Then for the smooth angle* $\beta_\xi$ *of the tangent, that is* $e^{i\beta_\xi} = \xi'/|\xi'|$ *, we have*

$$|\beta_\xi(s_1)| \leq \pi/2 \Longrightarrow \beta_\xi(s_1) \geq \beta_\xi(s_2).$$

**Proof:**
We may assume that $\xi$ is parametrized by hyperbolic length. On contrary, we may assume that $|\beta_\xi(s_1)| \leq \pi/2$ , but

$$\beta_\xi(s_1) < \beta_\xi(s) \quad \forall s \in ]s_1, s_2] \tag{4.28}$$

In case $|\beta_\xi(s_1)| < \pi/2$ , we have from (4.27) that

$$\beta_\xi'(s_1) = \kappa_\xi(s_1) - \cos\beta_\xi(s_1) < 0,$$

which contradicts (4.28). In case $|\beta_\xi(s_1)| = \pi/2$ , we have from (4.27) that

$$\beta_\xi' \leq -\cos\beta_\xi \leq C(\beta_\xi - \beta_\xi(s_1)),$$

hence

$$e^{-C(s_2-s_1)}(\beta_\xi(s_2) - \beta_\xi(s_1)) \leq 0,$$

which contradicts (4.28).

///

**Remark:**
Switching the orientation of $\gamma$ , we see for $\kappa_\xi \geq 0$ on $[s_1, s_2]$ that

$$|\beta_\xi(s_1) + \pi| \leq \pi/2 \Longrightarrow \beta_\xi(s_1) \leq \beta_\xi(s_2).$$

□

We use this to determine the sign in (4.13).

**Proposition 4.6** *In addition to (4.13), we have*

$$\gamma'(s_0) = \gamma_2(s_0) \cdot \gamma(s_0). \tag{4.29}$$

**Proof:**
To simplify the computations, we work with $\tilde\gamma = \Phi(\gamma)$ . We know that $\tilde\gamma$ intersects the $y-$axis perpendicularly at its inflection points $(2n+1)s_0$ in $\lambda^{n+(1/2)}i$ for $n \in \mathbb{Z}$ , see (4.20), in particular $\tilde\gamma'(s_0) \in \{\pm\lambda^{1/2}e_1\}$ , as $\gamma$ is parametrized by hyperbolic arc-length. We exclude $\tilde\gamma'(s_0) = \lambda^{1/2}e_1$ . Indeed otherwise we can choose the smooth angle $\tilde\beta$ of the tangent of $\tilde\gamma$ as in (4.23) with $\tilde\beta(s_0) = 0$ . As $\kappa \leq 0$ on $[s_0, 3s_0]$ by (4.6) and (4.8), we get from Lemma 4.5 that $\tilde\beta \leq 0$ on $[s_0, 3s_0]$ . As $|\tilde\gamma|$ is increasing by (4.15) and $\tilde\gamma_2 > 0$ by $\tilde\gamma \in \mathcal{H}$ , no tangent of $\tilde\gamma$ is pointing vertically downwards, hence $\tilde\beta \neq -\pi/2$ on $\mathbb{R}$ . Together $-\pi/2 < \tilde\beta \leq 0$ on $[s_0, 3s_0]$ , in particular $\tilde\gamma_1' > 0$ on $[s_0, 3s_0]$ . But this is not true, as $\tilde\gamma_1(3s_0) = 0 = \tilde\gamma_1(s_0)$ by (4.20), hence we have $\tilde\gamma'(s_0) = -\lambda^{1/2}e_1$ , in particular $\tilde\gamma_1'(s_0) < 0$ . This implies

$$\mp Re(\tilde\gamma)(s_0 \pm \tau) > 0 \quad \text{for } 0 < \tau < \tau_0$$

for some small $\tau_0 > 0$ . By elementary calculation, we get $\Phi^{-1}(w) = (w-1)/(w+1) = -1/\Phi(w)$ , and we see by (4.16) that

$$|\Phi^{-1}(w)|^{-2} = |\Phi(w)|^2 = 1 + \frac{4Re(w)}{|1-w|^2} \quad \text{for } w \in \mathcal{H},$$

hence

$$|\gamma(s_0 - \tau)| < 1 < |\gamma(s_0 + \tau)| \quad \text{for } 0 < \tau < \tau_0.$$

This excludes $\gamma'(s_0) = -\gamma(s_0)\gamma_2(s_0)$ , and we get $\gamma'(s_0) = \gamma(s_0)\gamma_2(s_0)$ by (4.13).

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After these preliminaries, we determine the asymptotics for $s_0$ for $\kappa_0 \searrow 2$ , and in the following we abbreviate for $a, b > 0$ defined for $\kappa_0 > 2$ that

$$\begin{aligned} a \approx b :&\Longleftrightarrow \quad \lim_{\kappa_0 \searrow 2} a/b = 1, \\ a \sim b :&\Longleftrightarrow \quad 0 < \liminf_{\kappa_0 \searrow 2} a/b \leq \limsup_{\kappa_0 \searrow 2} a/b < \infty. \end{aligned}$$

More precisely, we will use the $O-$ and $o-$calculus for $\kappa_0 \searrow 2$ . Clearly

$$\begin{aligned} \varepsilon^2 := \frac{1}{p^2} - 1 = \frac{2(\kappa_0^2 - 2) - \kappa_0^2}{\kappa_0^2} &= (1 + O(\kappa_0 - 2))(\kappa_0 - 2) \approx (\kappa_0 - 2), \\ r - 1 = \frac{r^2 - 1}{r + 1} = \frac{\kappa_0^2 - 4}{2(r+1)} &= (1 + O(\kappa_0 - 2))(\kappa_0 - 2) \approx (\kappa_0 - 2). \end{aligned} \tag{4.30}$$

**Proposition 4.7** *We have*

$$s_0 = \frac{1}{2}\log\frac{1}{\kappa_0 - 2} + 2\log 2 + o(1), \quad \textit{for } \kappa_0 \searrow 2, \tag{4.31}$$

*in particular*

$$|s_0 - \frac{1}{2}\log\frac{1}{\kappa_0 - 2}| \le C \quad \textit{for } \kappa_0 \textit{ close to } 2. \tag{4.32}$$

**Proof:**
We calculate

$$s_0 = \frac{1}{r}x_p(\pi/2) = \frac{1}{r}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-p^2\sin^2\varphi}} = \frac{1}{r}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{(1-p^2)+p^2\cos^2\varphi}} =$$

$$= \frac{1}{pr}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{((1/p^2)-1)+\sin^2\varphi}} = \frac{2}{\kappa_0}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}. \tag{4.33}$$

