Benjamini-Schramm vs Plancherel convergence

Giacomo Gavelli

joint work with Claudius Kamp

Universität Tübingen

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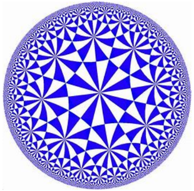

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- $M_n = \Gamma_n \setminus X$ associated sequence of locally simmetric spaces

Overview

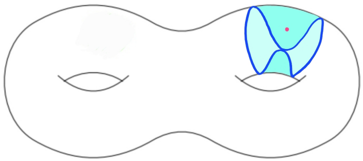
Benjamini-Schramm Convergence (geometrical)

$$\updownarrow$$

Plancherel Convergence (spectral)

Benjamini-Schramm convergence

Let $M = \Gamma \setminus X$. The injectivity radius at a point p is the radius of the largest ball in M centered at p which is isometric to a ball in X.



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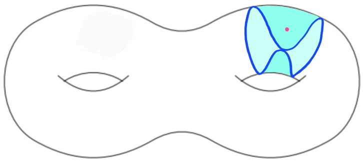
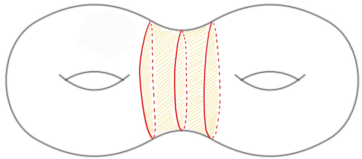


For $R > 0$, the *R-thin part* of M is

$$S_{<R} = \left\{ p \in X \ : \ \mathrm{InjRad}(p) < R \right\}.$$



Benjamini-Schramm convergence

Definition (Benjamini-Schramm convergence)

We say that a sequence $(M_n)_{n\in\mathbb{N}}$ of locally symmetric spaces given by $M_n = \Gamma_n \setminus X$ is *Benjamini–Schramm convergent* to X if for every $R > 0$

$$\frac{\mathrm{vol}\left((M_n)_{<R}\right)}{\mathrm{vol}(M_n)} \longrightarrow 0, \qquad \text{as } n \to \infty.$$

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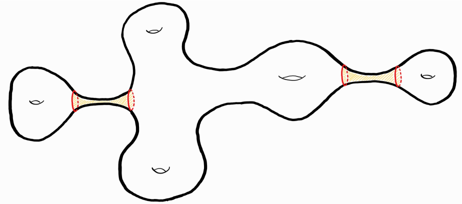
Equivalently, $(M_n)_{n\in\mathbb{N}}$ is BS-convergent if for any $R > 0$, the probability of a ball in M_n of radius R being isometric to a ball in X is asimptotically 1.



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$\mathrm{InjRad}(M_n) \xrightarrow{n \to \infty} \infty \implies$ the R-thin part of M_n is empty for large enough $n \implies M_n$ is Benjamini-Schramm convergent to X.

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$\hat{f} : \pi \mapsto \operatorname{tr} \pi(f)$ is the (scalar) Fourier transform of f.

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For $f \in C_c^\infty(G)$,

$$\mu_\Gamma(\hat{f}) = \sum_{\pi \in \widehat{G}} m(\pi, \Gamma)\hat{f}(\pi), \qquad \mu_{Pl}(\hat{f}) := \int_{\widehat{G}} \hat{f}(\pi) d\mu_{Pl}(\pi) = f(1).$$

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A sequence $(\Gamma_n)_{n\in\mathbb{N}}$ of cocompact lattices in G, or equiv. the sequence $(M_n)_{n\in\mathbb{N}}$ of associated locally symmetric spaces, is called a *Plancherel sequence* if for every $f \in C_c^\infty(G)$

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i.e. $(\Gamma_n)_{n\in\mathbb{N}}$ has the *limit multiplicity property*.

Asymptotics of the Laplace spectrum

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For all $0 < a < b < \infty$, $N(\Delta, M_n, a, b) = \mu_{\Gamma_n}((a, b))$.

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For all $0 < a < b < \infty$, $N(\Delta, M_n, a, b) = \mu_{\Gamma_n}((a,b))$.

