K-theory

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1 Vector bundles

1.1 Definitions

Definition 1.1.1. We use the symbol $\mathbb{K}$ for either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1.2. A (real or complex) **vector bundle** over a topological space X consists of the following data:

(a) a continuous map $p : E \to X$, where E is a topological space,

(b) on every fibre $E_x = p^{-1}(x)$ the structure of a finite-dimensional $\mathbb{K}$ vector space,

(c) for every $x \in X$ an open neighbourhood U and an $n \in \mathbb{N}_0$, as well as a commutative diagram

$$\begin{array}{ccc} E_U & \xrightarrow{\alpha_U} & U \times \mathbb{K}^n \\ {\scriptstyle p}\downarrow & \swarrow & \\ U & & \end{array}$$

where $E_U = p^{-1}(U)$ and the map $U \times \mathbb{K}^n \to U$ is the projection onto the first coordinate. We insist that the map α_U is a homeomorphism and linear on every fibre.

The map α_U is called a **local trivialisation**.

Definition 1.1.3. The dimension of the fibre $n = n(x)$ depends on x. It is called the **rank**. The ensuing map $X \to \mathbb{N}_0$, $x \mapsto n(x)$ is locally constant.

A **line bundle** is a vector bundle of constant rank $n = 1$.

It follows that the structure maps of the vector spaces E_b are continuous. More precisely, the scalar multiplication

$$\mathbb{K} \times E \to E$$

and the addition

$$\Big\{(e, f) \in E \times E : p(e) = p(f)\Big\} \to E$$

are continuous maps.

Examples 1.1.4. (a) The **trivial bundle** $X \times \mathbb{K}^n \to X$.

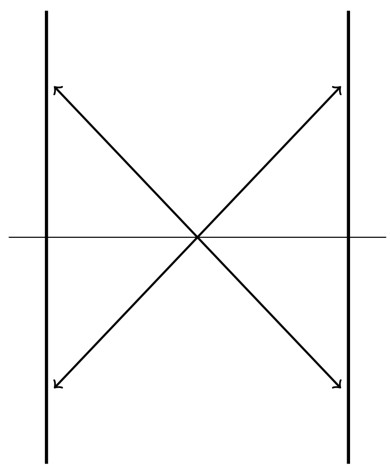
(b) If X is a smooth manifold, then the **tangent bundle** $TX \to X$ is a vector bundle over X.

(c) The **Moebius strip** is a real vector bundle over $X = S^1 = \mathbb{R}/\mathbb{Z}$. As a space, we set $E = \mathbb{R}^2/\mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}^2$ by

$$k(x, y) = (x + k, (-1)^k y).$$

The bundle projection $p : E \to \mathbb{R}/\mathbb{Z}$ is given by the projection onto the first variable, so $p\big(\mathbb{Z}(x, y)\big) = x + \mathbb{Z}$.

The Moebius strip can be constructed from a strip $[0, 1] \times \mathbb{R}$ by crosswise identification:



Definition 1.1.5. A **vector bundle homomorphism** $\phi : (E \to X) \to (F \to X)$ is a continuous map $\phi : E \to F$, such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & F \\ & {\scriptstyle p_E}\searrow \quad \swarrow{\scriptstyle p_F} & \\ & X & \end{array}$$

commutes and such that ϕ is linear on fibres.

An **isomorphism of vector bundles** is a homomorphism ϕ, which is bijective and such that ϕ^{-1} is continuous, too. Then ϕ^{-1} is a bundle homomorphism.

Examples 1.1.6. (a) The zero map, which sends each $v \in E_b$ to zero in F_b, is a vector bundle homomorphism.

(b) Note that we only consider vector bundle homomorphisms between bundles on the same base space.

(c) Complexification is a bundle homomorphism: Let $E \xrightarrow{p} X$ be a real vector bundle and let

$$E_{\mathbb{C}} = \bigsqcup_{x \in X} E_x \otimes_{\mathbb{R}} \mathbb{C},$$

where $\bigsqcup$ is the disjoint union. Let the projection $p_{\mathbb{C}}$ be defined by

$$p_{\mathbb{C}}(v_x \otimes \lambda) = x,$$

when $v_x \in E_x$. The space $E = \bigsqcup_{x \in X} E_x$ can be viewed as subset of $E_{\mathbb{C}}$. We equip $E_{\mathbb{C}}$ with a topology as follows: Let $\alpha_U : p^{-1}(U) \to U \times \mathbb{R}^n$ be a local trivialisation. Then there is exactly one fibre-wise $\mathbb{C}$-linear extension $\alpha_{\mathbb{C},U} : p_{\mathbb{C}}^{-1}(U) \to U \times \mathbb{C}^n$, which again is bijective. On $E_{\mathbb{C},U}$ one chooses the topology making $\alpha_{\mathbb{C},U}$ a homeomorphism and on $E_{\mathbb{C}}$ the topology, which is generated by all open sets in $E_{\mathbb{C},U}$, where U varies.

Definition 1.1.7. A bundle, which is isomorphic to a trivial bundle, is called **trivialisable**.

Example 1.1.8. The complexification of the Moebius strip is a trivialisable complex line bundle.

Proof. The complexification E of the Moebius strip is isomorphic to $(\mathbb{R} \times \mathbb{C})/\mathbb{Z}$, where $\mathbb{Z}$ acts by

$$k(x, z) = (x + k, (-1)^k z)$$

operiert. We now give an explicit trivialisation $\phi : E \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{C}$,

$$\phi(\mathbb{Z}(x, z)) = (x + \mathbb{Z}, e^{\pi i x} z). \qquad \square$$

Definition 1.1.9. A **section** of a vector bundle $p : E \to X$ is a continuous map $s : X \to E$ such that $p(s(x)) = x$ for every $x \in X$.

The **zero section** s_0 maps each point p to the zero in the vector space E_p.

The set of all sections is denoted by $\Gamma(E)$. It is a vector space.

Examples 1.1.10. (a) A section s of the trivial bundle $p : X \times \mathbb{K} \to X$ is a map $s : x \mapsto (x, f(x))$, where $f : X \to \mathbb{K}$ can be any continuous map.

(b) A section of the Moebius strip is a map $s(x) = (x, \alpha(x))$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\alpha(x + 1) = -\alpha(x).$$

Such a function α necessarily has a zero hence every section of the Moebius strip has a zero. This implies that the Moebius strip is not trivialisable.

Definition 1.1.11. Let $A \subset X$ be a subset and $p : E \to X$ a vector bundle. Then the **restriction** $E_A = p^{-1}(A) \to A$ is a vector bundle over A.

Definition 1.1.12. More generally, if $f : Y \to X$ is a continuous map and $p : E \to X$ a vector bundle, then the **pullback** $f^*E \to Y$ is a vector bundle defined by

$$f^*E = \Big\{(y, v) \in Y \times E : f(y) = p(v)\Big\}$$

The topology is the one of $Y \times E$ and the bundle projection is the projection onto the first coordinate. For the fibre one has $f^*E_y \cong E_{f(y)}$.

Example 1.1.13. Let $X = \mathbb{P}^n(\mathbb{K})$ be the **projective space** of dimension n over $\mathbb{K}$. One can view X as the set of all 1-dimensional subspaces of $\mathbb{K}^{n+1}$. The identification $X \cong \left(\mathbb{K}^{n+1} \smallsetminus \{0\}\right)/\mathbb{K}^\times$ yields a topology on X. Let $E \subset X \times \mathbb{K}^{n+1}$ be the set of all pairs (L, v) with $v \in L$. The projection onto the first coordinate is a line bundle $p : E \to X$, called the **tautological bundle**.

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1.2 Construction of vector bundles

Definition 1.2.1. Let E and F be vector bundles over X. Set

$$E \oplus F = \bigsqcup_{x \in X} E_x \oplus F_x.$$

We equip $E \oplus F$ with a topology as follows: Since $E_x \oplus F_x = E_x \times F_x$, one can identify $E \oplus F$ with the set of all $(v, w) \in E \times F$, for which $p_E(v) = p_F(w)$. This is a closed subset of $E \times F$ and we equip it with the subspace topology. Let $U \subset X$ be an open set on which both bundles are triovialisable. One identifies $(E \oplus F)_U$ with the set of all $(x, u, y, v) \in U \times \mathbb{K}^n \times U \times \mathbb{K}^m$, for which one has $x = y$. The latter can be identified with $U \times \mathbb{K}^n \times \mathbb{K}^m$. In this way one gets local trivialisations. We get a vector bundle $E \oplus F$ with fibres $(F \oplus F)_x = E_x \oplus F_x$.

In similarly one constructs

$$E \otimes F,$$
$$E^*$$
$$\mathrm{Hom}(E,F) = E^* \otimes F$$
$$\bigwedge^k E.$$

There are natural isomorphisms

$$E \oplus F \cong F \oplus E,$$
$$E \otimes F \cong F \otimes E,$$
$$E \otimes (F \oplus G) \cong (E \otimes F) \oplus (E \otimes G),$$
$$\bigwedge^k (E \oplus F) \cong \bigoplus_{i+j=k} \bigwedge^i E \otimes \bigwedge^j F.$$

Every homomorphism $\phi : E \to F$ yields a **dual homomorphism** $\phi^* : F^* \to E^*$, which on each fibre is the dual map. There is a natural isomorphism $E \to E^{**}$ and so ϕ^{**} can be identified with ϕ.

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1.3 Sub-bundles and quotients

Definition 1.3.1. A **sub-bundle** of a vector bundle E over X is a subset $T \subset E$, which is a vector bundle over X with the induced structures

Examples 1.3.2. (a) The trivial bundle $X \times \mathbb{K}^k$ is a sub-bundle of $X \times \mathbb{K}^{k+l}$.

(b) The Moebius strip is a sub-bundle of the trivial bundle $S^1 \times \mathbb{R}^2 \cong S^1 \times \mathbb{C}$.

Lemma 1.3.3. *A homomorphism ϕ surjective if and only if its dual ϕ^* is injective.*

If $\phi : E \to F$ is injective, then $\phi(E)$ is a sub-bundle and $\phi : E \to \phi(E)$ is an isomorphism.

Proof. Let $\phi : E \to F$. Covering X with open sets, on which both bundles trivialise, one sees that both assertions only need to be shown for trivial bundles $E = X \times V$ and

$F = X \times W$. In this case $\phi(x, v) = (x, \eta(x)v)$ for some continuous map $\eta : X \to \mathrm{Lin}(V, W)$ to the set of linear maps $V \to W$. Then the lemma follows from Linear Algebra. □

Definition 1.3.4. Let F be a sub-bundle of E. We define the **quotient bundle** E/F to be the quotient set $E/\sim$, where $v \sim w$ holds iff v and w lie in the same fibre and $v - w \in F$. The quotient topology turns E/F into a vector bundle and the projection $E \to E/F$ is a surjective bundle homomorphism.

Remark 1.3.5. If $\phi : E \to F$ is a homomorphism, then the function $x \mapsto \dim(\ker(\phi))$ needs not be locally constant. An example is given by $X = [0, 1]$ the unit interval and $E = F = X \times \mathbb{K}$ as well as $\phi(x, v) = (x, xv)$.

Definition 1.3.6. A homomorphism $\phi : E \to F$ is called a **strict homomorphism**, if $x \mapsto \dim\big(\ker(\phi_x)\big)$ is locally constant.

Lemma 1.3.7. *If $\phi : E \to F$ is strict, then*

(a) $\ker(\phi)$ *is a sub-bundle of E,*

(b) $\mathrm{im}(\phi)$ *is a sub-bundle of F,*

(c) $\mathrm{coker}(\phi)$ *is a bundle.*

Proof. (b) implies (c), we show (a) and (b). The question is local, so one can assume all bundles to be trivial and $\dim(\ker(\phi))$ to be constant. Then the assertions are clear by Linear Algebra. □

Lemma 1.3.8. *Let $\phi : E \to F$ be a homomorphism. The map $x \mapsto \dim\ker(\phi_x)$ is* **lower semicontinuous**, *i.e., for every $x_0 \in X$ there exists a neighbourhood U such that*

$$\dim\ker(\phi_x) \leq \dim\ker(\phi_{x_0})$$

for every $x \in U$.

Proof. The assertion can be formulated by saying that for every $k \in \mathbb{N}_0$ the set

$$\Big\{x \in X : \dim\ker(\phi_x) \geq k\Big\} = \Big\{x \in X : \mathrm{Rang}(\phi_x) \leq (\dim E_x - k)\Big\}$$

is closed. Since the question is local, one can assume all bundles trivial. Then the assertion is equivalent to saying that for every $k \in \mathbb{N}_0$ the set

$$A_k = \{x \in X : \mathrm{Rang}(\phi_x) \leq k\}$$

is closed. We view $x \mapsto \phi_x$ as a continuous map $X \to \mathrm{Lin}(V, W) \cong \mathrm{M}_{n\times m}(\mathbb{K})$. Note that for a matrix A, the rank equals the maximal dimension of a square submatrix of determinant $\neq 0$ ist. Hence we have $\mathrm{Rang}(A) \le k$ iff all $(k+1)\times(k+1)$ quadratic submatrices B satisfy $\det(B) = 0$. Let $D : \mathrm{M}_{n\times m} \to [0, \infty)$,

$$D(A) = \sum_B |\det(B)|,$$

where the sum runs over all $(k+1)\times(k+1)$ submatrices of A. The map D is continuous and A_k is the zero set of $x \mapsto D(\phi_x)$, hence A_k is closed. □

Definition 1.3.9. Let $E \to X$ be a vector bundle. A **projection** on E is a homomorphism $P : E \to E$ such that $P^2 = P$.

Remark 1.3.10. Note that for a projection P one has $\dim\ker(P_x) + \dim\ker(1 - P_x) = \dim E_x$, and the latter is locally constant. It is easy to see that for two lower semicontinuous functions $f, g : X \to \mathbb{N}_0$, such that $f + g$ is locally constant, both functions f, g have to be locally constant.

Therefore $\dim\ker(P_x)$ is locally constant, every projection is strict and we have the direct sum decomposition

$$E = PE \oplus (1 - P)E.$$

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1.4 Paracompact Hausdorff spaces

Definition 1.4.1. (a) A covering $X = \bigcup_{i\in I} U_i$ is called **locally-finite**, if every $x \in X$ has an open neighbourhood U, with meets only finitely many of the U_i.

(b) A **refinement** of a covering $(U_i)_{i\in I}$ is a covering $(V_j)_{j\in J}$ such that for every $j \in J$ there is an $i \in I$ with $V_j \subset U_i$.