We abbreviate the integral on the right hand side and calculate for $M > 0$ that

$$I_\varepsilon := \int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} =$$

$$= \int_0^{M\varepsilon}\frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} + \int_{M\varepsilon}^{\pi/2}\frac{\mathrm{d}\varphi}{\sin\varphi} + \Big(\int_{M\varepsilon}^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} - \int_{M\varepsilon}^{\pi/2}\frac{\mathrm{d}\varphi}{\sin\varphi}\Big) =$$

$$=: I_{1,\varepsilon}^M + I_{2,\varepsilon}^M - I_{3,\varepsilon}^M.$$

Now

$$I_{1,\varepsilon}^M = \int_0^M\frac{\mathrm{d}\varphi}{\sqrt{1+\varepsilon^{-2}\sin(\varepsilon\varphi)^2}} \to \int_0^M\frac{\mathrm{d}\varphi}{\sqrt{1+\varphi^2}} =: I_1^M \quad \text{for } \varepsilon \to 0,$$

$$I_{2,\varepsilon}^M = \Big[\log\Big(\frac{\sin\varphi}{1+\cos\varphi}\Big)\Big]_{M\varepsilon}^{\pi/2} = \log\frac{1+\cos(M\varepsilon)}{\sin(M\varepsilon)},$$

hence

$$I_{2,\varepsilon}^M - \log\frac{1}{\varepsilon} = \log\frac{\varepsilon(1+\cos(M\varepsilon))}{\sin(M\varepsilon)} \to \log\frac{2}{M} =: I_2^M \quad \text{for } \varepsilon \to 0, \tag{4.34}$$

and

$$0 \le I_{3,\varepsilon}^M = \int_{M\varepsilon}^{\pi/2}\frac{\sqrt{\varepsilon^2+\sin^2\varphi}-\sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot\sin\varphi}\,\mathrm{d}\varphi =$$

$$= \int_{M\varepsilon}^{\pi/2}\frac{\varepsilon^2}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot\sin\varphi\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)}\,\mathrm{d}\varphi \le$$

$$\leq C\varepsilon^2 \int\limits_{M\varepsilon}^{\infty} \frac{\mathrm{d}\varphi}{\varphi^3} = C/M^2 =: I_3^M,$$

as $(2/\pi)\varphi \leq \sin\varphi$ for $0 \leq \varphi \leq \pi/2$ . Further

$$I_1^M = \int\limits_0^M \frac{\mathrm{d}\varphi}{\sqrt{1+\varphi^2}} = \int\limits_0^{\arctan M} \frac{\tan'\zeta}{\sqrt{1+\tan^2\zeta}}\,\mathrm{d}\zeta = \int\limits_0^{\arctan M} \frac{1}{\cos\zeta}\,\mathrm{d}\zeta =$$

$$= \left[\log\Big(\frac{1+\sin\zeta}{\cos\zeta}\Big)\right]_0^{\arctan M} = \left[\log\Big(\sqrt{1+\tan^2\zeta}+\tan\zeta\Big)\right]_0^{\arctan M} = \log(\sqrt{1+M^2}+M), \tag{4.35}$$

hence

$$I_1^M + I_2^M = \log\frac{2(\sqrt{1+M^2}+M)}{M} \to \log 4 = 2\log 2 \quad \text{for } M \to \infty.$$

As $I_3^M \to 0$ for $M \to \infty$ , we obtain

$$\lim_{\varepsilon\to 0}(I_\varepsilon - \log\frac{1}{\varepsilon}) = 2\log 2. \tag{4.36}$$

Then (4.30) and (4.33) yield

$$s_0 = \frac{2}{\kappa_0} I_\varepsilon = \log\frac{1}{\varepsilon} + 2\log 2 + o(1) + O((\frac{2}{\kappa_0}-1)\log\frac{1}{\varepsilon}) =$$

$$= \frac{1}{2}\log\frac{1}{\kappa_0-2} + 2\log 2 + o(1) \quad \text{for } \kappa_0 \searrow 2, \tag{4.37}$$

in particular

$$|s_0 - \frac{1}{2}\log\frac{1}{\kappa_0-2}| \leq C \quad \text{for } \kappa_0 \text{ close to } 2, \tag{4.38}$$

which are (4.31) and (4.32).

///

We proceed with determining the asymptotics for the hyperbolic distance $\mu := d_{\mathcal{H}}(i,\gamma(\pm s_0))$ in (4.19) of the free elastica for $\kappa_0 \searrow 2$ .

**Proposition 4.8** *We have*

$$\mu/\sqrt{\kappa_0-2} = \frac{1}{2}\log\frac{1}{\kappa_0-2} + O(1) \quad \textit{for } \kappa_0 \searrow 2 \tag{4.39}$$

*and*

$$s_0 - \log\frac{1}{\mu} - \log\log\frac{1}{\kappa_0-2} = \log 2 + o(1) \quad \textit{for } \kappa_0 \searrow 2. \tag{4.40}$$

**Proof:**
$\mu$ being defined in (4.19) as half of the hyperbolic distance of two consecutive inflection

points of a wavelike free elastica was calculated in [LaSi84b] Proposition 5.1, see also [Ei16], [Ei17] and (B.1), as

$$\mu/\sqrt{\kappa_0 - 2} = \frac{\kappa_0\sqrt{\kappa_0+2}}{r} \int\limits_0^{\pi/2} \frac{\cos^2\varphi}{\kappa_0^2 - 4\sin^2\varphi} \cdot \frac{\mathrm{d}\varphi}{\sqrt{1-p^2\sin^2\varphi}} =$$

$$= \frac{\sqrt{\kappa_0+2}}{\kappa_0 r} \int\limits_0^{\pi/2} \frac{\cos^2\varphi}{1-(4/\kappa_0^2)\sin^2\varphi} \cdot \frac{\mathrm{d}\varphi}{\sqrt{1-p^2\sin^2\varphi}} =$$

$$= \frac{\sqrt{\kappa_0+2}}{\kappa_0 r} \int\limits_0^{\pi/2} \frac{\cos^2\varphi}{((1-(4/\kappa_0^2)) + (4/\kappa_0^2)\cos^2\varphi) \cdot \sqrt{(1-p^2)+p^2\cos^2\varphi}} \,\mathrm{d}\varphi =$$

$$= \frac{\sqrt{\kappa_0+2}}{2} \int\limits_0^{\pi/2} \frac{\sin^2\varphi}{((\kappa_0^2/4)-1)+\sin^2\varphi) \cdot \sqrt{((1/p^2)-1)+\sin^2\varphi}} \,\mathrm{d}\varphi$$

with (4.7). Abbreviating

$$\delta^2 := \frac{\kappa_0^2}{4} - 1 = \frac{\kappa_0+2}{4}(\kappa_0-2) \approx \kappa_0 - 2, \tag{4.41}$$

$$J_{\delta,\varepsilon} := \int\limits_0^{\pi/2} \frac{\sin^2\varphi}{(\delta^2+\sin^2\varphi)\cdot\sqrt{\varepsilon^2+\sin^2\varphi}} \,\mathrm{d}\varphi,$$

we see

$$\mu/\sqrt{\kappa_0-2} = \frac{\sqrt{\kappa_0+2}}{2} J_{\delta,\varepsilon}.$$

As with (4.7) by smoothness

$$\Big|\frac{\sqrt{\kappa_0+2}}{2} - 1\Big| \le C(\kappa_0-2) \quad \text{for } \kappa_0 \text{ close to 2},$$

we get

$$\mu/\sqrt{\kappa_0-2} = (1+O(\kappa_0-2))J_{\delta,\varepsilon}. \tag{4.42}$$

We calculate

$$J_{\delta,\varepsilon} = \int\limits_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} - \int\limits_0^{\pi/2} \frac{\delta^2}{(\delta^2+\sin^2\varphi)\cdot\sqrt{\varepsilon^2+\sin^2\varphi}} \,\mathrm{d}\varphi =: I_\varepsilon - R_{\delta,\varepsilon}$$

with $I_\varepsilon$ defined in the proof of Proposition 4.7. As by (4.30) and (4.41) that

$$0 \le R_{\delta,\varepsilon} \le \int\limits_0^{\varepsilon} \frac{\mathrm{d}\varphi}{\varepsilon} + C\int\limits_\varepsilon^{\infty} \frac{\delta^2}{\varphi^3}\,\mathrm{d}\varphi \le C(1+\delta^2/\varepsilon^2) = O(1),$$

we get with (4.30), (4.36), (4.41), and (4.42) that

$$\mu/\sqrt{\kappa_0-2} = (1+O(\kappa_0-2))J_{\delta,\varepsilon} =$$

$$= (\log\frac{1}{\varepsilon} + O(1)) + O\Big((\kappa_0 - 2)(\log\frac{1}{\varepsilon} + O(1))\Big) =$$

$$= \frac{1}{2}\log\frac{1}{\kappa_0 - 2} + O(1) \quad \text{for } \kappa_0 \searrow 2,$$

which is (4.39), in particular

$$\log(\mu/\sqrt{\kappa_0 - 2}) = \log\log\frac{1}{\kappa_0 - 2} - \log 2 + o(1) \quad \text{for } \kappa_0 \searrow 2$$

and combining with (4.31) that

$$s_0 - \log\frac{1}{\mu} - \log\log\frac{1}{\kappa_0 - 2} = \log 2 + o(1) \quad \text{for } \kappa_0 \searrow 2,$$

which is (4.40).