The limit multiplicity property translates to:

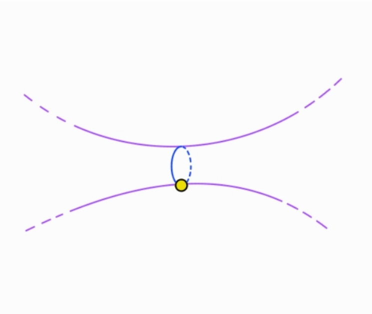
$$\frac{N(\Delta, M_n, a, b)}{\mathrm{vol}(M_n)} \xrightarrow{n\to\infty} \mu_{Pl}((a,b)).$$

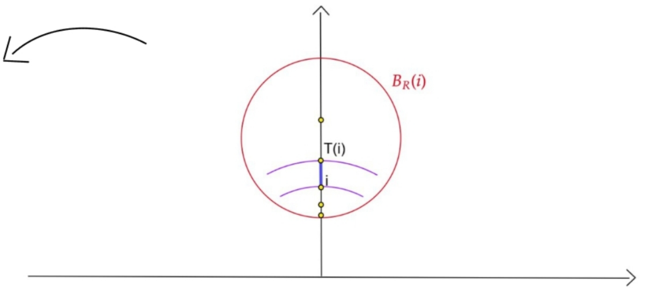
Geometrical interpretation

Theorem (Deitmar, 2018)

$(M_n)_{n\in\mathbb{N}}$ is a Plancherel sequence if and only if for every $R > 0$

$$\frac{1}{\mathrm{vol}(M_n)} \int_{\mathcal{F}_n} \sharp\left(\Gamma_n^{\star} \cdot x \cap \overline{B_R(x)}\right) dx \xrightarrow{n\to\infty} 0.$$





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Definition (Uniform Discreteness)

We call a sequence of locally symmetric spaces $(M_n)_{n\in\mathbb{N}}$ *uniformly discrete* if there is a uniform lower bound on the injectivity radius of $M_n = \Gamma_n \setminus X$ for $n \in \mathbb{N}$.

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Theorem (ABBGNRS, 2017)

A uniformly discrete Benjamini-Schramm convergent sequence is Plancherel.

Benjamini-Schramm vs Plancherel convergence

Benjamini-Schramm + Uniformly Discrete

$\Downarrow \quad \not\Uparrow$

Plancherel

$\Downarrow$

Benjamini-Schramm

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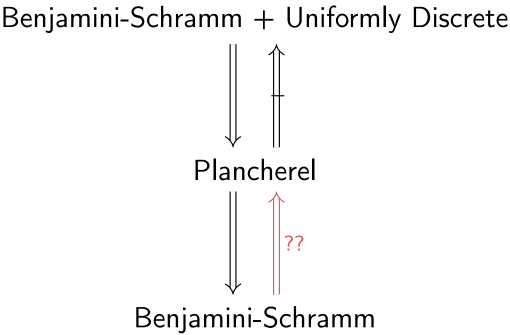

Theorem (G., Kamp, 2024)

There exists a Benjamini-Schramm convergent sequence of compact hyperbolic surfaces which is <u>not</u> Plancherel convergent.

Asymptotics for the Length Spectrum

$(M_n)_{n\in\mathbb{N}}$ has the *closed geodesics property* if for all $R > 0$

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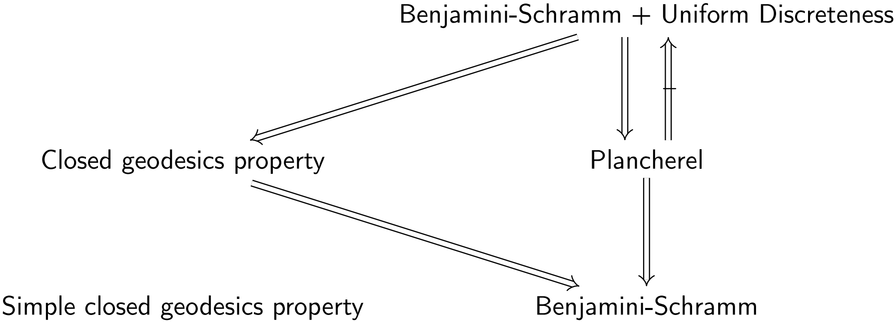

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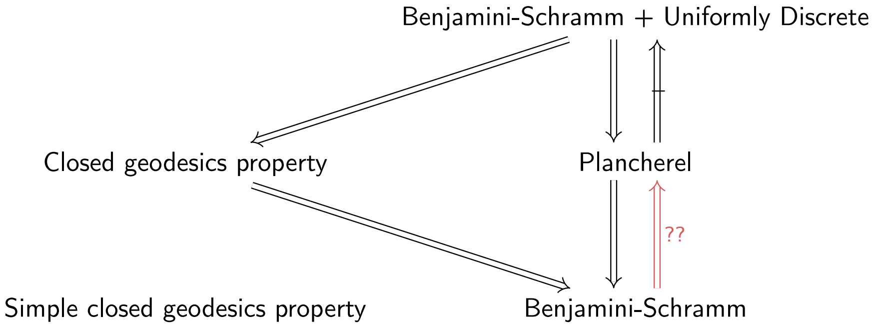