(c) A topological space X is called **paracompact**, if every open covering admits a locally-finite refinement.

Examples 1.4.2. (a) Compact spaces are paracompact.

(b) Metric spaces are paracompact.

(c) Locally compact Hausdorff spaces are paracompact.

(d) CW-complexes are paracompact.

(For proofs see for instance Stephen Willard: General Topology. Addison-Wesley 1970)

Theorem 1.4.3 (Tietze's Extension Theorem)**.** *Let X be a paracompact Hausdorff space, $A \subset X$ a closed subset and $f : A \to \mathbb{K}$ a continuous function. Then there exists a continuous extension of f to all of X.*

Proof. Willard: General Topology. □

Theorem 1.4.4 (Partition of unit)**.** *Let X be a paracompact Hausdorff-space and let $(U_i)_{i\in I}$ be an open covering of X. Then there are continuous functions $u_i : X \to [0,1]$, such that the support of u_i is contained in U_i and*

$$\sum_{i\in I} u_i \equiv 1$$

on all of X, where the sum is locally finite, i.e., for every $p \in X$ there is an open neighbourhood U, such that the set

$$\left\{ i \in I : u_i|_U \neq 0 \right\}$$

is finite. It follows, that for every compact subset $K \subset X$ the set

$$\left\{ i \in I : u_i|_K \neq 0 \right\}$$

is finite. The family (u_i) is called a partition of unity *attached to the covering (U_i).*

Proof. Stephen Willard: General Topology. Addison-Wesley 1970. □

From now on all spaces are supposed to be paracompact Hausdorff spaces.

Lemma 1.4.5. *Let $E \to X$ be a vector bundle over a paracompact Hausdorff space. Then every section $s : A \to E_A$ on a closed subset $A \subset X$ can be extended to a section t on all of X.*

Proof. Let $(U_\omega)_{\omega\in\Omega}$ be a locally finite open covering of X, such that E is trivialisable on every U_ω. Let (u_ω) be a partition of unity for this covering. Then the $\mathbb{K}^n$-valued function $s|_{A\cap U_\omega}$ can be extended by Tietze's Theorem to a section t_ω on U_ω. The section $u_\omega t_\omega$ can be extended (by zero) to a section on X. The section

$$t = \sum_{\omega\in\Omega} u_\omega t_\omega$$

satisfies the assertion of the lemma. □

Lemma 1.4.6. *Let E, F be vector bundles over a paracompacten Hausdorff space X and let $A \subset X$ be a closed subset. Let $\phi : E_A \to F_A$ be an isomorphism. Then there exists an open neighbourhood U of A and an extension of ϕ to an isomorphism $\phi_U : E_U \to F_U$.*

Proof. The map ϕ is a section of the bundle $\mathrm{Hom}(E, F)_A$, so it can be extended to a section of $\mathrm{Hom}(E, F)$. Let U be the set of all $x \in X$, such that $\phi(x)$ is an isomorphism. Then U is open and contain A. □

Lemma 1.4.7. *Let X be paracompact and let (U_ω) be an open covering. Then there exists a countable open covering (V_k), such that every V_k is a disjoint union of open sets, which each lie in some of the U_ω.*

Proof. Let $(u_\omega)_{\omega\in\Omega}$ be a partition of unity for this covering. For every finite subset $E \subset \Omega$ let V_E be the set of all $x \in X$ such that $u_\omega(x) > u_\alpha(x)$ for every $\omega \in E$ and every $\alpha \notin E$. Since for every $x \in X$ only finitely many of the u_ω are non-zero, there is a neighbourhood of x, in which V_E is defined by finitely many inequalities, so V_E is open. Further, V_E lies in U_ω for every $\omega \in E$. Let V_k denote the union of all open subsets V_E, such that E has exactly k elements. Then V_k is a disjoint union of these V_E. The family $\left(V_k\right)_{k\in\mathbb{N}}$ is an open covering. □

Theorem 1.4.8. (a) *Let X be a paracompact Hausdorff space and let $E \to I \times X$ be a vector bundle. Then the restrictions of E to $X \cong \{0\} \times X$ and $\{1\} \times X$ are isomorphic.*

(b) *Let Y be a paracompact Hausdorff space, $f_0, f_1 : Y \to X$ two freely homotopic maps and E a vector bundle over X. Then*

$$f_0^* E \cong f_1^* E.$$

Proof. We first make two remarks:

(1) A vector bundle $p : E \to [a, b] \times X$ is trivial, if the restrictions on $[a, c] \times X$ and $[c, b] \times X$ are trivial for some $c \in [a, b]$.
To see this, let E_1 and E_2 denote these restrictions. Let $\phi_1 : E_1 \to [a, c] \times X \times \mathbb{K}^n$ and $\phi_2 : E_2 \to [c, b] \times X \times \mathbb{K}^n$ be isomorphisms. Replace ϕ_2 be the composition with the isomorphism $[c, b] \times X \times \mathbb{K}^n \to [c, b] \times X \times \mathbb{K}^n$, which on $\{t\} \times X \times \mathbb{K}^n$ is given by

$$\phi_1 \phi_2^{-1} : \{t\} \times X \times \mathbb{K}^n \cong \{c\} \times X \times \mathbb{K}^n \to \{c\} \times X \times \mathbb{K}^n \cong \{t\} \times X \times \mathbb{K}^n.$$

Then ϕ_1 and ϕ_2 agree on $E_1 \cap E_2$, so they define a trivialisation of E.

(2) For a vector bundle $p : E \to I \times X$ there is an covering $(U_\omega)_{\omega \in \Omega}$ of X, such that every restriction on $I \times U_\omega$ is trivial.
Since I is compact ist, for every $x \in X$ there are open neighbourhoods $U_{x,1}, \ldots, U_{x,k}$ and $0 = t_0 < t_1 < \cdots < t_k = 1$ such that the bundle is trivial over $[t_{j-1}, t_j] \times U_{x,j}$. By remark (1) the bundle is trivial on $I \times U_\omega$, where $U_\omega = U_{x,1} \cap \cdots \cap U_{x,k}$.

Now for the proof of (a). Let $(U_\omega)_{\omega \in \Omega}$ be an open covering, such that E is trivial on $I \times U_\omega$. Assume first, that X is compact. Then finitely many U_ω will do. Write these as $U_1, \ldots U_m$. Let (u_j) be a compatible partition of unity and set $\psi_j = u_1 + \cdots + u_j$. Let $X_j \subset I \times X$ be the graph of ψ_j and let $p_j : E_j \to X_j$ be the restriction of E. Since E is trivial on $I \times U_i$, the projection $X_j \to X_{j-1}$ lifts to a homeomorphism $h_j : E_j \to E_{j-1}$, which is the identity outside of $p^{-1}(I \times U_j)$ and which maps every fibre of E_j isomorphic to the corresponding fibre of E_{j-1}. Explicitely, one can put $h_j(\psi_j(x), x, v) = (\psi_{j-1}(x), x, v)$. The composition $h = h_1 h_2 \cdots h_m$ then is an isomorphism of $E_{\{1\} \times X}$ to $E_{\{0\} \times X}$.

In the case of X only being paracompact, by Lemma 1.4.7 there is a countable open covering $(V_k)_{k \in \mathbb{N}}$, such that every V_k is a disjoint union of open sets, which each lie in one of the U_ω. Then E is trivial on every $I \times V_j$. Let $(u_k)_{k \in \mathbb{N}}$ be a compatible partition of unity and let $\psi_j = u_1 + \cdots + u_j$ as well as $p_j : E_j \to X_j$ be the restriction of E to den graph X_j of ψ_j. As before let $h_j : E_j \to E_{j-1}$, which is possible, as E is trivial on $I \times V_j$.

The composition $h_1 h_2 \cdots$ is a well-defined isomorphism, since in the neighbourhood of every point only finitely many of the u_k are non-zero.

(b) Let $h : I \times X \to Y$ be a homotopy of f_0 to f_1. Then by part (a) the bundles $f_0^* E = h^* E|_{\{0\}\times X}$ and $f_1^* E = h^* E|_{\{1\}\times X}$ are isomorphic. □

Definition 1.4.9. Let Vect(X) the set of all isomorphy classes of vector bundles over X. This is an abelian monoid with the direct sum $\oplus$.

Here a **monoid** is a set A with a composition $A \times A \to A$, $(a, b) \mapsto ab$, which is associative: $(ab)c = a(bc)$ and which has a neutral element $e \in A$ with $ae = ea = a$ for every $a \in A$. The neutral element ist uniquely determined, as for a second e' one has $e' = e'e = e$. The monoid is called **abelian**, if one has $ab = ba$. Abelian monoids cn be written additively.

Definition 1.4.10. A continuous map $f : X \to Y$ is called **homotopy equivalence**, if there is a continuous map $g : Y \to X$, such that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X.

Lemma 1.4.11. *Let X and Y be paracompact Hausdorff spaces.*

(a) *If $f : X \to Y$ is a homotopy equivalence, then $f^* : \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$ is an isomorphism.*

(b) *If X is contractible, then every bundle over X is trivialisable and* $\mathrm{Vect}(X) \cong (\mathbb{N}_0, +)$.

Proof. Clear. □

Corollary 1.4.12. *A real line bundle over S^1 is either trivial or isomorphic to the Moebius strip.*

Proof. Let $\pi : L \to S^2 = \mathbb{R}/\mathbb{Z}$ be a line bundle and let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection. Then $\tilde{L} = p^* L$ is trivialisable. Now $(\mathbb{Z}, +)$ acts on $\tilde{L}$ by bundle homomorphisms and $L = \tilde{L}/\mathbb{Z}$.Replace $\tilde{L}$ by its trivialisation $\mathbb{R} \times \mathbb{R}$ and so $\mathbb{Z}$ acts on the bundle $\mathbb{R} \times \mathbb{R}$ by bundle homomorphisms extending the translation action on $\mathbb{R}$. Hence $k.(x, v) = (x + k, \alpha^k x)$ for some $\alpha \in \mathbb{R}^\times$. The map $(x, v) \mapsto (x, |\alpha|^x v)$ is a bundle isomorphism which translates this action to the same action with $|\alpha| = 1$, i.e., $\alpha = \pm 1$.If $\alpha = 1$, the bundle L is trivial. If $\alpha = -1$, it is the Moebius strip. □

Definition 1.4.13. A **space pair** is a pair (X, A), consisting of a paracompact Hausdorff space X and a closed subset $A \neq \emptyset$. One gets a **quotient** X/A by shrinking A to a point.

More precisely, X/A is the quotient space $X/\sim$, where $\sim$ is the equivalence

$$x \sim y \quad \Leftrightarrow \quad \begin{cases} x = y \text{ or} \\ x, y \in A. \end{cases}$$

A **pointed space** is a space pair of the form $(X, x_0) = (X, \{x_0\})$. Note that $(X/A, A)$ is a pointed space.

A **map of space pairs** $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ with $f(A) \subset B$.

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1.5 Gluing of vector bundles

Definition 1.5.1. Let X be a locally compact Hausdorff space and let A_1, A_2 be closed subsets with $X = A_1 \cup A_2$. Further let $A = A_1 \cap A_2$ and for $i = 1, 2$ we have a vector bundle E_i over A_i with an isomorphism $\phi : E_1|_A \xrightarrow{\cong} E_2|_A$. One can cook up a vector bundle E with these data, which is isomorphic to E_i on each A_i. As topological space we set $E = E_1 \cup_\phi E_2$, which is

$$E = E_1 \sqcup E_2/\sim,$$

where $v \sim w$ holds iff $v = w$ or $v = \phi(w)$ or $w = \phi(v)$. The projection $p : E \to X$ is given by the two projections of E_1 and E_2, their compatibility over A is clear, since ϕ is a bundle map and thus commutes with the projections. It remains to show that E is locally trivial. Since A is closed, this is clear outside of A. So let $a \in A$ and let U_1 be an open neighbourhood of $a \in A_1$ on which E_1 is trivial, so there is an isomorphism

$$\theta_1 : E_1|_{U_1} \xrightarrow{\cong} U_1 \times \mathbb{K}^n.$$

The restriction to A yields an isomorphism

$$\theta_1^A : E_1|_{A \cap U_1} \xrightarrow{\cong} (A \cap U_1) \times \mathbb{K}^n.$$

Precomposition with ϕ^{-1} yields

$$\theta_2^A : E_2|_{A \cap U_1} \xrightarrow{\cong} (A \cap U_1) \times \mathbb{K}^n.$$

By Lemma 1.4.6 one can extend this to an isomorphism

$$\theta_2 : E_2|_{U_2} \xrightarrow{\cong} U_2 \times \mathbb{K}^n,$$

where U_2 a neighbourhood of a. The pair (θ_1, θ_2) defines an isomorphism

$$\theta_1 \cup_\phi \theta_2 : E|_{U_1 \cup U_2} \to (U_1 \cup U_2) \times \mathbb{K}^n.$$

The local triviality is proven and this finishes the construction. We call the bundle E the **gluing** of the bundles E_1 and E_2 and we call the triple (E_1, E_2, ϕ) **gluing data**.

Lemma 1.5.2. (a) *If E is a vector bundle on X and $E_i = E|_{A_i}$, then* $\mathrm{Id} : E_1|_A \xrightarrow{\cong} E_2|_A$ *is an isomorphism and one has*

$$E \cong E_1 \cup_{\mathrm{Id}} E_2.$$

(b) *If $\beta_i : E_i \to E_i'$ are isomorphisms over A_i and if $\phi'\beta_1 = \beta_2\phi$, then*

$$E_1 \cup_\phi E_2 \cong E_1' \cup_{\phi'} E_2'.$$

(c) *Gluing is compatible with the vector bundle operations as follows: Let (E_i, ϕ) and (E_i', ϕ') be gluing data. then*

$$\begin{aligned}(E_1 \cup_\phi E_2) \oplus (E_1' \cup_{\phi'} E_2') &\cong (E_1 \oplus E_1') \cup_{\phi\oplus\phi'} (E_2 \oplus E_2'),\\ (E_1 \cup_\phi E_2) \otimes (E_1' \cup_{\phi'} E_2') &\cong (E_1 \otimes E_1') \cup_{\phi\otimes\phi'} (E_2 \otimes E_2'),\\ (E_1 \cup_\phi E_2)^* &\cong E_1^* \cup_{\phi^{-*}} E_2^*.\end{aligned}$$

(d) *The isomorphy class of $(E_1 \cup_\phi E_2)$ only depends on ϕ up to homotopy.*

Proof. (a) - (c) are clear. A homotopy of isomorphisms $\phi_t : E_1|_A \to E_2|_A$ is a continuous map $\phi : I \times E_1|_A \to E_2|_A$ such that for every $t \in I$ the map $\phi(t, .)$ is a bundle isomorphism. This induces an isomorphism

$$\Phi : \pi^* E_1|_{I\times A} \xrightarrow{\cong} \pi^* E_2|_{I\times A},$$

where $\pi : I \times X \to X$ is the projection, since one can define Φ by $\Phi(t, v, x) = (t, \phi_t(v), x)$. Conversely, every Φ comes from a homotopy ϕ_t. Let $f_t : X \to I \times X$ be defined by $f_t(x) = (t, x)$. Then

$$E_1 \cup_{\phi_t} E_2 \cong f_t^*\big(\pi^* E_1 \cup_\Phi \pi^* E_2\big).$$

Since f_0 and f_1 are homotopic, the claim follows. □

We consider the special case of gluing trivial bundles. So let $E_i = A_i \times \mathbb{K}^n$. Then an isomorphism $\phi : E_1|_A \to E_2|_A$ is given by a continuous map $\phi : A \to \mathrm{GL}_n(\mathbb{K})$.