///

In particular, we see $\mu \to 0$ for $\kappa_0 \searrow 2$ by (4.39), hence by (4.4), (4.18) and (4.19) that

$$|\gamma(s_0) - i| \approx d_{\mathcal{H}}(\gamma(s_0), i) = \mu \quad \text{and} \quad \gamma_1(s_0) > 0, \tag{4.43}$$

hence, as $|\gamma(s_0)| = 1$ by (4.13), that

$$\gamma_1(s_0) \approx \mu \quad \text{and} \quad \gamma_2(s_0) = 1 + o(\mu)$$

and, as these are smooth in $\mu$, that

$$\gamma_1(s_0) = \mu + O(\mu^2) \quad \text{and} \quad \gamma_2(s_0) = 1 - O(\mu^2). \tag{4.44}$$

Next by (4.29) that

$$\frac{\gamma_2'(s_0)}{\gamma_2(s_0)} = \gamma_2(s_0) = 1 - O(\mu^2) > 0 \quad \text{and} \quad \gamma_1'(s_0) > 0 \tag{4.45}$$

for $\kappa_0$ close to 2, hence for the smooth angle of the tangent $\beta$ in (4.23), we have $\cos\beta(s_0) = \gamma_1'(s_0)/\gamma_2(s_0) > 0$. Then again by (4.29) and, as $|\gamma(s_0)| = 1$ by (4.13), we get from (4.22) and (4.26) that

$$f(s_0) = focal(\gamma(s_0), \beta(s_0) + \pi) = \frac{\gamma_1(s_0)}{1 + \sqrt{1 - \gamma_1(s_0)^2}} =$$

$$= \gamma_1(s_0)(\frac{1}{2} + O(\gamma_1(s_0)^2)) = \frac{1}{2}\mu + O(\mu^2). \tag{4.46}$$

We continue estimating the slope of $\gamma$ in order to ensure that the slope of touching in (4.23) is steep.

**Lemma 4.9** *For* $0 < \sigma < 1/2, 0 < \hat{\sigma} < (1/2) - \sigma$ *and*

$$0 \leq s \leq \hat{\sigma}\log\frac{1}{\kappa_0 - 2} + C \ll s_0 \tag{4.47}$$

*with*

$$f(s_0+s)\geq 0, \tag{4.48}$$

*we have*

$$\left.\begin{array}{c}\dfrac{\gamma_2'(\tau)}{\gamma_2(\tau)}\geq 1-C(\kappa_0-2)^{2\sigma},\\ \gamma_1'(\tau)>0,\end{array}\right\} \quad \textit{for } s_0\leq\tau\leq s_0+s \tag{4.49}$$

*when* $\kappa_0$ *is close to* $2$ *.*

**Proof:**
Clearly the last estimate in (4.47) is true for $\kappa_0$ close to $2$ by (4.31). As $\kappa\leq 0$ on $[s_0,2s_0]$ by (4.6) and (4.8), we know from Lemma 4.4 that

$$f(\tau)\geq f(s_0+s)\geq 0 \quad \forall s_0\leq\tau\leq s_0+s. \tag{4.50}$$

First we assume additionally

$$\gamma_2'(\tau)\geq 0 \quad \text{for } s_0\leq\tau\leq s_0+s. \tag{4.51}$$

From (4.45), we know $\gamma_1'(s_0),\gamma_2'(s_0)>0$ , hence we can choose the smooth angle $\beta$ of the tangent in (4.23) with $0<\beta(s_0)<\pi/2$ and conclude $\beta\geq 0$ on $[s_0,s_0+s]$ by (4.51). As $\kappa\leq 0$ on $[s_0,3s_0]$ by (4.6) and (4.8), we get from Lemma 4.5 that $\beta\leq\beta(s_0)<\pi/2$ on $[s_0,3s_0]$ . Together

$$0\leq\beta(\tau)\leq\beta(s_0)<\pi/2 \quad \text{for } s_0\leq\tau\leq s_0+s, \tag{4.52}$$

in particular $\cos\beta(\tau),\gamma_1'(\tau)>0$ . Then we get for any $s_0\leq\tau\leq s_0+s$ from (4.25) and (4.50) that

$$0\leq f(\tau)=\gamma_1(\tau)-\gamma_2(\tau)\frac{\cos\beta(\tau)}{1+\sin\beta(\tau)}$$

or likewise

$$\frac{\gamma_2(\tau)}{\gamma_1(\tau)}\leq\frac{1+\sin\beta(\tau)}{\cos\beta(\tau)}\leq\frac{2}{\cos\beta(\tau)}. \tag{4.53}$$

Clearly (4.43) and (4.51) imply

$$\gamma_2(\tau)\geq\gamma_2(s_0)\geq 1/2, \tag{4.54}$$

for $\kappa_0$ close to $2$ , as $\mu\to 0$ for $\kappa_0\searrow 2$ . Further $\gamma_2'\leq\gamma_2$ , as $\gamma$ is parametrized by hyperbolic arc-length, hence

$$\log\gamma_2(\tau)\leq s+\log\gamma_2(s_0)$$

and by (4.43) and (4.47) that

$$|\log\gamma_2(\tau)|\leq\hat{\sigma}\log\frac{1}{\kappa_0-2}+C$$

in particular

$$\gamma_2(\tau)\leq C(\kappa_0-2)^{-\hat{\sigma}} \tag{4.55}$$

and $\kappa_0$ close to 2 . Then

$$|\gamma(\tau)| \leq C(\kappa_0 - 2)^{-\hat{\sigma}} \cdot d_{\mathcal{H}}(\gamma(\tau), \gamma(s_0)) + 1 \leq C(\kappa_0 - 2)^{-\hat{\sigma}} \cdot \log \frac{1}{\kappa_0 - 2} \tag{4.56}$$

for $\kappa_0$ close to 2 . Combining with (4.39), we get

$$(|\gamma(\tau)| + 1)\mu \leq C(\kappa_0 - 2)^{(1/2)-\hat{\sigma}} \cdot \Big(\log \frac{1}{\kappa_0 - 2}\Big)^2 \leq C(\kappa_0 - 2)^{\sigma} \tag{4.57}$$

for $\kappa_0$ close to 2 . On the other hand by (4.16) and (4.21), we know

$$1 < 1 + \frac{4\gamma_1(\tau)}{|1 - \gamma(\tau)|^2} \leq \lambda^3$$

when recalling $0 \leq s \leq s_0$ , hence with (4.19) that

$$0 < \log \Big(1 + \frac{4\gamma_1(\tau)}{|1 - \gamma(\tau)|^2}\Big) \leq 3 \log \lambda = 6\mu.$$