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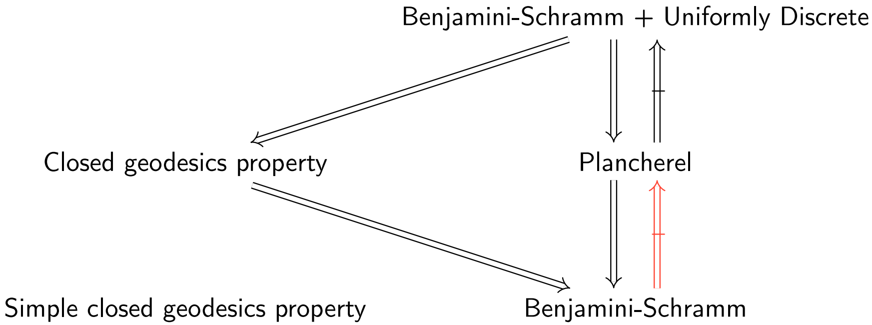

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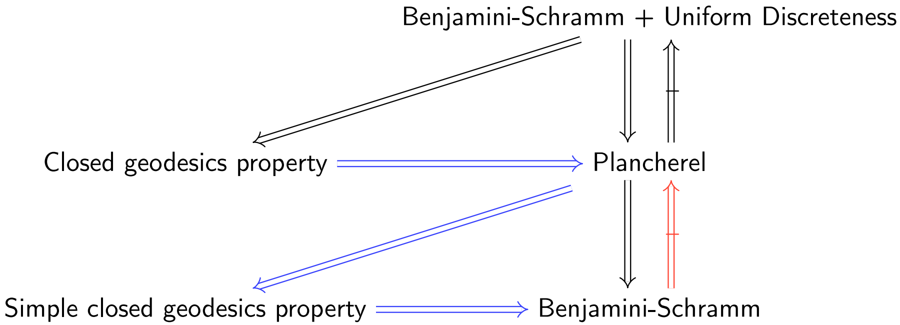

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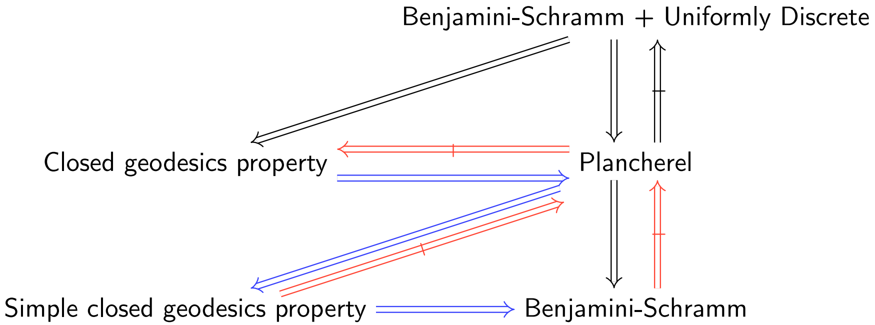
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Construction of the example

Consider the principal congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \; : \; a, d \equiv 1 \pmod{N}, \; b, c \equiv 0 \pmod{N} \right\}.$$

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We denote by $X(N) = \Gamma(N) \setminus \mathbb{H}$ the congruence surface of level N.

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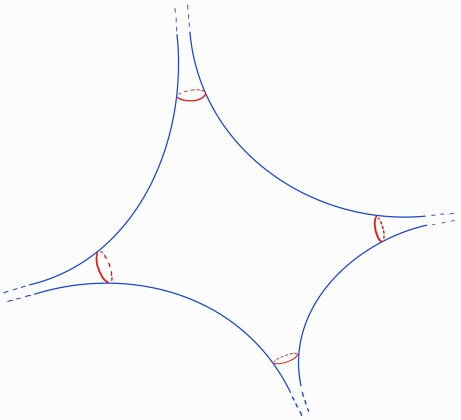

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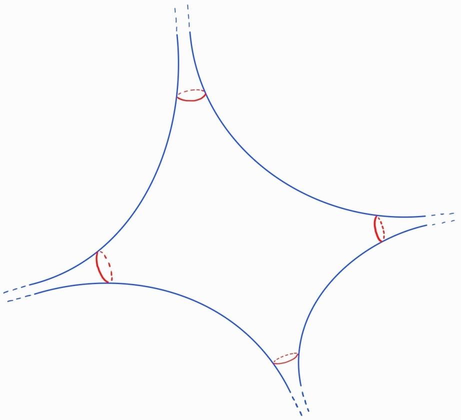