Example 1.5.3. Let H be the tautological complex line bundle over $X = \mathbb{P}^1(\mathbb{C}) \cong S^2$. The space X is the union of an upper and a lower hemisphere, $X = X_1 \cup X_2$, where the intersection A is S^1. In homogeneous coordinates X_1 is the set of all $[z,1]$ with $|z| \le 1$ and X_2 is the set of all $[1,z]$ with $|z| \le 1$. The tautological bundle can be described in these coordinates by $H_{[a,b]} = \mathbb{C}\begin{pmatrix} a \\ b \end{pmatrix}$. So for $|z| = 1$, $H_{[z,1]} = \mathbb{C}\begin{pmatrix} z \\ 1 \end{pmatrix}$. We trivialise this bundle on X_1 by the section $s_1([z,1]) = \begin{pmatrix} z \\ 1 \end{pmatrix}$ and on X_2 by the section $s_2([1,z]) = \begin{pmatrix} 1 \\ z \end{pmatrix}$. On $A = X_1 \cap X_2 = \{[z,1] : |z| = 1\}$ we transform $s_1([z,1])$ into $s_2([z,1]) = s_2([1,\bar{z}])$ by $\phi([z,1]) = \bar{z}$, or $\phi([1,z]) = z$.

Definition 1.5.4. We write $[X,Y]$ for the of homotopy classes of continuous maps $X \to Y$.

Examples 1.5.5. (a) $[\{0\},X] \cong \pi_0(X)$ is the set of path-connected components of X.

(b) $[S^1,S^1] \cong \mathbb{Z}$, since every homotopy class of continuous maps $S^1 \to S^1$ is determined by its winding number.

Definition 1.5.6. For $n \in \mathbb{N}_0$ let $\mathrm{Vect}_{n,\mathbb{K}}(X)$ be the set of isomorphy classes of $\mathbb{K}$-vector bundles of rank n.

Proposition 1.5.7. (a) $\mathrm{GL}_n(\mathbb{C})$ *is path-connected, while* $\mathrm{GL}_n(\mathbb{R})$ *is not.*

(b) *Let $X = X_1 \cup X_2$, $A = X_1 \cap X_2$ with closed X_1 and X_2. Assume that X_1 and X_2 are contractible, then there is a canonical bijection*

$$\mathrm{Vect}_{n,\mathbb{C}}(X) \xrightarrow{\cong} [A, \mathrm{GL}_n(\mathbb{C})].$$

This assertion only holds over the field $\mathbb{C}$.

Proof. (a) Let $A \in \mathrm{GL}_n(\mathbb{C})$. Define the polynomial $P(x) = \det(A + x(A - I))$. Then $P(0) = \det A$ and $P(1) = 1$. Let $\gamma : [0,1] \to \mathbb{C}$ be a path joining 0 to 1 which avoids the zeros of $P(x)$ and let $\Gamma(t) = A + \gamma(t)(I - A)$ then $\Gamma(t)$ joins A to I inside $\mathrm{GL}_n(\mathbb{C})$.

The set $\mathrm{GL}_n(\mathbb{R})$ is not connected, because if it where, the set $\det\big(\mathrm{GL}_n(\mathbb{R})\big) = \mathbb{R}^\times$ would be connected.

(b) Every vector bundle E is the gluing of its restrictions to X_1 and X_2, which both are trivialisable. The only thing we have to show is, that if two bundles are isomorphic, then their gluing maps are homotopic. This is done as follows: Let the bundles $E = (X_1 \times \mathbb{C}^n) \cup_\phi (X_2 \times \mathbb{C}^n)$ and $F = (X_1 \times \mathbb{C}^n) \cup_\psi (X_2 \times \mathbb{C}^n)$ be isomorphic. We have to show that ϕ and ψ are homotopic. For this let $T : E \to F$ be an isomorphism, Then $T|_{X_1} : X_1 \times \mathbb{C}^n \to X_1 \times \mathbb{C}^n$ is given by a continuous map $T_1 : X_1 \to \mathrm{GL}_n(\mathbb{C})$. Likewise, $T|_{X_2}$ is given by a continuous map $T_2 : X_2 \to \mathrm{GL}_n(\mathbb{C})$. The compatibility with the gluings yields for every $a \in A$, that $\psi(a)T_1(a) = T_2(a)\phi(a)$, or

$$\psi(a) = T_2(a)\phi(a)T_1(a)^{-1}.$$

Since X_1 is contractible, there is a continuous map $h : I \times X_1 \to X_1$ with $h(0, x) = x_0$ and $h(1, x) = x$. Set $T_{1,t}(x) = T_1(h(t, x))$. Combining this with a continuous path in $\mathrm{GL}_n(\mathbb{C})$, which joins $T_1(x_0)$ with the unit matrix, one gets a homotopy $T_{1,t} : A \to \mathrm{GL}_n(\mathbb{C})$ with $T_{1,1}(a) = T_1(a)$ and $T_{1,0}(a) = 1$. Doing the same for T_2 yields a homotopy $T_{2,t} : A \to \mathrm{GL}_n(\mathbb{C})$ with $T_{2,1}(a) = T_2(a)$ and $T_{2,0}(a) = 1$. Therefore the homotopy $H_t : A \to \mathrm{GL}_n(\mathbb{C})$:

$$H_t(a) = T_{2,t}(a)\phi(a)T_{1,t}(a)^{-1},$$

satisfies $H_0(a) = \phi(a)$ and $H_1(a) = \psi(a)$, so it is a homotopy between ϕ and ψ. □

Remark 1.5.8. Over the reals, the situation is more complicated, since $\mathrm{GL}_n(\mathbb{R})$ is not connected. In this case, homotopic maps make isomorphic bundles, but if the bundles E_ϕ and E_ψ are isomorphic, one can only conclude that ϕ is homotopic to one of the following: ψ, $S\psi$, ψT or $S\psi T$ for any given matrices S, T of negative determinant.

* * *

1.6 Metrics

Definition 1.6.1. A **hermitean metric** on a vector bundle E is a family $(\langle ., . \rangle_x)_{x \in X}$, where $\langle ., . \rangle_x$ is an inner product E_x, such that for any two continuous sections $s, t \in \Gamma(E)$ the function $x \mapsto \langle s(x), t(x) \rangle_x$ is continuous.

Lemma 1.6.2. *For every vector bundle on a paracompact Hausdorff space X there exists a hermitean metric.*

Proof. On every trivial bundle there is a hermitean metric. So let (U_i) be a trivialising covering and h_i a hermitean metric on $E|_{U_i}$. Let (u_i) be a partition of unity to the covering (U_i). Then

$$h = \sum_{i \in I} h_i u_i$$

is a hermitean metric on E. $\square$

Definition 1.6.3. A sequence $\cdots \to E^{i-1} \to E^i \to E^{i+1} \dots$ is called **exact**, if for every $x \in X$ the sequence of vector spaces $\cdots \to E_x^{i-1} \to E_x^i \to E_x^{i+1} \dots$ is exact.

Theorem 1.6.4. *Let X be a paracompact Hausdorff space. Then every short exact sequence of vector bundles splits over X.*

This means that, for every exact sequence $0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} G \to 0$ there is a homomorphism $s : G \to F$, such that $\beta \circ s = \mathrm{Id}_G$. In this case it follows
$F = \alpha(E) \oplus s(G) \cong E \oplus G$.

$$0 \longrightarrow E \longrightarrow F \underset{s}{\overset{\beta}{\rightleftarrows}} G \longrightarrow 0$$

Proof. Equip F with a hermitean metric and write $E_x^\perp$ for the orthogonal space of $\alpha(E_x)$ in F_x. Then $E^\perp$ is a sub-bundle of F and β induces an isomorphism $E^\perp \to G$. The map $s = (\beta|_{E^\perp})^{-1}$ satisfies the claim. For the direct sum decomposition note that $\beta s = \mathrm{Id}$ implies that s is injective, so the image $s(G)$ is a bundle isomorphic to G. Since $\alpha(E)$ is the kernel of β, the sum $\alpha(E) + \sigma(G)$ is direct. By dimension reasons, is equals the whole of F. $\square$

Definition 1.6.5. A subspace $V \subset \Gamma(E)$ is called **ample**, if the map

$$X \times V \to E,$$
$$(x, s) \mapsto s(x)$$

is surjective. This means that at every $x \in X$ the set $\{s(x) : s \in V\}$ spans the vector space E_x.

Lemma 1.6.6. (a) *A bundle E admits a finite-dimensional ample space $V \subset \Gamma(E)$ iff there exists a surjective bundle homomorphism $T \to E$, where T is a trivial bundle.*

(b) *For a bundle of rank n, every ample space has dimension $\geq n$.*

(c) *A bundle E of rank n is trivial iff there exists an ample space $V \subset \Gamma(E)$ with* $\dim V = n$.

(d) *If X is a compact Hausdorff space then every vector bundle E admits a finite dimensional ample space $V \subset \Gamma(E)$.*

Proof. (a) Let V be ample and let $s_1, \dots, s_n$ be a basis of V. Let $T = X \times \mathbb{K}^n$. Then the homomorphism $\phi : T \to E$, $\phi\left(\sum_{j=1}^n c_j e_j\right) = \sum_{j=1}^n c_j s_j$ is the asked for homomorphism. Conversely, let $\phi : X \times \mathbb{K}^n \to E$ be surjective, then the set of all maps $s : X \to E$, $x \mapsto \phi(x, v)$ for $v \in \mathbb{K}^n$ is an ample space.

(b) Let $V \subset \Gamma(E)$ be ample, then for each $x \in X$ the map $s \mapsto s(x)$ is a linear surjection onto an n-dimensional space.

(c) Let $E = X \times \mathbb{K}^n$. Let $s_j(x) = (x, e_j)$, where e_j is the j-th standard basis vector. Then the span of $s_1, \dots, s_n$ is an ample space of dimension n.

(cd) By (b) the assertion follows for a trivial bundle. Let $(U_i)_{i \in I}$ be a finite trivialising covering and let $V_i \subset \Gamma(E|_{U_i})$ be a finite-dimensional ample space. Let (u_i) be a partition of unity to (U_i). Then

$$V = \sum_{i \in I} u_i V_i$$

is a finite-dimensional ample space. □

Definition 1.6.7. Recall that two spaces X, Y are **homotopy-equivalent**, if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that fg and gf are homotopic to the identity maps on X and Y.

Theorem 1.6.8. *Let X be a topological space, which is homotopy equivalent to a compact Hausdorff space. For any given vector bundle E there exists a vector bundle F, such that $E \oplus F$ is trivial.*

Proof. By Lemma 1.4.11 we can assume that X is a compact Hausdorff space. Let E be a vector bundle let $V \subset \Gamma(E)$ be a finite-dimensional ample space. Then there is a surjective homomorphism from the trivial bundle $X \times V$ to E. Let F be the kernel. We have a short exact sequence

$$0 \to F \to X \times V \to E \to 0.$$

By Theorem 1.6.4 the bundle $E \oplus F$ is trivial. □

* * *

1.7 Projective C(X)-modules

Let R be a ring with unit, not necessarily commutative. A module P of R is called **projective**, if for every surjective morphism $M \twoheadrightarrow N$, every morphism $\phi : P \to N$ can be lifted to M, which means that there exists a ψ making the diagram

$$\begin{array}{ccc} & & M \\ & \overset{\psi}{\nearrow} & \downarrow\!\!\downarrow \\ P & \longrightarrow & N \end{array}$$

commutative.

Lemma 1.7.1. (a) *Free modules are projective.*

(b) *Let R be a ring. A module P is projective iff every short exact sequence $0 \to A \to B \to P \to 0$ splits.*

(c) *A module P is projective iff there is a module Q, such that $P \oplus Q$ is free. The module Q then is projective, too. If P is finitely generated and projective, then Q can be chosen in a way that $P \oplus Q \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$. The module Q then is finitely generated, too.*

Proof. (a) Let $F = \bigoplus_{i \in I} R\alpha_i$ be free and $M \twoheadrightarrow N$ surjective and $\phi : F \to N$ a homomorphism. For every $i \in I$ choose a preimage m_i in M of $\phi(\alpha_i)$. Then there is exactly one morphism $\psi : F \to M$ with $\psi(\alpha_i) = m_i$ for every i. This is a lift to ϕ.

(b) Let P be projective and let $0 \to A \to B \xrightarrow{\eta} P \to 0$ be exact, then there exists a lift of the identity map $P \to P$ to a map $s : P \to B$ such that $\eta \circ s = \mathrm{Id}_P$, i.e., the sequence splits.

The other way round assume that P makes every such sequence split and assume one has a diagram

$$\begin{array}{ccc} & & M \\ & & \downarrow\!\!\downarrow \alpha \\ P & \xrightarrow[\beta]{} & N \end{array}$$

Let $M \times_N P = \left\{(p, m) \in P \times M : \alpha(p) = \beta(m)\right\}$. Then the solid arrow diagram

$$\begin{array}{ccc} M \times_N P & \xrightarrow{p} & M \\ {\scriptstyle q}\downarrow \;\; \nwarrow {\scriptstyle s} & & \downarrow\!\!\downarrow {\scriptstyle \alpha} \\ P & \xrightarrow[\beta]{} & N \end{array}$$

commutes, where p and q are the coordinate projections. As α is surjective, the definition of $M \times_N M$ implies that q is surjective. By the property of P, there exists $s : P \to M \times_N P$ with $qs = \mathrm{Id}_P$. We claim that ps lifts β. For this we compute $\alpha ps = \beta qs = \beta$, as claimed.