This implies

$$\frac{4\gamma_1(\tau)}{|1 - \gamma(\tau)|^2} \to 0 \quad \text{for } \kappa_0 \searrow 2$$

uniformly for $s_0 \leq \tau \leq s_0 + s$ , hence

$$0 < \frac{\gamma_1(\tau)}{|1 - \gamma(\tau)|^2} \leq \frac{1}{2} \log \Big(1 + \frac{4\gamma_1(\tau)}{|1 - \gamma(\tau)|^2}\Big) \leq 3\mu \tag{4.58}$$

for $\kappa_0$ close to 2 . Combining with (4.57), we get

$$0 < \gamma_1(\tau) \leq 3\mu(|\gamma(\tau)| + 1)|\gamma(\tau) - 1| \leq C(\kappa_0 - 2)^{\sigma}|\gamma(\tau) - 1| \leq$$
$$\leq o(1)\gamma_1(\tau) + C(\kappa_0 - 2)^{\sigma}(\gamma_2(\tau) + 1),$$

hence by (4.54) that

$$0 < \gamma_1(s) \leq C(\kappa_0 - 2)^{\sigma}(\gamma_2(\tau) + 1) \leq C(\kappa_0 - 2)^{\sigma}\gamma_2(\tau).$$

Plugging into (4.53) yields

$$\cos \beta(\tau) \leq 2\gamma_1(\tau)/\gamma_2(\tau) \leq C(\kappa_0 - 2)^{\sigma},$$

hence recalling (4.52) that

$$0 < (\pi/2) - \beta(\tau) \leq C(\kappa_0 - 2)^{\sigma}$$

for $\kappa_0$ close to 2 . This implies

$$\frac{\gamma_2'(\tau)}{\gamma_2(\tau)} = \frac{\gamma_2'(\tau)}{|\gamma'(\tau)|} = \sin \beta(\tau) \geq 1 - C(\kappa_0 - 2)^{2\sigma} \quad \text{for } s_0 \leq \tau \leq s_0 + s$$

as $\gamma$ is parametrized by hyperbolic length, and gives (4.49), as we already know $\gamma_1'(\tau) > 0$ by (4.52). Therefore we have established (4.49) under the additional assumption (4.51). Without assumption (4.51), we define

$$s_1 := \sup\{0 \leq t \leq s \mid \gamma_2'(\tau) \geq 0 \text{ for } s_0 \leq \tau < s_0 + t\ \}.$$

As $\gamma_2'(s_0) > 0$ by (4.45), we know $0 < s_1 \le s$ . Now we can apply the first part to any $0 \le t < s_1$ and get

$$\frac{\gamma_2'(\tau)}{\gamma_2(\tau)} \ge 1 - C(\kappa_0 - 2)^{2\sigma} \quad \text{for } s_0 \le \tau < s_0 + s_1 \tag{4.59}$$

and $\kappa_0$ close to 2 , hence letting $\tau \nearrow s_0 + s_1$ that

$$\frac{\gamma_2'(s_0 + s_1)}{\gamma_2(s_0 + s_1)} \ge 1 - C(\kappa_0 - 2)^{2\sigma} > 0$$

for $\kappa_0$ close to 2 . In particular we get $\gamma_2'(s_0 + s_1) > 0$ and conclude $s_1 = s$ . This establishes (4.51) and in turn implies (4.49).

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We proceed with estimating the energy of wavelike free elastica

$$\mathcal{F}_p := \int_0^{s_0} \kappa_p(s)^2 \, \mathrm{d}s \quad \text{for } 1/2 < p < 1$$

and abbreviate

$$\mathcal{F}_p(s) := \int_0^{s} \kappa_p(\tau)^2 \, \mathrm{d}\tau \quad \text{for } s \ge 0, 1/2 < p < 1.$$

**Lemma 4.10** *Fixing*

$$\varphi_0(s) := (\pi/2) - x_p^{-1}(r(s_0 - s)) \quad \textit{for } 0 \le s \le s_0, \tag{4.60}$$

*we have*

$$\mathcal{F}_p(s_0 + s) - \mathcal{F}_p = 2\kappa_0 \int_0^{\varphi_0(s)} \frac{\sin^2 \varphi}{\sqrt{\varepsilon^2 + \sin^2 \varphi}} \, \mathrm{d}\varphi = \tag{4.61}$$

$$= 2\kappa_0(1 - \cos \varphi_0(s)) + O(\kappa_0 - 2)s \tag{4.62}$$

*and*

$$4 - \mathcal{F}_p = (\kappa_0 - 2) \log \frac{1}{\kappa_0 - 2} + O(\kappa_0 - 2). \tag{4.63}$$

*For* $0 < \sigma < 1/2, 0 < \hat{\sigma} < (1/2) - \sigma$ *and*

$$0 \le s \le \hat{\sigma} \log \frac{1}{\kappa_0 - 2} + C, \tag{4.64}$$

*we get*

$$\sqrt{\varepsilon^2 + \varphi_0(s)^2} + \varphi_0(s) = (1 + O(\kappa_0 - 2)^{2\sigma})(\kappa_0 - 2)^{1/2} e^s. \tag{4.65}$$

**Proof:**
We calculate for $1/2 < p < 1, 0 \le s \le s_0$ with (4.7), (4.8), (4.30) and (4.60) that

$$\mathcal{F}_p(s_0 + s) - \mathcal{F}_p = \int_{s_0}^{s_0+s} \kappa_0^2 \cdot cn(r\tau, p)^2 \, \mathrm{d}\tau = \int_{s_0-s}^{s_0} \kappa_0^2 \cdot cn(r\tau, p)^2 \, \mathrm{d}\tau = \frac{\kappa_0^2}{r} \int_{r(s_0-s)}^{rs_0} cn(\tau, p)^2 \, \mathrm{d}\tau =$$

$$= 2\kappa_0 p \int\limits_{x_p^{-1}(r(s_0-s))}^{\pi/2} \frac{\cos^2\varphi}{\sqrt{1-p^2\sin^2\varphi}}\,\mathrm{d}\varphi = 2\kappa_0 \int\limits_{x_p^{-1}(r(s_0-s))}^{\pi/2} \frac{\cos^2\varphi}{\sqrt{((1/p^2)-1)+\cos^2\varphi}}\,\mathrm{d}\varphi =$$

$$= 2\kappa_0 \int\limits_0^{(\pi/2)-x_p^{-1}(r(s_0-s))} \frac{\sin^2\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi = 2\kappa_0 \int\limits_0^{\varphi_0(s)} \frac{\sin^2\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi,$$

which is (4.61). We continue

$$\mathcal{F}_p(s_0+s) - \mathcal{F}_p = 2\kappa_0 \int\limits_0^{\varphi_0(s)} \frac{\sin^2\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi =$$

$$= 2\kappa_0 \int\limits_0^{\varphi_0(s)} \Big(\frac{\sin^2\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} - \sin\varphi\Big)\,\mathrm{d}\varphi + 2\kappa_0 \int\limits_0^{\varphi_0(s)} \sin\varphi\,\mathrm{d}\varphi =$$

$$= 2\kappa_0 \int\limits_0^{\varphi_0(s)} \sin\varphi \cdot \frac{\sin\varphi - \sqrt{\varepsilon^2+\sin^2\varphi}}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi + 2\kappa_0(1-\cos\varphi_0(s)) =$$

$$=: 2\kappa_0(1-\cos\varphi_0(s)) - 2\kappa_0 I(s). \tag{4.66}$$

For the integral, we see

$$0 \le I(s) = \int\limits_0^{\varphi_0(s)} \sin\varphi \cdot \frac{\sqrt{\varepsilon^2+\sin^2\varphi} - \sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi =$$

$$= \int\limits_0^{\varphi_0(s)} \frac{\varepsilon^2\sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)}\,\mathrm{d}\varphi \le \frac{\varepsilon^2}{2} \int\limits_0^{\varphi_0(s)} \frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}.$$

Next by the definition of $s_0$ in (4.9) and of $x_p$ in (4.5), we calculate as in the proof of (4.61) above that

$$rs = x_p(\pi/2) - r(s_0-s) = \int\limits_{x_p^{-1}(r(s_0-s))}^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1-p^2\sin^2\varphi}} = \frac{1}{p} \int\limits_0^{\varphi_0(s)} \frac{1}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi, \tag{4.67}$$

hence with (4.7) that

$$0 \le I(s) \le \frac{\kappa_0}{4}\varepsilon^2 s,$$

which gives (4.62) by (4.66) and (4.30).