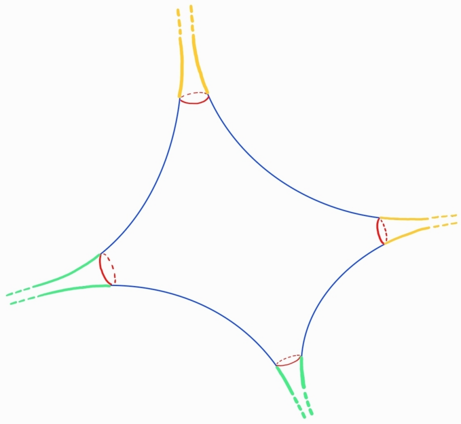

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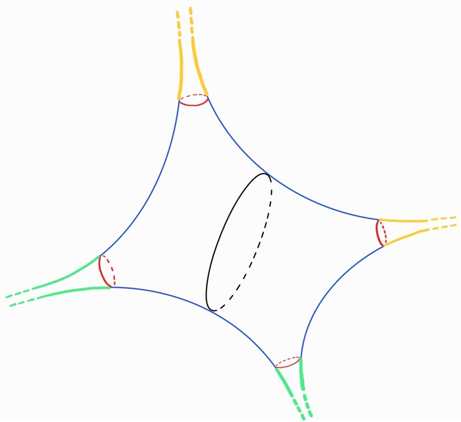

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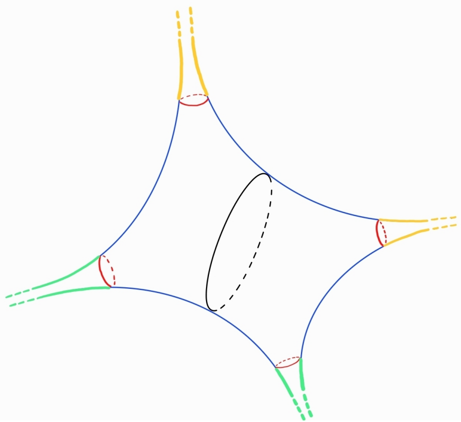
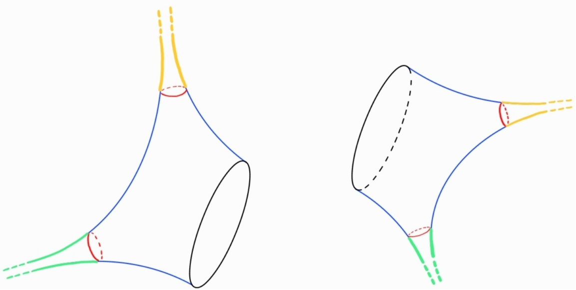
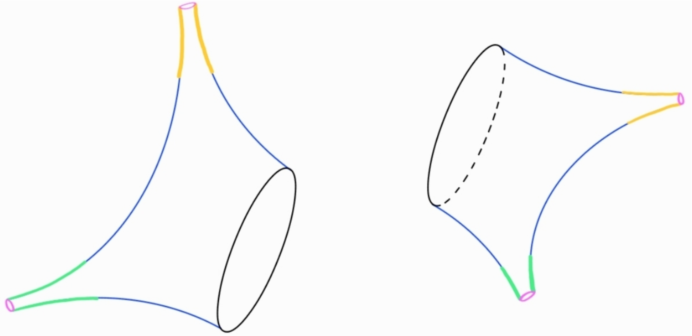




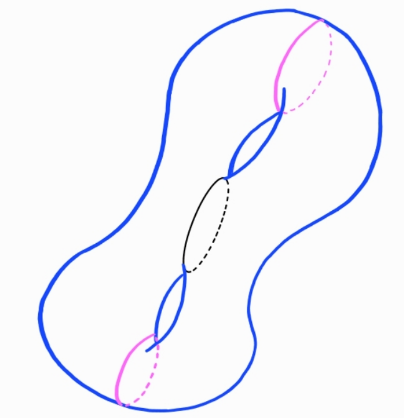
Construction of the example

We keep the boundary geodesics and replace each puncture by a geodesic of length $t > 0$.



Construction of the example

Reassemble these pieces using the old identifications. Since the number of cusps of $X(N)$ is even, we can identify the remaining geodesics in pairs. This yields a closed hyperbolic surface $X_t(N)$.



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- The sequence $(X_n)_{n\in\mathbb{N}}$ is Plancherel convergent if and only if t_n^{-1} grows sub-exponentially in n.
- The sequence $(X_n)_{n\in\mathbb{N}}$ is Benjamini–Schramm convergent.

Thank you for your attention!