(c) Let P be projective. Choosing generators of P, one gets a surjective morphism

$$\phi : \bigoplus_{i \in I} R\alpha_i = F \twoheadrightarrow P$$

form a free module F onto P. By part (b) there is a module homomorphism $s : P \to F$ with $\phi s = \mathrm{Id}$. Let Q be the kernel of ϕ, then the map $Q \oplus P \to F$, $(q, p) \mapsto q + s(p)$ is an isomorphism. If P is finitely generated, one can choose F as finite-free, i.e., isomorphic to R^n. The module Q is the image of the projection $R^n = P \oplus Q \to Q$ and therefore is finitely generated.

Conversely, let P and Q be modules and $\alpha : P \oplus Q \to F$ an isomorphism, where F is free. Let i be the inclusion map $P \hookrightarrow P \oplus Q$. Consider morphisms $M \twoheadrightarrow^{\eta} N$ and $\phi : P \to N$. Let $\tilde{\phi} : F \to N$ be the projection ont P followed by ϕ. Since free modules are projective, there is a lift $\tilde{\psi} : F \to M$ of $\tilde{\phi}$. Then $\psi = \tilde{\psi} \circ \alpha \circ i$ is a lift of $P \to M$. □

Definition 1.7.2. For two topological spaces X, Y let $C(X, Y)$ denote the set of all continuous maps $f : X \to Y$. Then $C(X, \mathbb{K})$ is a $\mathbb{K}$-algebra by the pointwise addition and multiplication.

Theorem 1.7.3. *Let X be homotopy equivalent to a compact Hausdorff space. Then the sections functor Γ yields an equivalence of categories between the category of $\mathbb{K}$-vector bundles over X and the category of the finitely generated projective $C(X, \mathbb{K})$-modules.*

Proof. If E is a trivial bundle, then $\Gamma(E)$ is a finitely generated free $C(X, \mathbb{K})$-module. If E is arbitrary, there exists by Theorem 1.6.8 a $C(X, \mathbb{K})$-module M, such that $\Gamma(E) \oplus M$ is finitely generated and free. Hence $\Gamma(E)$ is finitely generated projective.

Conversely, let M be a finitely generated projective $C(X, \mathbb{K})$-module. Then there exists a module N, such that $M \oplus N$ is finitely generated free, say $M \oplus N \cong C(X, \mathbb{K})^n \cong \Gamma(X \times \mathbb{K}^n)$. This means that we view M as a submodule of $\Gamma(X \times \mathbb{K}^n)$. Let $E \subset X \times \mathbb{K}^n$ be the union of all images $s(X)$ for $s \in M$ and let F be the same for N. We claim that E and F are vector bundles over X. For this let $x_0 \in X$ and $k = \dim E_{x_0}$. Choose sections $s_1, \dots, s_k$ in M such that $s_1(x_0), \dots, s_k(x_0)$ is a basis of the space E_x. Becaus $M \oplus N = \Gamma(X \times \mathbb{K}^n)$ we have $\dim(F_{x_0}) = n - k$ and there are sections $s_{k+1}, \dots, s_n$ in N, such that $s_{k+1}(x_0), \dots, s_n(x_0)$ is a basis of F_{x_0}. There is an open neighbourhood U of x_0 such that for all $x \in U$ the vectors $s_1(x), \dots, s_n(x)$ are linearly independent. These give the desired local trivialisation. □

* * *

2 Topological K-theory

2.1 Definitions

Definition 2.1.1. An **abelian monoid** is a set M with a composition $\oplus : M \times M \to M$, $(m, n) \mapsto m \oplus n$, which is associative and commutative and possesses a neutral element 0. We shall write abelian monoids additively.

Examples 2.1.2.

- An abelian group is an abelian monoid.
- $(\mathbb{N}_0, +)$ is an abelian monoid.

Proposition 2.1.3. (a) *Let $(A, \oplus)$ be an abelian monoid. Then there exists an abelian group $K(A)$ and a monoid morphism $\alpha : A \to K(A)$, such that for every group G and every monoid morphism $f : A \to G$ there is exactly one group homomorphism $K(A) \to G$, making the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & K(A) \\ & {\scriptstyle f}\searrow & \vdots\,{\scriptstyle \exists!} \\ & & G \end{array}$$

commutative. The group $K(A)$ and the homomorphism α are uniquely determined up to isomorphy.

(b) *Every element of the group $K(A)$ can be written in the form $a - b$ for two elements $a, b \in A$.*

(c) *For two elements $a, b \in A$ one has $\alpha(a) = \alpha(b)$ if and only if there is $c \in A$, such that $a \oplus c = b \oplus c$ holds in A.*

Definition 2.1.4. The group $K(A)$ is called the **Quotient group** of the monoid.

Proof. (a) Let $F(A)$ be the free abelian group generated by the elements of A and let N be the subgroup generated by all element of the form $a + a' - (a \oplus a')$, where $\oplus$ is the Addition in A. Set $K(A) = F(A)/N$, then the universal property is clear.

(b) Let $\alpha = \sum_{a \in A} k_a a \in F(A)$ with $k_a \in \mathbb{Z}$. Further let $\alpha_+ = \sum_{k_a > 0} k_a a$ and $\alpha_- = \sum_{k_a < 0} (-k_a) a$. Then one has $\alpha = \alpha_+ - \alpha_-$, so it suffices to show that an element with positive coefficients in $K(A)$ coincides with the image of an element in A. In $K(A)$ one has $ka = a + \cdots + a = a \oplus \cdots \oplus a \in A$ and $a + b = a \oplus b \in A$ for $a, b \in A$, which implies the claim.

To show (c), we need another construction of $K(A)$. On $A \times A$ consider the equivalence relation $\sim$ given by

$$(a,b) \sim (c,d) \quad \Leftrightarrow \quad \exists_{x \in A} : a \oplus d \oplus x = c \oplus b \oplus x.$$

Let $K'(A) = A \times A/ \sim$. We show that the addition $[a,b] + [c,d] = [a \oplus c, b \oplus d]$ is well-defined and turns $K'(A)$ into a group. So let $[a,b] = [a',b']$ and $[c,d] = [c',d']$, say $a + b' + x = a' + b + x$ and $c + d' + y = c' + d + y$. We want to show $[a + c, b + d] = [a' + c', b' + d']$. This follows from $a + c + b' + d' + x + y = a' + c' + b + d + x + y$. The group axioms are easily verified, the neutral element is $[0,0]$ and the inverse of $[a,b]$ is $[b,a]$. Let $\eta : A \to K'(A)$ be given by $a \mapsto (a,0)$. This is a monoid morphism. If $\phi : A \to G$ is a homomorphism to a group G, then see $\phi' : K'(A) \to G$, $\tilde{\phi}(a,b) = \phi(a) - \phi(b)$. This is the unique group homomorphism with the property, that it extends ϕ fortsetzt. It follows, that the map $K'(A) \to K(A)$, $(a,b) \mapsto a - b$ is an isomorphism, so (c) follows. □

Definition 2.1.5. For a topological space X the set $\mathrm{Vect}_{\mathbb{K}}(X)$ of isomorphy classes of $\mathbb{K}$ vector bundles over X is an abelian monoid under the direct sum. We write

$$K_{\mathbb{K}}(X) = K\Big(\mathrm{Vect}_{\mathbb{K}}(X)\Big)$$

and call $K_{\mathbb{K}}(X)$ the real or complex **K-group** of X. As long as we don't specify $\mathbb{K}$, we simply write $K(X)$.

For a bundle E, we write $[E]$ for the class of E in $K(X)$. Every element of $K(X)$ can be written in the form $[E] - [F]$ for vector bundles E and F. We write $\underline{n}$ for the trivial bundle of rank n, i.e.,

$$\underline{n} = X \times \mathbb{K}^n.$$

Remark 2.1.6. Suppose that X has the homotopy type of a compact Hausdorff space, then for a given bundle F there exists a bundle G with $F \oplus G = \underline{n}$ and so one has

$$[E] - [F] = [E \oplus G] - [F \oplus G] = [E \oplus G] - \underline{n}.$$

This implies that every element of $K(X)$ can be written in the form $[E] - \underline{n}$.

Definition 2.1.7. Two bundles E and F are called **stably equivalent**, if there is a bundle G, such that $E \oplus G \cong F \oplus G$.

Lemma 2.1.8. *For two bundles E, F one has $[E] = [F]$ if and only if E and F are stably equivalent.*

Proof. This assertion follows from Proposition 2.1.3 (c). □

Lemma 2.1.9. (a) *The tensor product* $[E][F] = [E \otimes_{\mathbb{K}} F]$ *turns the abelian group* $K(X)$ *into a commutative ring with unit. The unit is given by the trivial bundle* $\underline{1}$.

(b) *If* $f : Y \to X$ *is a continuous map, then* $f^* : K(X) \to K(Y)$ *is a unital ring homomorphism. In particular,* $K(Y)$ *is a* $K(X)$*-Algebra.*

Proof. Clear. □

> **Theorem 2.1.10.** *One has* $K_{\mathbb{R}}(S^1) = \mathbb{Z}[X]/I$, *where* I *is the ideal generated by* $X^2 - 1$ *and* $2X - 2$. *The class* X *is given by the Moebius strip.*
>
> *In the complex case one has* $K_{\mathbb{C}}(S^1) = \mathbb{Z}$.

Proof. Let E be a vector bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ and let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ the projection. Then p^*E is trivial, so $E \cong (\mathbb{R} \times \mathbb{R}^n)/\mathbb{Z}$, where $\mathbb{Z}$ acts by $k(x, v) = (x + k, A^k v)$ for some $A \in \mathrm{GL}_n(\mathbb{R})$. As in Example 1.1.8 one sees, that for $n \geq 2$ there exists a zero-free section $s \in \Gamma(E)$. This generates a trivial line bundle $\underline{1} \cong L \subset E$. Choose a hermitean metric and decompose $E = \underline{1} \oplus L^{\perp}$. We iterate this process to get that in $K(S^1)$ one has $E = \underline{n} + L$ for a line bundle and some $n \in \mathbb{N}_0$. If $K = \mathbb{C}$, then L is trivialisable as in Example 1.1.8, so we get the claim in the complex case.

Now for $\mathbb{K} = \mathbb{R}$. By Corollary 1.4.12, the line bundle L is either trivialisable or isomorphic to the Moebius strip. Further, $2M = M \oplus M$ is trivialisabel, as $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{R})$ can be joined by a path with the unit matrix. Hence $K_{\mathbb{R}}(S^1)$ is a Quotient of $Z[X]/I$. The elements of this ring are of the form $a + \varepsilon X$, where $a \in \mathbb{Z}$ and $\varepsilon \in \{0, 1\}$. Since X itself is not trivialisable, we conlude that $K_{\mathbb{R}}(S^1)$ equals this ring. □

> **Theorem 2.1.11.** *Let* H *be the tautological line bundle over* $S^2 = \mathbb{P}^1(\mathbb{C})$, *also called the* **Hopf bundle**. *Then*
>
> $$K_{\mathbb{C}}(S^2) \cong \mathbb{Z}[H]/(H - 1)^2.$$

Proof. In the next section. □

* * *

2.2 The Product Theorem

Definition 2.2.1. Let $E \to X$ and $F \to Y$ be vector bundles. The **exterior product** $E \boxtimes F \to X \times Y$ is defined as

$$p_1^* E \otimes p_2^* F,$$

where p_1 and p_2 are the two projection.

Then $E \boxtimes F$ is a vector bundle over $X \times Y$ with fibre $(E \boxtimes F)_{(x,y)} = E_x \otimes F_y$.

The exterior product defines a ring homomorphism

$$K(X) \otimes K(Y) \to K(X \times Y).$$

Let H be the tautological $\mathbb{C}$-bundle over $S^2 = \mathbb{P}^1(\mathbb{C})$. In Example 1.5.3 we have shown, that $(H \otimes H) \oplus 1 \cong H \oplus H$, so in the K-ring one has $H^2 + 1 = 2H$ or $(H-1)^2 = 0$. We get a ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$. Let μ the ring homomorphism

$$\mu : K_{\mathbb{C}}(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K_{\mathbb{C}}(X) \otimes K_{\mathbb{C}}(S^2) \to K_{\mathbb{C}}(X \times S^2),$$

where the second map is the exterior tensor product.

Theorem 2.2.2. *If X is a compact Hausdorf space, then μ is an isomorphism*

$$K_{\mathbb{C}}(X) \otimes \mathbb{Z}[H]/(H-1)^2 \cong K_{\mathbb{C}}(X \times S^2).$$

In particular, the map $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ is an isomorphism of rings.

The proof will take the rest of the section.

Writing S^2 as union of two hemispheres with intersection S^1 we see by Proposition 1.5.7, that the isomorphy classes of vector bundles over S^2 are given by homotopy classes of maps $S^1 \to \mathrm{GL}_n(\mathbb{C})$. Let $p : E \to X$ be any vector bundle and let $f : E \times S^1 \to E \times S^1$ be an automorphism of the vector bundle $p \times 1 : E \times S^1 \to X \times S^1$. Then for every $z \in S^1$ the map f defines an automorphism $f(x, z) : p^{-1}(x) \to p^{-1}(x)$.

We construct a bundle over $X \times S^2$, by writing S^2 as union of two hemispheres with S^1 as their intersection and by using f as gluing map. We write this bundle as $[E, f]$.

Every vector bundle is of this form and we can normalise f in a way that f is the identity over $X \times \{1\}$.

Definition 2.2.3. A **polynomial gluing map** is a gluing map $p : E \times S^1 \to E \times S^1$ as above of the special form

$$p(x,z) = \sum_{j=0}^{n} z^j p_j(x),$$

for some $n \in \mathbb{N}$ and $p_j(x) : E_x \to E_x$ endomorphisms of the vector space E_x.

Lemma 2.2.4. *Let X be a compact Hausdorff space. Every vector bundle $[E, f]$ over $X \times S^2$ is isomorphic to a bundle $[E, p]$ with a polynomial gliung map p. If two polynomial gluing maps p and q are homotopic in the space of all gluing maps, then they are already homotopic in the space of all polynomial gluing maps.*

Proof. We equip a given bundle E over $X \times S^1$ with a metric and get a norm on the space of all endomorphisms, and so on the set of all gluing maps:

$$\|T\| = \sup_{y \in X \times S^1} \left\| T_y \right\|_{\text{op}},$$

where $\left\| T_y \right\|_{\text{op}}$ is the operator norm. We show that the polynomial gluing maps are dense in the set of all gluing maps.