Moreover we have for $s = s_0$ that $\varphi_0(s_0) = \pi/2$ and get

$$0 \le I(s_0) = \int\limits_0^{\pi/2} \sin\varphi \cdot \frac{\sqrt{\varepsilon^2+\sin^2\varphi} - \sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi =$$

$$= \int_0^{\pi/2} \frac{\varepsilon^2 \sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)}\,\mathrm{d}\varphi =$$

$$= \int_0^{\varepsilon} \frac{\varepsilon^2 \sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)}\,\mathrm{d}\varphi +$$

$$+\frac{\varepsilon^2}{2}\int_\varepsilon^{\pi/2}\frac{\mathrm{d}\varphi}{\sin\varphi} + \int_\varepsilon^{\pi/2}\Big(\frac{\varepsilon^2 \sin\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)} - \frac{\varepsilon^2}{2\sin\varphi}\Big)\,\mathrm{d}\varphi =$$

$$=: \varepsilon^2(I_1+I_2-I_3).$$

Similarly as in Proposition 4.7, in particular (4.34), we estimate

$$0 \le I_1 \le \int_0^{\varepsilon}\frac{1}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi \le 1,$$

$$2I_2 - \log\frac{1}{\varepsilon} = \log\frac{\varepsilon(1+\cos\varepsilon)}{\sin\varepsilon} = O(1)$$

and

$$0 \le I_3 = \int_\varepsilon^{\pi/2}\frac{\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi) - 2\sin^2\varphi}{2\sqrt{\varepsilon^2+\sin^2\varphi}\cdot(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)\sin\varphi} \le$$

$$\le \int_\varepsilon^{\pi/2}\frac{\varepsilon^2+\sin^2\varphi+\sqrt{\varepsilon^2+\sin^2\varphi}\cdot\sin\varphi - 2\sin^2\varphi}{4\sin^3\varphi} \le$$

$$\le \int_\varepsilon^{\pi/2}\frac{\varepsilon^2}{4\sin^3\varphi} + \int_\varepsilon^{\pi/2}\frac{\sqrt{\varepsilon^2+\sin^2\varphi}-\sin\varphi}{4\sin^2\varphi} =$$

$$= \int_\varepsilon^{\pi/2}\frac{\varepsilon^2}{4\sin^3\varphi} + \int_\varepsilon^{\pi/2}\frac{\varepsilon^2}{4\sin^2\varphi(\sqrt{\varepsilon^2+\sin^2\varphi}+\sin\varphi)} \le C\varepsilon^2\int_\varepsilon^{\infty}\frac{\mathrm{d}\varphi}{\varphi^3} \le C,$$

as $(2/\pi)\varphi \le \sin\varphi$ for $0 \le \varphi \le \pi/2$ . Together, we get using (4.30) that

$$I(s_0) = \varepsilon^2\Big(\frac{1}{2}\log\frac{1}{\varepsilon} + O(1)\Big) = \frac{1}{4}(\kappa_0-2)\log\frac{1}{\kappa_0-2} + O(\kappa_0-2),$$

hence with (4.66) that

$$\mathcal{F}_p = \mathcal{F}_p(2s_0) - \mathcal{F}_p = 2\kappa_0(1-\cos(\pi/2)) - 2\kappa_0 I(s_0) =$$

$$= 4 - (\kappa_0-2)\log\frac{1}{\kappa_0-2} + O(\kappa_0-2),$$

which is (4.63).

Returning to (4.67), we see with (4.35) that

$$\kappa_0 s/2 = prs = \int\limits_0^{\varphi_0(s)} \frac{1}{\sqrt{\varepsilon^2+\sin^2\varphi}}\,\mathrm{d}\varphi = \int\limits_0^{\varphi_0(s)/\varepsilon} \frac{1}{\sqrt{1+\varepsilon^{-2}\sin^2(\varepsilon\tau)}}\,\mathrm{d}\tau \geq$$

$$\geq \int\limits_0^{\varphi_0(s)/\varepsilon} \frac{1}{\sqrt{1+\tau^2}}\,\mathrm{d}\tau = \log\Big(\sqrt{1+(\varphi_0(s)/\varepsilon)^2}+(\varphi_0(s)/\varepsilon)\Big) \geq \log(2\varphi_0(s)/\varepsilon),$$

hence

$$\varphi_0(s) \leq (\varepsilon/2)e^{\kappa_0 s/2}$$

and for $\sigma < \sigma' < (1/2) - \hat{\sigma}$ under the assumption (4.64) and with (4.30) that

$$\varphi_0(s) \leq C(\kappa_0-2)^{(1/2)-\kappa_0\hat{\sigma}/2} = O((\kappa_0-2)^{\sigma'}) \quad \text{for } \kappa_0 \text{ close to } 2.$$

Then by the elementary expansion

$$\sin\varphi = (1+O(\varphi^2))\varphi = (1+O((\kappa_0-2)^{2\sigma'}))\varphi \quad \text{for } 0 \leq \varphi \leq \varphi_0(s)$$

and from (4.67) with (4.35) that

$$\kappa_0 s/2 = prs = \int\limits_0^{\varphi_0(s)} \frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\sin^2\varphi}} \leq (1+O((\kappa_0-2)^{2\sigma'})) \int\limits_0^{\varphi_0(s)} \frac{\mathrm{d}\varphi}{\sqrt{\varepsilon^2+\varphi^2}} =$$

$$= (1+O((\kappa_0-2)^{2\sigma'})) \int\limits_0^{\varphi_0(s)/\varepsilon} \frac{1}{\sqrt{1+\tau^2}}\,\mathrm{d}\tau =$$

$$= (1+O((\kappa_0-2)^{2\sigma'}))\log\Big(\sqrt{1+(\varphi_0(s)/\varepsilon)^2}+(\varphi_0(s)/\varepsilon)\Big),$$

hence

$$\log\Big(\sqrt{1+(\varphi_0(s)/\varepsilon)^2}+(\varphi_0(s)/\varepsilon)\Big) = (1+O((\kappa_0-2)^{2\sigma'}))\kappa_0 s/2 = (1+O((\kappa_0-2)^{2\sigma'}))s.$$

Observing with (4.64) that

$$O((\kappa_0-2)^{2\sigma'})s = O\Big((\kappa_0-2)^{2\sigma'}\log\frac{1}{\kappa_0-2}\Big) = O((\kappa_0-2)^{2\sigma})$$

for $\kappa_0 \searrow 2$ , we continue with (4.30) to

$$\sqrt{\varepsilon^2+\varphi_0(s)^2}+\varphi_0(s) = (1+O((\kappa_0-2)^{2\sigma}))\varepsilon e^s =$$

$$= (1+O((\kappa_0-2)^{2\sigma}))(\kappa_0-2)^{1/2}e^s,$$

which is (4.65), and the lemma is proved.

///

This enables to estimate the energy before touching takes place.

**Proposition 4.11** *For* $0 < \sigma < 1/2, 0 < \hat{\sigma} < (1/2) - \sigma$ *and*

$$0 \leq s \leq \hat{\sigma} \log \frac{1}{\kappa_0 - 2} + C \ll s_0 \tag{4.68}$$

*with*

$$f(s_0 + s) \geq 0 \quad \textit{for } s_0 \leq \tau \leq s_0 + s, \tag{4.69}$$

*we have*

$$\mathcal{F}(\gamma|[0, s_0 + s]) < 4 \quad \textit{for } \kappa_0 \textit{ close to } 2. \tag{4.70}$$

**Proof:**
First we get from Lemma 4.9 with (4.68) and (4.69) and choosing $\hat{\sigma} < (1/2) - \sigma_1 < (1/2) - \sigma$ that

$$\left. \begin{aligned} \frac{\gamma_2'(\tau)}{\gamma_2(\tau)} \geq 1 - C(\kappa_0 - 2)^{2\sigma_1} =: \alpha, \\ \gamma_1'(\tau) > 0, \end{aligned} \right\} \quad \text{for } s_0 \leq \tau \leq s_0 + s \tag{4.71}$$

when $\kappa_0$ is close to 2 . From Proposition 4.4, we get

$$\int_{s_0}^{s_0+s} |\kappa| \frac{\gamma_2^2}{\gamma_2 + \gamma_2'} = - \int_{s_0}^{s_0+s} d'(\tau)\ \mathrm{d}\tau = f(s_0) - f(s_0 + s) \leq f(s_0). \tag{4.72}$$

Now we get from (4.71) and (4.44) that

$$\gamma^2(\tau) \geq \gamma^2(s_0) e^{\alpha(\tau - s_0)} \geq (1 - O(\mu^2)) e^{\alpha(\tau - s_0)} \quad \text{for } s_0 \leq \tau \leq s_0 + s$$

and

$$\frac{\gamma_2}{\gamma_2 + \gamma_2'}(\tau) = \frac{1}{1 + (\gamma_2'/\gamma_2)}(\tau) \geq \frac{1}{2} + O((\kappa_0 - 2)^{2\sigma_1}) \quad \text{for } s_0 \leq \tau \leq s_0 + s,$$

hence

$$\frac{\gamma_2^2}{\gamma_2 + \gamma_2'}(\tau) \geq \frac{1}{2}(1 + O((\kappa_0 - 2)^{2\sigma}))(1 - O(\mu^2)) \cdot e^{\alpha(\tau - s_0)} \quad \text{for } s_0 \leq \tau \leq s_0 + s.$$

We estimate by (4.32) that

$$(1 - \alpha)(\tau - s_0) \leq C(\kappa_0 - 2)^{2\sigma_1} s_0 \leq C(\kappa_0 - 2)^{2\sigma_1} \Big( \frac{1}{2} \log \frac{1}{\kappa_0 - 2} + C \Big) = O((\kappa_0 - 2)^{2\sigma})$$

and further by (4.39) assuming $0 < \sigma < 1/4$ that

$$\mu^2 \leq (\kappa_0 - 2) \Big( \frac{1}{2} \log \frac{1}{\kappa_0 - 2} + C \Big)^2 = O((\kappa_0 - 2)^{2\sigma}).$$

Together

$$\frac{\gamma_2^2}{\gamma_2 + \gamma_2'}(\tau) \geq \frac{1}{2}(1 + O((\kappa_0 - 2)^{2\sigma})) e^{\tau - s_0} \quad \text{for } s_0 \leq \tau \leq s_0 + s,$$

and with (4.72) that

$$\int_{s_0}^{s_0+s} |\kappa(\tau)| \cdot e^{\tau - s_0} \, \mathrm{d}\tau \leq (1 + O((\kappa_0 - 2)^{2\sigma})) \cdot 2f(s_0).$$

Further with (4.8) and (4.10) that

$$\int_{s_0}^{s_0+s} |\kappa(\tau)| \cdot e^{\tau - s_0} \, \mathrm{d}\tau = \int_{s_0-s}^{s_0} |\kappa(2s_0 - \tau)| \cdot e^{s_0 - \tau} \, \mathrm{d}\tau = \int_{s_0-s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \, \mathrm{d}\tau,$$

hence

$$\int_{s_0-s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \, \mathrm{d}\tau \leq (1 + O((\kappa_0 - 2)^{2\sigma})) \cdot 2f(s_0).$$

Then by (4.46) that

$$\int_{s_0-s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \, \mathrm{d}\tau \leq (1 + O((\kappa_0 - 2)^{2\sigma}))(1 + O(\mu))\mu,$$

hence, as by (4.39) that

$$\mu = O((\kappa_0 - 2)^{1/2} \log \frac{1}{\kappa_0 - 2}) = O((\kappa_0 - 2)^{2\sigma}) \quad \text{for } 0 < \sigma < 1/4,$$

we get

$$\int_{s_0-s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \, \mathrm{d}\tau \leq (1 + O((\kappa_0 - 2)^{2\sigma}))\mu. \tag{4.73}$$

We calculate the integral on the left hand side with (4.8) and (4.7) and the definition of $\varphi_0$ in (4.60) and (4.65) as

$$(1 + O((\kappa_0 - 2)^{2\sigma}))(\kappa_0 - 2)^{1/2} \int_{s_0-s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \, \mathrm{d}\tau \geq$$

$$\geq \kappa_0 \int_{s_0-s}^{s_0} cn(r\tau, p) \cdot 2\varphi_0(s_0 - \tau) \, \mathrm{d}\tau = 2\kappa_0 \int_{s_0-s}^{s_0} cn(r\tau, p) \cdot (((\pi/2) - x_p^{-1}(r\tau)) \, \mathrm{d}\tau =$$

$$= \frac{2\kappa_0}{r} \int_{r(s_0-s)}^{rs_0} cn(\tau, p) \cdot ((\pi/2) - x_p^{-1}(\tau)) \, \mathrm{d}\tau = 4p \int_{x_p^{-1}(r(s_0-s))}^{\pi/2} \frac{\cos\varphi}{\sqrt{1 - p^2 \sin^2\varphi}} \cdot ((\pi/2) - \varphi) \, \mathrm{d}\varphi =$$

$$= 4 \int_{(\pi/2)-\varphi_0(s)}^{\pi/2} \frac{\cos\varphi}{\sqrt{((1/p^2) - 1) + \cos^2\varphi}} \cdot ((\pi/2) - \varphi) \, \mathrm{d}\varphi =$$

$$= 4 \int_0^{\varphi_0(s)} \frac{\sin \varphi}{\sqrt{\varepsilon^2 + \sin^2 \varphi}} \cdot \varphi \ \mathrm{d}\varphi \geq 4 \int_0^{\varphi_0(s)} \frac{\sin^2 \varphi}{\sqrt{\varepsilon^2 + \sin^2 \varphi}} \ \mathrm{d}\varphi.$$