By the Theorem of Stone-Weierstrass every continuous function $X \times S^1 \to \mathbb{C}$ is a uniform limit of polynomials of the form $\sum_{j=0}^{n} z^j a_j(x)$ with continuous maps $a_j : X \to \mathbb{C}$. Let $X = \bigcup_{i=1}^{r} U_i$ be a finite open covering such that E trivialises on each closure $\overline{U_i}$. Locally we write the endomorphisms as matrices, approximate the entries by polynomials and thus approximate the given gluing map by polynomial gluing maps on each $\overline{U_i}$. Then we use a partition of unity to write every f as a uniform limit of polynomial gluing maps.

Using compactness of X, one sees that for a given gluing map f there is a polynomial gluing map p close enough to f so that $g_t = (1-t)f + tp$ is a gluing map for every $t \in [0,1]$. This means that f is homotopic to p and so the induced line bundles are isomorphic.

The second assertion follows by the same argument, as one can approximate a given homotopy by a homotopy in the space of polynomial homotopies. □

Lemma 2.2.5. *If q is a polynomial gluing map of degree $\leq n$, then $[E, q] \oplus [nE, 1] \cong [(n+1)E, L_q]$, where L_q is a gluing map of degree 1.*

Proof. Let $q(x,z) = a_0(x) + a_1(x)z + \cdots + a_n(x)z^n$. The matrices

$$A = \begin{pmatrix} 1 & -z & 0 & \dots & 0 \\ 0 & 1 & -z & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 1 & -z \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}, \quad B = \begin{pmatrix} I_n & 0 \\ 0 & q \end{pmatrix}$$

We want to instal a homotopy from A as functions from $X \times S^1$ to $\mathrm{GL}_n(\mathbb{C})$. To do this, we transform A into B by elementary row and column transformations: First we add z times the first column to the second, then the same with the second and third and so on. In this way all entries above the diagonal vanish and in the lower right we have q. Then one subtracts a suitable multiple of the j-th row from the last one to finally end up at B. Each of these transformation was given by a matrix $I + E_{i,j}(\lambda)$ with $i \neq j$ and $E_{i,j}(\lambda)$ being the matrix with λ at the i, j the position and zero elsewhere. Then $t \mapsto I + E_{i,j}(t\lambda)$ is a homotopy from the unit matrix. Hence we find that A and B are homotopic. The matrix B defines the gluing map of $[nE, 1] \oplus [E, q]$ and A defines a gluing map of degree 1, which we call L. □

Lemma 2.2.6. (a) *Let Y be a compact Hausdorff space and $p : E \to Y$ a $\mathbb{C}$-vector bundle. If $b : E \to E$ is an endomorphism, such that no eigenvalue of $b_y : E_y \to E_y$ ever lies in $\mathbb{T}$, then there is a uniquely determined, b-stable decomposition $E = E_0 \oplus E_1$ such that for every eigenvalue λ of $b|_{E_0}$ one has $|\lambda| < 1$ and every eigenvalue μ of $b|_{E_1}$ satisfies $|\mu| > 1$.*

(b) *Let the bundle $[E, a(x)z + b(x)]$ over $X \times S^1$ with a gluing map of degree 1 be given. Then there exists a decomposition $E \cong E_0 \oplus E_1$ with $[E, a(x)z + b(x)] \cong [E_0, 1] \oplus [E_1, z]$.*

Proof. (a) If V is a finite-dimensional $\mathbb{C}$ vector space and $A : V \to V$ an endomorphism without eigenvalues in $\mathbb{T}$, then we define V_0 to be the sum of all principal spaces to eigenvalues $|\lambda| < 1$ and V_1 the same for the eigenvalues $|\mu| > 1$. Then $V = V_0 \oplus V_1$ is an A-stable decomposition which is uniquely determined by A-stability and the position of eigenvalues inseide or outside of $\mathbb{T}$. In a vector bundle, one decomposes on local trivialisations $E|_{U_i}$ and gets a unique decomposition. This uniqueness guarantees that these decompositions coincide for different trivialisations and thus can be glued to a global decomposition of the bundle.

(b) First we reduce to the case $a(x) = \mathrm{Id}$. For this consider the expression

$$f_t = (1+tz)\Big(a(x)\underbrace{\frac{z+t}{1+tz}}_{\in\mathbb{T}} + b(x)\Big) = \Big(a(x)+tb(x)\Big)z + ta(x) + b(x).$$

Since $a(x)z + b(x)$ is a gluing map, the endomorphism $a(x)z + b(x)$ is invertible for every $z \in \mathbb{T}$ and hence so is f_t for every $t < 1$. In particular, $a(x) + b(x)$ is invertible and by compactness of X it follows that there is $0 < t_0 < 1$ such that $a(x) + tb(x)$ is invertible for every $t_0 \le t \le 1$.

If g is an automorphism of E, then $[E, f] \cong [E, fg]$ and therefore $[E, f_{t_0}] \cong [E, z + (t_0a(x) + b(x))(a(x) + t_0b(x))^{-1}]$, so that we can assume $a(x) = 1$, i.e., the gluing map is of the form $z + b(x)$. Now $z + b(x)$ is invertible for every $z \in \mathbb{T}$ and kence $b(x)$ ha no eigenvalues in $\mathbb{T}$.

By part (a) we have a decomposition $[E, z + b(x)] = [E_0, z + b_0(x)] \oplus [E_1, z + b_1(x)]$. Since all eigenvalues of $b_0(x)$ lie inside the unit circle, the map $t \mapsto z + tb_0(x)$ is a homotopy of gluing maps between z and $z + b_0(x)$. Since all eigenvalues of $b_1(x)$ lie outside of the unit circle, $t \mapsto tz + b_1(x)$ is a homotopy from $b_1(x)$ to $z + b_1(x)$. It follows that $[E_0, z + b_0] \cong [E_0, z]$ and $[E_1, z + b_1] \cong [E_1, b_1] \cong [E_1, 1]$. The proof of Lemma 2.2.6 is finished. □

Next we prove Theorem 2.2.2. In $K_{\mathbb{C}}(X \times S^2)$ we have

$$\begin{aligned} [E, f] &\cong [E, q] \\ &\cong [(n+1)E, L_q] - [nE, 1] \\ &\cong [((n+1)E)_0, 1] + [((n+1)E)_1, z] - [nE, 1] \\ &\cong ((n+1)E)_0 + ((n+1)E)_1 \otimes H - nE. \end{aligned}$$

This lies in the image of μ, hence μ is surjective.

One can read the last line of the previous computation as an element of $K_{\mathbb{C}}(X) \otimes \mathbb{Z}[H]/(H-1)^2$. Then the computation also yields the invers map of μ, which maps $[E, z^{-m}q]$ ot the last line of the computation. Theorem 2.2.2 is proven. □

* * *

2.3 Higher K-theory

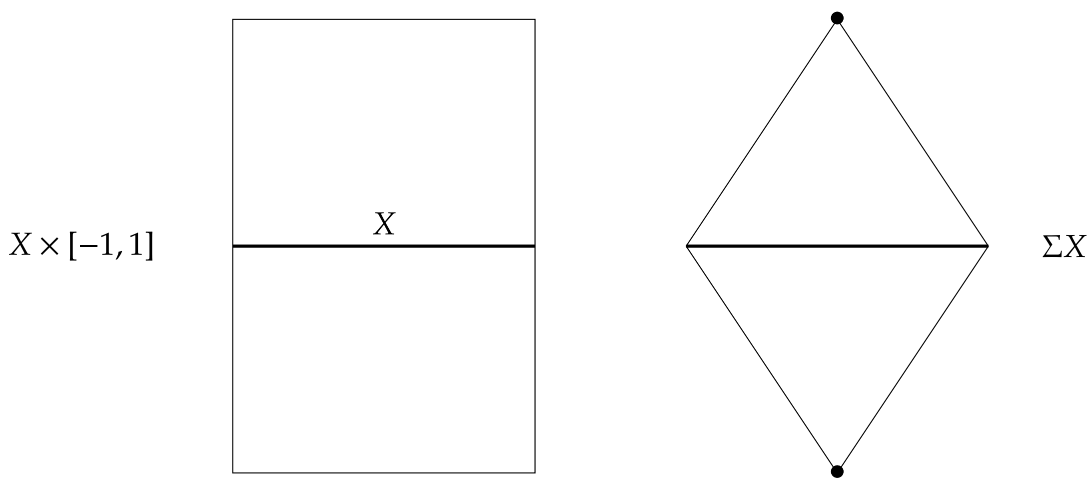
Definition 2.3.1. The **suspension** of a topological space X is defined as

$$\Sigma X = X \times [-1,1]/\sim,$$

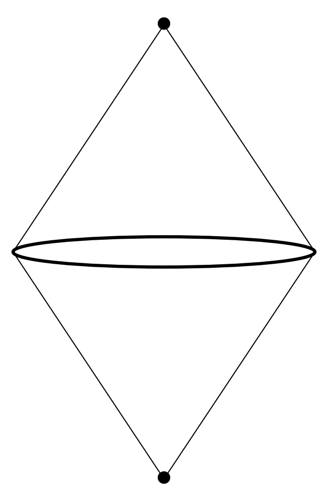
where $\sim$ the Aequivalenzrelation generated von

$$(x,-1) \sim (y,-1), \qquad (x,1) \sim (y,1)$$

for all $x, y \in X$.



Example 2.3.2. ΣS^n is homemorphic to S^{n+1}.



Definition 2.3.3. (a) There is a natural continuous map $X \to \Sigma X$, $x \mapsto \big(0, x\big)$.

(b) If $f : X \to Y$ is continuous, then there is a natural continuous map

$$\Sigma f : \Sigma X \to \Sigma Y$$

given by $\Sigma f(x,t) = (f(x),t)$. This means that $X \mapsto \Sigma X$ is a functor on the category of topological spaces.

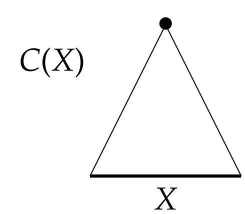
(c) Let $\Sigma^n X$ denote the n-fold iteration of the suspension functor Σ.

Definition 2.3.4. For $n \geq 0$ set

$$K^n(X) := K(\Sigma^n X).$$

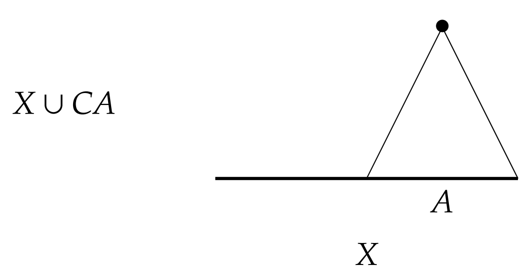
Definition 2.3.5. For a topological space X let the **cone** over X be defined as

$$C(X) := I \times X / \big(\{1\} \times X\big).$$



Let TOP be the category of topological spaces and let TOP_* be the category of pointed spaces. The cone construction is a functor $\text{TOP} \to \text{TOP}_*$. We identify X with the subspace $X \times \{0\}$. One has $C(X)/X \cong \Sigma X$.

Definition 2.3.6. Let (X, A) be a pair of spaces. We consider the union $X \cup CA$ of X and the cone of A.



We make $X \cup CA$ a pointed space by taking the special point of CA the base point. There is a natural homeomorphism

$$X \cup CA/X \cong CA/A \cong \Sigma A.$$

* * *

2.4 The long exact sequence

Definition 2.4.1. Let X be a space. Define the group

$$\tilde{K}(X) = K(X)/\mathbb{Z},$$

where $\mathbb{Z}$ stands for the trivial bundles.

Lemma 2.4.2. *Let $x_0 \in X$ be a point. Then $\mathrm{aug}_{x_0} : K(X) \to \mathbb{Z}$, $E \mapsto \dim E_{x_0}$ is a ring homomorphism. Let $J_{x_0} = \ker(\mathrm{aug}_{x_0})$ be the kernel.*

Then J_{x_0} is an ideal, hence closed under multiplication and the map

$$J_{x_0} \hookrightarrow K(X) \to \tilde{K}(X)$$

is an isomorphism of abelian groups. One identifies $\tilde{K}(X)$ with J_{x_0}.

Proof. If E is a vector bundle of rank n, then $[E] - n \in J_{x_0}$ and therefore

$$K(X) = J_{x_0} \oplus \mathbb{Z}.$$

The claim follows. □

Lemma 2.4.3. *Let (X, A) be a pair of spaces, let $i : A \to X$ be the inclusion and $q : X \to X/A$ the projection.*

(a) *Let $E \to X$ be a vector bundle, which is trivialisable over A. Then every trivialisation $\alpha : E|_A \to A \times \mathbb{K}^n$ gives rise to a bundle E/α over X/A, such that $q^*(E/\alpha) \cong E$. If A is contractibvle, then every bundle over A is trivialisable and $E/\alpha \cong E/\beta$ for any two trivialisations α, β.*

(b) *The inclusion $i : A \to X$ and the projection $q : X \to X/A$ induce exact sequences of abelian groups:*

$$\tilde{K}(X/A) \xrightarrow{q^*} K(X) \xrightarrow{i^*} K(A)$$

and

$$\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$$

(c) *If A is contractible, then the map $q : X \to X/A$ yields isomorphisms*

$$K(X/A) \xrightarrow{\cong} K(X) \quad \textit{and} \quad \tilde{K}(X/A) \xrightarrow{\cong} \tilde{K}(X).$$

Proof. (a) For a trivialisation $\alpha : E|_A \to A \times \mathbb{R}^n$ we define

$$E/\alpha = E \Big/ \Big[\alpha^{-1}(a, v) \sim \alpha^{-1}(b, v)\Big]$$

The projection $E \to X \to X/A$ factors through E/α, so it induces $\pi : E/\alpha \to X/A$. In order to show that this is a vector bundle, we have to find a local trivialisation around the point A/A. Since E is trivialisable over A, by Lemma 1.4.6 it is trivialisable over some open neighbourhood U of A. Such a trivialisation induces a trivialisation of E/α over U/A, so that E/α is a vector bundle. It remains to show that $E \cong q^*(E/\alpha)$. In the commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E/\alpha \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & \xrightarrow{q} & X/A \end{array}$$

the map $E \to E/\alpha$ is an isomorphism on the fibres (by definition of E/α), hence this map gives an isomorphism $E \cong q^*(E/\alpha)$.