Then by (4.61) and (4.30) that

$$(1 + O((\kappa_0 - 2)^{2\sigma}))(\kappa_0 - 2)^{1/2} \int_{s_0 - s}^{s_0} \kappa(\tau) \cdot e^{s_0 - \tau} \ \mathrm{d}\tau \geq \frac{2}{\kappa_0}(\mathcal{F}_p(s_0 + s) - \mathcal{F}_p) =$$

$$= \mathcal{F}_p(s_0 + s) - \mathcal{F}_p + O(\kappa_0 - 2).$$

Plugging into (4.73), we get with (4.39) that

$$\mathcal{F}_p(s_0 + s) - \mathcal{F}_p \leq (1 + O((\kappa_0 - 2)^{2\sigma}))(\kappa_0 - 2)^{1/2} \mu + O(\kappa_0 - 2) =$$

$$= \frac{1}{2}(\kappa_0 - 2) \log \frac{1}{\kappa_0 - 2} + O(\kappa_0 - 2) + O((\kappa_0 - 2)^{2\sigma}))(\kappa_0 - 2)^{1/2} \mu =$$

$$= \frac{1}{2}(\kappa_0 - 2) \log \frac{1}{\kappa_0 - 2} + O(\kappa_0 - 2).$$

Adding (4.63), we obtain

$$\mathcal{F}(\gamma|[0, s_0 + s]) = \mathcal{F}_p(s_0 + s) = \mathcal{F}_p + \mathcal{F}_p(s_0 + s) - \mathcal{F}_p \leq$$

$$\leq 4 - \frac{1}{2}(\kappa_0 - 2) \log \frac{1}{\kappa_0 - 2} + O(\kappa_0 - 2) < 4 \quad \text{for } \kappa_0 \text{ close to } 2,$$

which is (4.70), and the proposition is proved.

///

Now we can prove Proposition 4.3.

**Proof of Proposition 4.3:**
We fix $0 < \sigma < 1/2, 0 < \hat{\sigma} < (1/2) - \sigma$ and see for $s = \hat{\sigma} \log 1/(\kappa_0 - 2)$ with (4.65) that

$$2\varphi_0(s) + \varepsilon \geq (1 + O(\kappa_0 - 2)^{2\sigma})(\kappa_0 - 2)^{1/2} e^s = (1 + O(\kappa_0 - 2)^{2\sigma})(\kappa_0 - 2)^{(1/2) - \hat{\sigma}},$$

in particular with (4.30) that

$$2\varphi_0(s) \geq (1 + O(\kappa_0 - 2)^{2\sigma})(\kappa_0 - 2)^{(1/2) - \hat{\sigma}} - C(\kappa_0 - 2)^{1/2} \geq c_0(\kappa_0 - 2)^{(1/2) - \hat{\sigma}}$$

for some $c_0 > 0$ . Next by (4.62) that

$$\mathcal{F}_p(s_0 + s) - \mathcal{F}_p = 2\kappa_0(1 - \cos \varphi_0(s)) + O(\kappa_0 - 2)s \geq$$

$$\geq c_0(\kappa_0 - 2)^{1 - 2\hat{\sigma}} + O((\kappa_0 - 2) \log \frac{1}{\kappa_0 - 2})$$

with adapted $c_0 > 0$ , hence with (4.63) that

$$\mathcal{F}(\gamma|[0, s_0 + s]) = \mathcal{F}_p(s_0 + s) = \mathcal{F}_p + \mathcal{F}_p(s_0 + s) - \mathcal{F}_p \geq$$

$$\geq 4 + c_0(\kappa_0 - 2)^{1-2\hat{\sigma}} + O((\kappa_0 - 2)\log\frac{1}{\kappa_0 - 2}) > 4 \quad \text{for } \kappa_0 \text{ close to 2.}$$

Then we conclude by Proposition 4.11 that $f(s_0 + s) < 0$ , and, as $f(s_0) > 0$ by (4.46), there exists

$$0 < s < \hat{\sigma}\log\frac{1}{\kappa_0 - 2} \quad \text{with} \quad f(s_0 + s) = 0 \quad \text{for } \kappa_0 \text{ close to 2,}$$

which is (4.22) for $s$ replaced by $s_0 + s$ . Moreover by Lemma 4.9 that

$$\frac{\gamma_2'(s_0 + s)}{\gamma_2(s_0 + s)} \geq 1 - C(\kappa_0 - 2)^{2\sigma} \quad \text{and} \quad \gamma_1'(s_0 + s) > 0,$$

hence

$$\beta_0 < \beta(s_0 + s) < \pi/2$$

for appropriate choice in (4.23) and $\kappa_0$ close to 2 , and by Proposition 4.11 that

$$\mathcal{F}(\gamma|[0, s_0 + s]) < 4,$$

which is (4.24), proving Proposition 4.3.

///

We have already shown after the statement of Proposition 4.3 that Proposition 4.3 implies Proposition 4.2, and finally, we are able to prove Theorem 1.1.

**Proof of Theorem 1.1:**
If $focal(x_-, r_-, \beta_-) \neq focal(x_+, r_+, \beta_+)$ , there exists by Proposition 3.9 a regular profile curve in the upper half plane which satisfies the boundary conditions (1.6) and minimizes the Willmore energy subject to these boundary conditions, hence Theorem 1.1 follows in this case. If $focal(x_-, r_-, \beta_-) = focal(x_+, r_+, \beta_+)$ , but none of the sphere caps is contained in the other, then

$$\mathcal{W}_{closed}(x_\pm, r_\pm, \beta_\pm) < 8\pi$$

by Proposition 4.2, and again Theorem 1.1 follows from Proposition 3.9.

///

# Appendix

## A An estimate for the catenoid touching two spheres

In this section, we consider the catenoid

$$cat := \{te_1 + \cosh t(0, \cos\theta, \sin\theta) \mid t, \theta \in \mathbb{R}\ \}$$

which is a minimal surface in $\mathbb{R}^3$ that is $\vec{\mathbf{H}}_{cat} = 0$ and $\mathcal{W}(cat) = 0$ . We follow and extend the computations in [NdSch14].

**Proposition A.1** *For two balls in* $\mathbb{R}^3$ *of same radius* $1$ *with their centers on the* $x-$*axis and of sufficiently small positive distance* $d > 0$ *apart, there exists a catenoid rotationally symmetric with respect to the* $x-$*axis and symmetric with respect to the balls that touches both balls and seperates on each ball a sphere cap of opening angle* $\beta(d) > 0$ *. Moreover* $\beta(d) \to 0$ *for* $d \searrow 0$ *and more precisely*

$$\frac{d}{\tan^2(\beta(d)/2)} \to \infty \quad \textit{for } d \searrow 0. \tag{A.1}$$

**Proof:**
As our construction will be rotational symmetric with respect to the $x-$axis, we work in the $(x, y)-$plane. We consider the sphere $S(t)$ with center on the $x-$axis which touches the catenoid from the right in $(t, \pm\cosh t)$ for $t \geq 0$ . Let $\xi(t)e_1$ be its center and $\varrho(t)$ be its radius. Then clearly

$$|(t, \cosh t) - (\xi(t), 0)| = \varrho(t), \quad (1, \sinh t) \perp (t - \xi(t), \cosh t),$$

and

$$\begin{aligned} \xi(t) &= \sinh t \cosh t + t, \\ \varrho(t)^2 = \cosh^2 t + (t - \xi(t))^2 &= \cosh^2 t + \sinh^2 t \cosh^2 t = \cosh^4 t. \end{aligned} \tag{A.2}$$

Likewise the sphere $-S(t)$ has center $-\xi(t)e_1$ , same radius and touches the catenoid from the left in $(-t, \pm\cosh t)$ . Rescaling their radius to $1$ , the rescaled distance of $\pm S(t)$ is

$$d(t) := 2\frac{\xi(t) - \varrho(t)}{\varrho(t)}, \tag{A.3}$$

provided that $\xi(t) > \varrho(t)$ , in particular $t > 0$ by (A.2). Therefore two balls in $\mathbb{R}^3$ of same radius $1$ with their centers on the $x-$axis and of positive distance $d > 0$ apart can be touched by a catenoid rotationally symmetric with respect to the $x-$axis and symmetric with respect to the balls, if $d = d(t) > 0$ for some $t > 0$ , and in this case, the opening angle $\beta(t) \in ]0, \pi[$ of the sphere caps separated on the two balls can be calculated by the reciprocal of the slope of catenoid at the touching point as

$$\tan(\beta(t)/2) = 1/\sinh t \in ]0, \infty[. \tag{A.4}$$

We calculate

$$\begin{aligned} d(t)/2 = \frac{\xi(t) - \varrho(t)}{\varrho(t)} &= \frac{\sinh t \cosh t + t - \cosh^2 t}{\cosh^2 t} = \\ = \frac{t}{\cosh^2 t} + \frac{\sinh t - \cosh t}{\cosh t} &= \frac{t}{\cosh^2 t} + \tanh t - 1 \end{aligned} \tag{A.5}$$

and get recalling that $\tanh' = 1/\cosh^2$ by differentiation

$$d'(t)/2 = -\frac{2t \sinh t}{\cosh^3 t} + \frac{2}{\cosh^2 t} = \frac{2}{\cosh^2 t}(1 - t \tanh t).$$

Since $(1/t) - \tanh t$ is strictly monotonically decreasing for $t > 0$ and tends to $\infty$ respectively to $-1$ for $t \searrow 0$ respectively for $t \to \infty$ , there is exactly one $t_0 > 0$ with $d'(t_0) = 0$ and $d' > 0$ on $[0, t_0[$ and $d' < 0$ on $]t_0, \infty[$ . Therefore

$$d(t_0) > \lim_{t \to \infty} d(t) = \lim_{t \to \infty} 2\frac{\xi(t) - \varrho(t)}{\varrho(t)} = 0$$

by (A.2), and for any $0 < d < d(t_0)$ there exists exactly one $t > t_0$ with $d = d(t)$ and hence there exists a catenoid as above touching the two balls. Clearly $t \to \infty$ for $d \searrow 0$ and by (A.4) and (A.5) that

$$\frac{d}{\tan^2(\beta(d)/2)} = 2\frac{\xi(t) - \varrho(t)}{\varrho(t)}\sinh^2 t = 2t\tanh^2 t - 2(1 - \tanh t)\sinh^2 t =$$

$$= 2t\tanh^2 t - 2\frac{(1 - \tanh^2 t)\sinh^2 t}{1 + \tanh t} = 2t\tanh^2 t - 2(\cosh^2 t - \sinh^2 t)\frac{\tanh^2 t}{1 + \tanh t} =$$

$$= 2t\tanh^2 t - 2\frac{\tanh^2 t}{1 + \tanh t} \to \infty \quad \text{for } t \to \infty, d \searrow 0,$$

which is (A.1), and the proposition is proved.

///

**Remark:**
As $d(0) = -2 < 0, d(t_0) > 0$ and $d' > 0$ on $[0, t_0[$ , hence there exists exactly one $\tilde{t} \in ]0, t_0[$ with $d(\tilde{t}) = 0$ , we remark for two balls of same radius touching each other there exists a catenoid symmetric with respect to the balls that touches both balls and seperates on each ball a sphere cap of opening angle $\beta := \beta(\tilde{t}) \in ]0, \pi[$ determined by (A.1). We conclude for two balls of same radius touching at the same focal point and boundary data of opening at least $\beta/2$ on each side that $\mathcal{W}_{closed} < 8\pi$ , more precsiely

$$\mathcal{W}_{closed}(-1 + e^{i\beta}, 1, \beta - \pi/2 \; ; \; 1 + e^{i(\beta+\pi/2)}, 1, \beta + \pi) < 8\pi. \tag{A.6}$$

□

# B The hyperbolic distance of the inflection points

Let $\gamma$ be a wavelike free elastica parametrized by hyperbolic arclength with curvature $\kappa$ and satisfying the following initial value problem $\gamma(0) = (0, h)$, $\gamma'(0) = (h, 0)$, $\kappa(0) = \kappa_0$, $\kappa'(0) = 0$ . Here $h > 0, \kappa_0 > 2$ and (4.8). The idea to specifically examine this initial value problem first came up in [Ei16, Lemma 4.10], was later refined in [MuSp20, Lemma 2.9] and in [Sl23, p. 20] such $\gamma$ were coined canonically parametrized. By [LaSi84b, Prop. 2.1] there exists a Killing vectorfield $J$ of the upper half plane and by [LaSi84b, Fig. 1a, rsp. p.8] (see also [Ei16, Lemma 6.1] rsp. [Ei17, Lemma 5.1] for more elaborate proofs) an integral cruve $\Sigma$ of $J$ crossing $\gamma$ perpendicularly exactly when $\kappa$ has a zero. This integral curve is also a geodesic w.r.t. hyperbolic metric and it crosses the $y$-axis perpendicularly (see e.g. [Ei17, Lemma 5.4] rsp. [Ei16, Lemma 7.1]).

Now half of the hyperbolic distance of to consecutive inflection points of the wavelike free elastica, abbreviated by $\mu$ in (4.19), was calculated in [LaSi84b, Prop. 5.1] as

$$\mu = \frac{\kappa_0}{r}\sqrt{\kappa_0 - 2}\sqrt{\kappa_0 + 2}\int_0^{\frac{\pi}{2}} \frac{\cos^2\varphi}{\kappa_0^2 - 4\sin^2\varphi}\frac{d\varphi}{\sqrt{1 - p^2\sin^2\varphi}}. \tag{B.1}$$

A proof with more details can be made with the results from [Ei16] rsp. [Ei17], which we sketch here for the readers convenience. Let $s_0 > 0$ be the smallest value, such that

$\kappa(s_0) = 0$, i.e. given by (4.9). For the flow $\Phi : \mathcal{H} \times \mathbb{R} \to \mathcal{H}$ of $J$ we find a $y_0 > 0$ such that the geodesic $\Sigma$ is parametrized by

$$t \mapsto \Phi((0, y_0), t).$$

Furthermore we find by definition of $\mu$ a $t_\gamma > 0$ with

$$\Phi((0, y_0), t_\gamma) = \gamma(s_0)$$

By [Ei17, Lemma 5.12] rsp. [Ei16, Lemma 7.8] we obtain

$$t_\gamma = \int_0^{s_0} \frac{\kappa^2(\ell)}{4\kappa^2(\ell) - 4\kappa_0^2 + \kappa_0^4} \, d\ell.$$

Since integral curves of Killingfields are parametrized proportional to arclength (see e.g. [Ei14, Remark 5.6] for a quick proof), we only need to calculate (see [Ei17, Remark 5.11] rsp. [Ei16, Equation 5.4])

$$|J(0, y_0)|_g^2 = |J(\gamma(s_0))|_g^2 = 4\kappa^2(s_0) - 4\kappa_0^2 + \kappa_0^4 = \kappa_0^2(\kappa_0 - 2)(\kappa_0 + 2)$$

and then we have

$$\mu = |J(0, y_0)|_g t_\gamma.$$

Using substitution with (4.9) the integral in $t_\gamma$ transforms to our desired result.

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