Finally let A be contractible. Then every bundle $p : E \to X$ over A is trivialisable. We claim, that the isomorphy class of E/α does not depend on the choice of α abhaengt. So let $\alpha_0, \alpha_1 : E|_A \to A \times \mathbb{R}^n$ be two trivialisations. Then $\alpha_1\alpha_0^{-1}$ is an automorphism of $A \times \mathbb{R}^n$, hence given by a continuous map $g : A \to \mathrm{GL}_n(\mathbb{R})$. Since A is contractible, g is homotopic to a constant map $x \mapsto M \in \mathrm{GL}_n(\mathbb{K})$. Writing $\alpha_1 = (\alpha_1\alpha_0^{-1}M^{-1})M\alpha_0$, one sees that, if one replaces α_0 by $M\alpha_0$, which doesn't change E/α_0, one can assume $M = \mathrm{Id}$. The homotopy of g to Id yields a homotopy h of α_0 to α_1. The vector bundle $F = I \times E \to I \times X$ is on $I \times A$ trivialised by h. The bundle E/h over $(I \times X)/(I \times A) = I \times (X/A)$ is on $\{0\} \times X/A$ equal to E/α_0 and over $\{1\} \times X/A$ equal to E/α_1. Hence $E/\alpha_0 \cong E/\alpha_1$.

(b) We choose a base point $a_0 \in A$. By the natural isomorphisms

$$K(X) \cong \tilde{K}(X) \oplus K(a_0),$$
$$K(A) \cong \tilde{K}(A) \oplus K(a_0),$$

the second assertion follows from the first. We show the first. The composition i^*q^* is given by $qi : A \to pt$, hence is zero. Let $\xi \in \ker(i^*)$. We can write ξ in the form $[E] - n$ with a vector bundle E over X. Then $E|_A = n \in K(A)$, so there is an $m \in \mathbb{N}$, such that $E \oplus m|_A = n \oplus m$. This means that there is a trivialisation α of $E \oplus m|_A$. By (a) this defines a bundle $E \oplus m/\alpha$ over X/A and hence an element $\eta = E + m - m - n$ of $\tilde{K}(X/A)$ with $q^*(\eta) = \xi$.

(c) This follows from (a). □

Definition 2.4.4. The **higher K-groups** are defined by

$$K^n(X) = K(\Sigma^n X), \qquad \tilde{K}^n(X) = \tilde{K}(\Sigma^n X).$$

Theorem 2.4.5. (a) *Let (X, A) be a pair. Then there is a natural infinite exact sequence*

$$\dots K^{n+1}(A) \xrightarrow{\delta} \tilde{K}^n(X/A) \xrightarrow{q^*} K^n(X) \xrightarrow{i^*} K^n(A) \xrightarrow{\delta} \tilde{K}^{n-1}(X/A) \to \dots$$
$$\dots \xrightarrow{q^*} K^0(X) \xrightarrow{i^*} K^0(A).$$

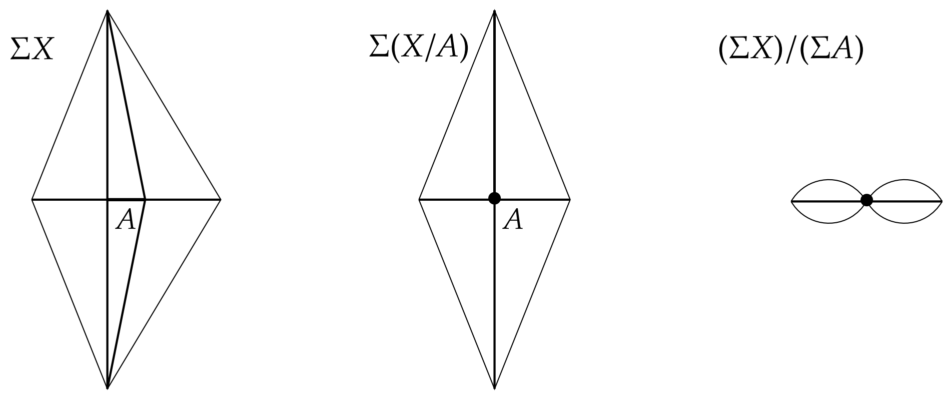
(b) *Further there is an exact sequence*

$$\dots \tilde{K}^{n+1}(A) \xrightarrow{\delta} \tilde{K}^n(X/A) \xrightarrow{q^*} \tilde{K}^n(X) \xrightarrow{i^*} \tilde{K}^n(A) \xrightarrow{\delta} \tilde{K}^{n-1}(X/A) \to \dots$$
$$\dots \xrightarrow{q^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A).$$

Proof. We choose a point $a_0 \in A$. Since $K(X) = \tilde{K}(X) \oplus \mathbb{Z}$ and since this decomposition is stable under f^* for every map, the second assertion is equivalent to the first. We show the first. Plugging in the definitions yields

$$\begin{array}{ccccccc} K^{n+1}(A) & \xrightarrow{\delta?} & \tilde{K}^n(X/A) & \longrightarrow & K^n(X) & \longrightarrow & K^n(A) \\ \| & & \| & & \| & & \| \\ K\big(\Sigma^{n+1}A\big) & \xrightarrow{\delta?} & \tilde{K}\big(\Sigma^n(X/A)\big) & \longrightarrow & K\big(\Sigma^n X\big) & \longrightarrow & K\big(\Sigma^n A\big) \end{array}$$

The space $\Sigma X/\Sigma A$ is constructed from $\Sigma(X/A)$ by collapsing the contractible space $[-1,1] \times A/A$. The following picture is an illustration.



By Lemma 2.4.3 part (b) this means that the spaces $\Sigma X/\Sigma A$ and $\Sigma(X/A)$ cannot be distinguished by K-theory. Therefore it suffices to show that for a pair (X, A) there exists a map δ making the sequence

$$K(\Sigma X) \xrightarrow{i^*} K(\Sigma A) \xrightarrow{\delta} \tilde{K}(X/A) \xrightarrow{q^*} K(X) \qquad (*)$$

exact.

We apply Lemma 2.4.3 to the pairs $(X \cup CA, X)$ and $(X \cup CA) \cup CX, X \cup CA)$. The pair $(X \cup CA, X)$ yields an exact sequence

$$\tilde{K}(X \cup CA/X) \xrightarrow{m^*} K(X \cup CA) \xrightarrow{k^*} K(X).$$

Since CA is contractible, the map $p : X \cup CA \to X \cup CA/CA = X/A$ gives an isomorphism

$$p^* : K(X/A) \xrightarrow{\cong} K(X \cup CA).$$

Let $\theta : K(X \cup CA/X) \xrightarrow{\cong} K(\Sigma A)$ be the isomorphism of the last section. Define

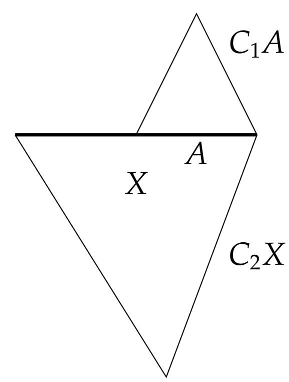
$\delta : K(\Sigma A) \to \tilde{K}(X/A)$ as the composition

$$K(\Sigma A) \xrightarrow{\theta^{-1}} K(X \cup CA/X) \xrightarrow{m^*} \tilde{K}(X \cup CA) \xrightarrow{(p^*)^{-1}} \tilde{K}(X/A)$$

The exact sequence above becomes

$$K(\Sigma A) \xrightarrow{\delta} \tilde{K}(X/A) \xrightarrow{k^*} K(X),$$

which is the right part of the desired sequence (*). We apply Lemma 2.4.3 to the pair $(X \cup C_1 A \cup C_2 X, X \cup CA)$, where we now have two cones, which for distinction have been numbered, see the picture



We get an exact sequence

$$\tilde{K}\left(\underbrace{\big(X \cup C_1 A \cup C_2 X\big)\big/\big(X \cup C_1 A\big)}_{\cong \Sigma X}\right) \to K(X \cup C_1 A \cup C_2 X) \to K(X \cup CA) \cong K(X/A)$$

and we want to show exactness of the sequence

$$\tilde{K}(\Sigma X) \xrightarrow{i^*} K(\Sigma A) \xrightarrow{\delta} K(X/A),$$

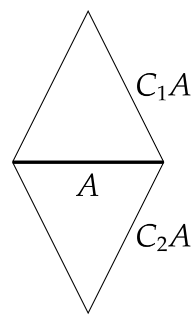
oder, equivalently, the sequence

$$\tilde{K}(\Sigma X) \xrightarrow{i^*} \tilde{K}(\Sigma A) \xrightarrow{\delta} \tilde{K}(X/A).$$

It suffices to show that the diagram

$$\begin{array}{ccc} \tilde{K}\Big(\big(X \cup C_1A \cup C_2X\big)\big/\big(X \cup C_1A\big)\Big) & \longrightarrow & \tilde{K}(X \cup C_1A \cup C_2X) \\ \| & & \uparrow \\ \tilde{K}(C_2X/X) & & \tilde{K}(C_1A/A) \\ \| & & \| \\ \tilde{K}(\Sigma X) & \xrightarrow{\quad i^* \quad} & \tilde{K}(\Sigma A) \end{array} \tag{A}$$

commutes up to sign. The problem is, that i^* is induced by the inclusion $C_2A \to C_2X$, but in the diagram we have C_1A, not C_2A. To handle this, we introduce the double cone $C_1A \cup C_2A \cong \Sigma A$:



We have the commutative diagram

$$\begin{array}{ccccc} X \cup C_1A \cup C_2X & & \longrightarrow & & C_1A/A = \Sigma A \\ \downarrow & \nwarrow & & \nearrow & \\ & & C_1A \cup C_2A & & \\ \downarrow & \swarrow & & \searrow & \\ C_2X/X & & \longleftarrow & & C_2A/A = \Sigma A. \end{array} \tag{B}$$

We see that the diagram (A) commutes up to sign, if the diagram, which is induced by (B),

$$\begin{array}{ccccc} & & K(C_1A/A) & \longleftarrow & K(\Sigma A) \\ & \swarrow & & & \| \\ K(C_1A \cup C_2A) & & & & \| \\ & \nwarrow & & & \| \\ & & K(C_2A/A) & \longleftarrow & K(\Sigma A) \end{array}$$

commutes up to sign. This follows from the next lemma. □

Lemma 2.4.6. *Let $\tau : \Sigma A \to \Sigma A$ be the map*

$$\tau(t, a) = (-t, a).$$

The map τ swaps $C_1 A$ and $C_2 A$ in $\Sigma A = C_1 A \cup C_2 A$.

One has $\tau^* E = -E$ for $E \in \tilde{K}(\Sigma A)$.

Proof. The lemma folllows from the next proposition. $\square$

Proposition 2.4.7. *Let $f : Y \to \mathrm{GL}_n(\mathbb{C})$ be a continuous map and let E_f be the the corresponding bundle over ΣY as in Proposition 1.5.7. Then the map $f \mapsto [E_f] - n$ induces a group isomorphism*

$$\lim_{n\to\infty} [Y, \mathrm{GL}_n(\mathbb{C})] \xrightarrow{\cong} \left(\tilde{K}(\Sigma Y), \oplus\right),$$

where the group structure on the left is induced by the one of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ is mapped to $\mathrm{GL}_{n+1}(\mathbb{C})$ via $A \mapsto \left(\begin{smallmatrix} A & \\ & 1 \end{smallmatrix}\right)$.

This means in particular, that the map $x \mapsto x^{-1}$ agrees on both sides, which proves Lemma 2.4.6.

Proof of Proposition 2.4.7. By Lemma 1.5.7 the map $f \mapsto E_f$ is a bijection of sets $\lim_{n\to\infty} [Y, \mathrm{GL}_n(\mathbb{C})] \leftrightarrow \tilde{K}(\Sigma Y)$. To see that it is a group homomorphism, one constructs a homotopy of the maps $\rho_0, \rho_1 : \mathrm{GL}_n(C) \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$ given by

$$\rho_0 : (A, B) \mapsto \begin{pmatrix} A & \\ & B \end{pmatrix},$$
$$\rho_1 : (A, B) \mapsto \begin{pmatrix} AB & \\ & 1 \end{pmatrix}.$$

Let $\gamma : [0, 1] \to \mathrm{GL}_{2n}(\mathbb{C})$ be a path with $\gamma(0) = \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $\gamma(1) = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$. Then the homotopy is

$$\rho_t(A, B) = \begin{pmatrix} A & \\ & 1 \end{pmatrix} \gamma(t) \begin{pmatrix} 1 & \\ & B \end{pmatrix} \gamma(t). \qquad \square$$

* * *

2.5 Smash and Cone

Definition 2.5.1. For pointed spaces $(X, x_0), (Y, y_0)$ define the **pointed sum** or **wedge product** as the following subset of $X \times Y$:

$$X \vee Y := \Big(\{x_0\} \times Y\Big) \cup \Big(X \times \{y_0\}\Big).$$

You ca also view the pointed sum as the disjoint union glued at the special.

Proposition 2.5.2. *The pointed sum is the the coporduct in the category* TOP$_*$ *of pointed spaces. This means that the canonical injections $X, Y \to X \vee Y$ induce functorial bijections*

$$\mathrm{Hom}(X \vee Y, Z) \to \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z).$$

Proof. For $\phi_X : X \to Z$ and $\phi_Y : Y \to Z$ define $\psi : X \vee Y \to Z$ by

$$\psi(x) = \begin{cases} \phi_X(x) & x \in X, \\ \phi_Y(x) & x \in Y. \end{cases}$$

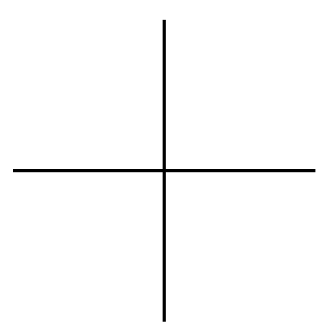
The map $(\phi_X, \phi_Y) \mapsto \psi$ is the wanted bijection. □

Lemma 2.5.3. *One has*

$$\tilde{K}(X \vee Y) = \tilde{K}(X) \oplus \tilde{K}(Y).$$

Proof. If E is a bundle over X, then on can extend it trivially on Y and one gets a bundle $\tilde{E}$. The map $[E] - n \mapsto [\tilde{E}] - n$ is an embedding of $\tilde{K}(X)$ into $\tilde{K}(X \vee Y)$. If E is a bundle over $X \vee Y$, then $[E] - n$ is the sum of the image of E_X and E_Y. □

Example 2.5.4. Let I be the unit interval with special point$x_0 = 1/2$. Then $I \vee I$ is the cross:



Definition 2.5.5. The **smash product** is

$$X \wedge Y := X \times Y / X \vee Y.$$

Note that $X \wedge \{pt\} = \{pt\}$.

Proposition 2.5.6. *If* $\mathrm{Hom}_{\mathrm{TOP}_*}(X, Y)$ *is equipped with the compact-open topology and* A *is locally compact, then there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{TOP}_*}(X \wedge A, Y) \cong \mathrm{Hom}_{\mathrm{TOP}_*}(X, \mathrm{Hom}_{\mathrm{TOP}_*}(A, Y)).$$

Das is called, the Funktor $X \mapsto X \wedge A$ *is linksadjungiert zu* $Y \mapsto \mathrm{Hom}_{\mathrm{TOP}_*}(A, Y)$.

Proof. Let $\mathrm{Abb}_*(X, Y)$ denote the set of all maps $f : X \to Y$ with $f(x_0) = y_0$. We define

$$\Phi : \mathrm{Abb}_*(X \wedge A, Y) \to \mathrm{Abb}_*(X, \mathrm{Abb}_*(A, Y))$$

by

$$\Phi(f)(x)(a) = f(x, a).$$

Then Φ is well-defined and bijective. Take some $f \in \mathrm{Abb}_*(X \wedge A, Y)$ and let $g = \Phi(f)$. We view f as a map on $X \times A$, which maps $X \vee A$ to y_0.

We show: if f is continuous, then g lies in $\mathrm{Hom}_{\mathrm{TOP}_*}\Big(X, \mathrm{Hom}_{\mathrm{TOP}_*}(A, Y)\Big)$.

Firstly it is clear that $g(x) : A \to Y \; a \mapsto f(x, a)$ is continuous. Next let $K \subset A$ be compact and $U \subset Y$ open. We need to show that $g^{-1}(L(K, U))$ is open, where

$$L(K, U) = \Big\{\alpha \in C(A, Y) : \alpha(K) \subset U\Big\}.$$

Let $a \in A$. One has

$$\begin{aligned} x \in g^{-1}(L(K, U)) &\Rightarrow g(x)(K) \subset U \\ &\Rightarrow f(x \times K) \subset U \\ &\Rightarrow x \times K \subset f^{-1}(U). \end{aligned}$$

Since $f^{-1}(U)$ is open and K is compact, there is an open neighbourhood V of a, such that $V \times K \subset f^{-1}(U)$, so

$$g(V)(K) \subset U, \quad \text{or} \quad g(V) \subset L(K, U).$$

Hence g is continuous, so it lies in $\mathrm{Hom}_{\mathrm{TOP}_*}\big(X, \mathrm{Hom}_{\mathrm{TOP}_*}(A, Y)\big)$.

We show: if $g \in \mathrm{Hom}_{\mathrm{TOP}_*}\big(X, \mathrm{Hom}_{\mathrm{TOP}_*}(A, Y)\big)$, then is f is continuous.

For this let $U \subset Y$ be open and let $(x_1, a_1) \in f^{-1}(U)$, so $g(x_1)(a_1) \in U$. Since $g(x_1)$ is continuous and A locally compact, there exists a compact neighbourhood $K \subset A$ of a_1, such that $g(x_1)(K) \subset U$, i.e., $g(x_1) \in L(K, U)$. Since g is continuous, there exists an open neighbourhood V of x_1, such that $g(V) \subset L(K, U)$, so $f(V \times K) = g(V)(K) \subset U$, which means that f is continuous. □

Remark 2.5.7. Let I^n be the n-dimensional cube. Let ∂I^n be its boundary. Then

$$I^n/\partial I^n \cong S^n$$

and

$$S^n \cong S^1 \wedge S^1 \wedge \cdots \wedge S^1$$

with n factors.

Proposition 2.5.8. *Let (X, x_0) be a pointed space. Then $A = [-1, 1] \times \{x_0\}$ is a subspace of the suspension ΣX. Then one has*

$$S^1 \wedge X \cong \Sigma X/A.$$

Since A is contractible, it follows

$$K(\Sigma X) = K(S^1 \wedge X).$$

Let $SX = S^1 \wedge X$ be the **reduced suspension***. We infer*

$$K^n(X) = K(S^n X).$$

Proof. Klar nach Definition and Lemma 2.4.3. □

* * *

2.6 Kuenneth formula and Bott periodicity

Definition 2.6.1. A closed subspace $A \subset X$ is called a **retract** of X, if there is a continuous map $f : X \to A$, such that $f|_A = \mathrm{Id}_A$. Such a map f is called a **retraction**

In this case let $a_0 \in A$ and let $i : A \to X$ be the inclusion map. Let $i_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ be the map induced by inclusion. Then $f_* i_* = \mathrm{Id}$, hence i_* is injective and f_* is surjective..

Examples 2.6.2. (a) S^1 is not a retract of S^2, since the fundamental group of S^2 is trivial, but the one of S^1 is not.

(b) The space X is a retract of $X \times Y$ and Y is a retract of $X \times Y/X$.

Theorem 2.6.3. *Let X and Y be pointed spaces.*

(a) *if A is a retract of X, then the exact sequence* $0 \to \tilde{K}^n(X/A) \xrightarrow{j^*} K^n(X) \xrightarrow{i^*} K^n(A) \to 0$ *splits and*

$$K^n(X) \cong \tilde{K}^n(X/A) \oplus K^n(A),$$

as well as

$$\tilde{K}^n(X) \cong \tilde{K}^n(X/A) \oplus \tilde{K}^n(A).$$

(b) (Kuenneth formula) *The projections $X \times Y \to X, Y$ induce isomorphisms*

$$\tilde{K}^n(X \times Y) \cong \tilde{K}^n(X \wedge Y) \oplus \tilde{K}^n(X) \oplus \tilde{K}^n(Y).$$

Proof. (a) Let $f : X \to A$ be a retraction, then $f^* : K^n(A) \to K^n(X)$ is a splitting, i.e., one has $i^* \circ f^* = \mathrm{Id}$. Hence i^* is surjectie. A part of the long exact sequence is

$$K^{n+1}(X) \xrightarrow{i^*} K^{n+1}(A) \xrightarrow{\delta} \tilde{K}^n(X/A) \xrightarrow{j^*} K^n(X) \xrightarrow{i^*} K^n(A)$$

und since i^* is surjective, we get $\delta = 0$. So we have a split exact sequence of the form

$$0 \to \tilde{K}^n(X/A) \xrightarrow{j^*} K^n(X) \xrightarrow{i^*} K^n(A) \to 0.$$

(b) Since X is a retract of $X \times Y$, we get from (a):

$$\tilde{K}^n(X \times Y) \cong \tilde{K}^n(X \times Y/X) \oplus \tilde{K}^n(X).$$

Since Y is a retract of $(X \times Y)/X$, it further follows that

$$\tilde{K}^n(X \times Y/X) \cong \tilde{K}^n(\underbrace{(X \times Y/X)/Y}_{=X \wedge Y}) \oplus \tilde{K}^n(Y)$$

und therefore the claim. □

Definition 2.6.4. The group $\tilde{K}(X \wedge Y) = \tilde{K}(X \times Y/X \vee Y)$ is the kernel of

$$i_X^* \oplus i_Y^* : \tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y).$$

The map $\tilde{K}(X) \to \tilde{K}(X \times Y) \to \tilde{K}(Y)$ is the zero map, hence
$\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y)$ is the zero map. We get a pairing

$$\tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y).$$

Since $S^m X \wedge S^n Y \cong S^{m+n}(X \wedge Y)$, we get a pairing

$$\tilde{K}^m(X) \otimes \tilde{K}^n(Y) \to \tilde{K}^{m+n}(X \wedge Y).$$

Replacing the spaces X and Y by $X^+ = X \sqcup \{\infty\}$ and Y^+, using $(X^+) \wedge (Y^+) = (X \times Y)^+$, we get a pairing

$$K^m(X) \otimes K^n(Y) \to K^{m+n}(X \times Y).$$

Theorem 2.6.5 (Bott Periodicity). (a) *For a pointed space Z there is a natural isomorphism*

$$\tilde{K}_{\mathbb{C}}(Z) \xrightarrow{\cong} \tilde{K}^2_{\mathbb{C}}(Z),$$

which comes from the tensor product with the pullback of $(H-1)$ on S^2 to $S^2 Z = S^2 \wedge Z$.

(b) *Let X be a space and $n \geq 0$. The map $K^2_{\mathbb{C}}(pt) \otimes K^n_{\mathbb{C}}(X) \to K^{n+2}_{\mathbb{C}}(X)$ induces an isomorphism*

$$\beta : K^n_{\mathbb{C}}(X) \xrightarrow{\cong} K^{n+2}_{\mathbb{C}}(X).$$

Proof. Part (b) follows from (a) by replacing Z with the space $S^n(X^+)$.

We show (a). I Theorem 2.2.2 we saw that the projections induce an isomorphism

$$K_{\mathbb{C}}(Z) \otimes K_{\mathbb{C}}(S^2) \cong K_{\mathbb{C}}(Z \times S^2).$$

By the Kuenneth formula we have $\tilde{K}_{\mathbb{C}}(Z \times S^2) = \tilde{K}_{\mathbb{C}}(Z \wedge S^2) \oplus \tilde{K}_{\mathbb{C}}(Z) \oplus \tilde{K}_{\mathbb{C}}(S^2)$. Writing $K = \tilde{K} \oplus \mathbb{Z}$ we get an isomorphism

$$\left(\tilde{K}_{\mathbb{C}}(Z) \otimes \tilde{K}_{\mathbb{C}}(S^2)\right) \oplus \tilde{K}_{\mathbb{C}}(Z) \oplus \tilde{K}_{\mathbb{C}}(S^2) \xrightarrow{\cong} \tilde{K}_{\mathbb{C}}(Z \times S^2) = \tilde{K}_{\mathbb{C}}(Z \wedge S^2) \oplus \tilde{K}_{\mathbb{C}}(Z) \oplus \tilde{K}_{\mathbb{C}}(S^2).$$

On the right hand side $\tilde{K}_{\mathbb{C}}(Z)$ stands for the image of $\tilde{K}_{\mathbb{C}}(Z)$ under pullback in $\tilde{K}_{\mathbb{C}}(Z \times S^2)$. Since the isomorphism in the middle is induced by pullbacks, too, $\tilde{K}_{\mathbb{C}}(Z)$ on the left is mapped to the same summand on the right and the same for S^2. Restriction the isomorphism therefore yields:

$$\tilde{K}_{\mathbb{C}}(Z) \otimes \tilde{K}_{\mathbb{C}}(S^2) \xrightarrow{\cong} \tilde{K}_{\mathbb{C}}(Z \wedge S^2).$$

Since $\tilde{K}_{\mathbb{C}}(S^2) \cong \mathbb{Z}$, the left hand side is isomorphic to $\tilde{K}_{\mathbb{C}}(Z)$ and the claim follows. □

Corollary 2.6.6. *Let $n \in \mathbb{N}$. Then gilt*

$$K_{\mathbb{C}}(S^n) = \begin{cases} \mathbb{Z} & n \text{ odd} \\ \mathbb{Z}[H]/(H^2 - 1) & n \text{ even.} \end{cases}$$

Proof. Clear by Bott periodicity and the Theorems 2.1.10 and 2.1.11. □

We can use this for another proof of the Brouwer Fixed Point Theorem:

Theorem 2.6.7. *Let $\mathbb{D}^n$ be the closed unit ball in $\mathbb{R}^n$. Then every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.*

Proof. Since $\tilde{K}^*_{\mathbb{C}}(\mathbb{D}^n) = 0$ and $\tilde{K}^*_{\mathbb{C}}(S^{n-1}) \neq 0$, the sphere S^{n-1} is not a retract $\mathbb{D}^n$. **Assume**, there is a continuous $f : \mathbb{D}^n \to \mathbb{D}^n$ with $f(x) \neq x$ for every x, then for every $x \in \mathbb{D}^n$ there exists exactly one real number $\alpha(x) > 0$ such that the vecto

$$g(x) = (1 - \alpha(x))f(x) + \alpha(x)x$$

lies on the boundary of $\mathbb{D}^n$.

The map g is readily seen to be continuous. If $x \in S^{n-1}$, then $g(x) = x$, so g is a retraction on S^{n-1}, **contradiction!** □

Proposition 2.6.8. *Let X be a finite CW-Komplex, all of whose celle have even dimensions. Then $\tilde{K}^1_{\mathbb{C}}(X) = 0$ and $\tilde{K}^0_{\mathbb{C}}(X)$ is a free abelian group, whose rank is at most equal to the number of cells.*

Proof. Induction by $d = \dim X$. If $d = 0$, then X is a finite set with the discrete topology and the claim is clear.

So let $d > 0$ be even and let $A \subset X$ denote the $(d-2)$-dimensional skeleton. The space X/A is a bundle of d-dimensional spheres, glued at the base point. Therfeore, by Lemma 2.5.3,

$$\tilde{K}(X/A) = \bigoplus_e \mathbb{Z}e,$$

where the sum runs over all cells of dimension d. By Theorem 2.4.5, together with Bott periodicity, we get the exact sequence

$$\underbrace{\tilde{K}^1(X/A)}_{=0} \xrightarrow{j^*} \tilde{K}^1(X) \xrightarrow{i^*} \tilde{K}^1(A) \xrightarrow{\delta} \underbrace{\tilde{K}^0(X/A)}_{=\bigoplus_e \mathbb{Z}e} \xrightarrow{j^*} \tilde{K}^0(X) \xrightarrow{i^*} \tilde{K}^0(A).$$

Induktively we can assume $\tilde{K}^1(A) = 0$, hence $\tilde{K}^1(X) = 0$. Further we have $\delta = 0$. By inductioin hypothesis the group $\tilde{K}^0(A)$ is free, so the image of i^* is free, hence the sequence $0 \to \tilde{K}^0(X/A) \to \tilde{K}^0(X) \to \mathrm{im}(i^*) \to 0$ splits and the claim follows. □

Theorem 2.6.9 (Bott periodicity in the real case)**.** *One has $K^n_{\mathbb{R}}(X) \cong K^{n+8}_{\mathbb{R}}(X)$ for every paracompact Hausdorff space.*

Proof. The proof will not be given here. □

Example 2.6.10. We list the real K-groups of the spheres:

$n \mod 8$	0	1	2	3	4	5	6	7
$KO(S^n)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

* * *

3 Algebraic K-theory

3.1 $K_0(R)$

Definition 3.1.1. A ring always mean a ring with unit. It can be non-commutative. Let R be a ring and let $P(R)$ be the set of all isomorphy classes of finitely-generated projective R-modules. Then $\big(P(R), \oplus\big)$ is an abelian monoid and we define the **algebraic K-theory** of the ring R as

$$K_0(R) := K\big(P(R)\big),$$

i.e., the quotient group of the monoid $P(R)$.

Examples 3.1.2. (a) If R is a principal ideal domain, then every finitely-generated projective module is free and these are klassified by dimension only, so then

$$K_0(R) \cong \mathbb{Z}.$$

(b) If X is a compact Hausdorff space and $R = C(X)$, then

$$K_0(R) = K^0(X).$$

K-group of a category

Definition 3.1.3. Let $\mathcal{A}$ be an additive category, which is equivalent to a small category (like Mod(R)). Let $C(\mathcal{A}) = \bigoplus_{[X]} \mathbb{Z}[X]$ be the free abelian group generated by all isomorphy classes $[X]$ of objects of $\mathcal{A}$. Let $N(\mathcal{A})$ be the subgroup generated by all elements of the form

$$X + Z - Y,$$

for which there is an exact sequence

$$0 \to X \to Y \to Z \to 0.$$

The **Grothendieck-group** or **K-group** of the category $\mathcal{A}$ is

$$K(\mathcal{A}) := C(\mathcal{A})/N(\mathcal{A}).$$

Proposition 3.1.4. *Let R be a ring and $P(R)$ the category of finitely-generated projective R-modules. Then*

$$K\big(P(R)\big) = K_0(R).$$

Proof. Dies follows from the fact, that or projective P fevery exact sequence of modules $0 \to A \to B \to P \to 0$ splits. □

Proposition 3.1.5. *K_0 is a functor of the category of rings to itself. The multiplikation on $K_0(R)$ is defined by the tensor product*

$$[P][Q] = [P \otimes_R Q].$$

Proof. Let $\phi : R \to S$ be a ring homomorphism (with $\phi(1) = 1$). Then one gets an induced map $\phi_* : K_0(R) \to K_0(S)$ by

$$\phi_*[P] = S \otimes_R P.$$

Then $S \otimes P$ is a finitely-generated S-module. It is projective, too, for there exists a module Q such that $P \oplus Q$ is free, say $P \oplus Q \cong R^n$. Then

$$\big(S \otimes P\big) \oplus \big(S \otimes Q\big) = S \otimes (P \oplus Q) \cong S \otimes R^n \cong S^n.$$

Therefore $S \otimes P$ is projective. Noq ϕ_* is additive and since

$$S \otimes_R P \otimes_R Q \cong (S \otimes_R P) \otimes_S (S \otimes_R Q),$$

the map ϕ_* is multiplikative, too. Finally it maps the unit = $[R]$ to the unit = $[S]$. □

Definition 3.1.6. Let R be a ring, $i : \mathbb{Z} \to R$ the uniquely determined ring homomorphism. We define the **reduced K-theory** of R as

$$\tilde{K}_0(R) = K_0(R)/i_* K_0(\mathbb{Z}).$$

Lemma 3.1.7. *Two projective modules P, Q define the same element in $K_0(R)$ iff there is $m \in \mathbb{N}$, such that*

$$P \oplus R^m \cong Q \oplus R^m.$$

Proof. We know that $[P] = [Q]$ iff there is a finitely-generated projective module M, such that

$$P \oplus M \cong Q \oplus M.$$

For given projective M there exists a module M' such that $M \oplus M' \cong R^m$ for an $m \in \mathbb{N}_0$, whence the claim. □

* * *

3.2 Idempotents

Definition 3.2.1. An **idempotent** or **idempotent element** of a ring R is an element $e \in R$ with the property

$$e^2 = e.$$

Example 3.2.2. Let $R = \mathrm{M}_n(K)$ for a field K. Then the Idempotents are the projection on subspaces of K^n.

Let P be a finitely-generated projective module of a ring R. Then there exists a module Q and an isomorphism $\alpha : P \oplus Q \to R^n$. We can view P and Q as submodules of R^n.

Definition 3.2.3. The group $\mathrm{GL}_n(R)$ of all invertible $n \times n$ matrices over R can be viewed as a subgroup of $\mathrm{GL}_{n+1}(R)$ by means of the embedding $A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$. Let

$$\mathrm{GL}(R) := \mathrm{GL}_\infty(R) = \bigcup_{n \in \mathbb{N}} \mathrm{GL}_n(R).$$

The elements of $\mathrm{GL}(R)$ may be visualised as infinite matrices, which differ from the unit matrix only at finitely many places.

Definition 3.2.4. By means of the (non-unital) ring homomorphism $A \mapsto \begin{pmatrix} A & \\ & 0 \end{pmatrix}$ the set $\mathrm{M}_n(R)$ of all matrices over R can be viewed as a subset of $M_{n+1}(R)$. Let

$$\mathrm{M}(R) := \mathrm{M}_\infty(R) = \bigcup_{n \in \mathbb{N}} \mathrm{M}_n(R).$$

The elements of $\mathrm{M}(R)$ are infinite matrices with only finitely many non-zero entries. Then $M(R)$ is a ring without unit. The group $\mathrm{GL}(R)$ acts by conjugation on the ring R.

For a given finitely-generated projective module P we choose a Q with $P \oplus Q = R^n$ and we let $p : R^n \to P$ denote the projection, then p is an idempotent in $\mathrm{M}_n(R) \subset \mathrm{M}(R)$.

Conversely, for an idempotent p in $\mathrm{M}(R)$ one has $p \in \mathrm{M}_n(R)$ for some n and $P = p(R^n)$ is a finitely-generated projective R-module.

Lemma 3.2.5. *Let p, q be idempotents in* $\mathrm{M}(R)$. *Then the finitely-generated projective modules $P = \mathrm{im}(p) = R^n p$ and $Q = \mathrm{im}(q) = R^m q$ are isomorphic iff p and q are* $\mathrm{GL}(R)$ *conjugate.*

Proof. If p and q are conjugate, i.e., $p = uqu^{-1}$, then
$P = \mathrm{im}(p) = \mathrm{im}(uqu^{-1}) = u\,\mathrm{im}(q) = uQ$, so u induces an isomorphism $Q \to P$, $q \mapsto uq$.

For the converse assume that $P, Q \subset R^n$ for some n. Let $\alpha : P \to Q$ be an isomorphism. Then P and Q both have complements in R^n and there are projections $p, q : R^n \to R^n$ with images P and Q. Since P and Q have complements in R^n, one can lift α to a homomorphism $a : R^n \to R^n$, so one gets a commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{a} & R^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ P & \xrightarrow{\alpha} & Q \end{array}$$

Doing the same with α^{-1}, one gets a lift $b : R^n \to R^n$. Since these lifts are R-linear, they are given by matrices $A, B \in \mathrm{M}_n(R)$, so $a(x) = xA$ and $b(x) = xB$, where we consider the elements of R^n as row vectors. This is because R could be non-commutative and we are considereing left modules. We obtain the equations

$$AB = p, \quad BA = q, \quad pA = A, \quad qB = B.$$

In $\mathrm{M}_{2n}(R)$ one therefore has

$$\begin{pmatrix} 1-p & A \\ B & 1-q \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1-p & A \\ B & 1-q \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-p & A \\ B & 1-q \end{pmatrix} = \begin{pmatrix} 1-p & A \\ B & 1-q \end{pmatrix}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

The matrix $\left(\begin{smallmatrix} 1-p & A \\ B & 1-q \end{smallmatrix}\right)$ therefore is invertible and conjugates p into q. □

Theorem 3.2.6. *Let R be a ring.*

(a) *The monoid $\overline{\mathrm{P}}(R)$ of all isomorphy classes of finitely-generated projective modules over R can be identified with the set of all $\mathrm{GL}(R)$-conjugacy classes of idempotents in $\mathrm{M}(R)$. The monoid composition is $(p, q) \mapsto \begin{pmatrix} p & \\ & q \end{pmatrix}$. $K_0(R)$ is the Grothendieck group of this monoid.*

(b) *There is a natural isomorphism*

$$K_0(R) \xrightarrow{\sim} K_0(\mathrm{M}_n(R)).$$

Proof. (a) Clear by the above.

(b) Let $\mathrm{Idem}(R)$ be the set of all idempotents in $\mathrm{M}(R)$. The natural identification $\mathrm{M}_k(\mathrm{M}_n(R)) \cong \mathrm{M}_{kn}(R)$ yields an identification $\mathrm{M}(R) \cong \mathrm{M}(\mathrm{M}_n(R))$ with $\mathrm{Idem}(R) = \mathrm{Idem}(M_n(R))$. Therefore (b) follows from (a). □

* * *

3.3 Local rings

Definition 3.3.1. (Localisation) Let R be a commutative ring and let $S \subset R$ be any subset with $0 \notin S$.. For each $s \in S$ fix a new variable X_s and set

$$S^{-1}R := R[S^{-1}] = R[X_s : s \in S] / \langle sX_s - 1 : s \in S \rangle .$$

This means that we take the polynomial ring in the variables X_s, $s \in S$ modulo the ideal generated by all elements of the form $sX_s - 1$.

In the ring $S^{-1}R$ every element of S becomes invertible and it is the smallest ring with this property.

Examples 3.3.2. (a) In the case $R = \mathbb{Z}$ and $S = \mathbb{N}$ one gets $S^{-1}R = \mathbb{Q}$.

(b) If R is an integral domain, then the map $R \mapsto S^{-1}R$ is injective and a localisation is the same as a subring R' of the quotient field $K = \mathrm{Quot}(R)$ with the property that $R \subset R' \subset K$.

(c) The natural map $R \to S^{-1}$ need not be injective. For instance if $R = \mathbb{Z} \oplus \mathbb{Z}$ and $S = 0 \oplus \mathbb{N}$, then $S^{-1}R = \mathbb{Q}$ and the map $R \to S^{-1}R$ sends $\mathbb{Z} \oplus 0$ to zero.

(d) In the case that S should contain two elements a, b with $ab = 0$, then in $S^{-1}R$ we have

$$1 = aa^{-1}bb^{-1} = (ab)(ab)^{-1} = 0,$$

hence $S^{-1}R$ is the zero ring.

(e) Let R be a commutative ring and let $p \subset R$ be a prime ideal,i.e., R/p is an integral domain. Let $S = R \smallsetminus p$, then the ring $R_p = S^{-1}R$ is called the **localisation at** p. In algebraic geometry, this is the ring, that contains all local information around a given point, whence the name "localisation". It also justifies the name "local ring" given below.

Proposition 3.3.3. *For a ring R with unit the following are equivalent:*

(a) *R has exactly one maximal right ideal and exactly one maximal left ideal and the two coincide.*

(b) *The set $I = R \smallsetminus R^\times$ is a two-sided ideal.*

Definition 3.3.4. Such a ring is called a **local ring**.

Proof. (a)⇒(b): Let M be the uniqiely determined maximal right- and left-ideal and let $I = R \smallsetminus R^\times$. Since $M \cap R^\times = \emptyset$ we have $M \subset I$. Let $x \in I$, then xR is a proper right-ideal, hence lies in M, which implies $x \in M$, so $I \subset M$, together $I = M$ and I is a two-sided ideal.

(b)⇒(a): Every proper one-sided ideal must be a subset of I. Therefor I is the uniquely determined maximal left- and right-ideal. □

Remark 3.3.5. If R is a commutative ring with unit and $p \subset R$ is a prime ideal, then the localisation $R_p = S_p^{-1}R$ is a local ring, where $S_p = R \smallsetminus p$. In algebraic geometry one views the ring as ring of global sections of a sheaf $\mathcal{F}$, the stalks $\mathcal{F}_x$ of which are exactly these local rings.

Corollary 3.3.6. *In a local ring every one-sided invertible element is already invertible.*

Proof. Let a be an element of the local ring R and b an element with $ab = 1$. If $a \notin R^\times$, then a lies in the maximal two-sided ideal M and then $1 = ab \in M = \mathbb{R} \smallsetminus R^\times$, which is a contradiction. □

Definition 3.3.7. Let R be a ring. The **radical** $\mathrm{Rad}(R)$ is the intersection of all maximal left-ideals.

Lemma 3.3.8. *A left-ideal I is maximal iff the R-module R/I is simple.*

Proof. Let I be a proper ideal and let $p : R \to R/I$ be the projection. The map $V \mapsto p^{-1}(V)$ is a bijection between the submodules V of R/I and the left-ideals $J \supset I$. □

Proposition 3.3.9. (a) *Let R be a ring The radical is a two-sided ideal.*

(b) *The radical* $\mathrm{Rad}(R)$ *equals the set*

$$\left\{x \in R : \forall_{a \in R}\ (1 - ax) \text{ has a left-inverse}\right\}$$

and it also equals the intersection of all right-ideals.

Proof. (a) If I is a maximal left-ideal, then the annihilator $\mathrm{Ann}_R(R/I)$ of R/I is contained in I, as for $a \in \mathrm{Ann}_R(R/I)$ one has $ax + I = I$ for all $x \in R$, so that for $x = 1$ one gets $a \in I$. Therefore

$$\bigcap_{I \text{ maximal left-ideal}} \mathrm{Ann}_R(R/I) \subset \bigcap_I I = \mathrm{Rad}(R).$$

We also have

$$\mathrm{Ann}_R(R/I) = \bigcap_{x \in R/I} \mathrm{Ann}_R(x).$$

The module $R/\mathrm{Ann}_R(x) \cong R(x + I)/I = R/I$ is simple, hence $\mathrm{Ann}_R(x)$ is a maximal left-ideal, which implies that $\mathrm{Ann}_R(R/I)$ is an intersection of maximal left-ideals and therefore $\mathrm{Rad}(R) \subset \mathrm{Ann}_R(R/I)$. Together we get

$$\mathrm{Rad}(R) = \bigcap_{I \text{ maximal left-ideal}} \mathrm{Ann}_R(R/I)$$

ergibt. Der Annihilator of an entire module, however, is a two-sided ideal. Therefore $\mathrm{Rad}(R)$ is a two-sided ideal.

(b) Let $x \in \mathrm{Rad}(R)$. Then Rx lies in every every maximal left-ideal. Let $a \in R$ and **assume**, that $1 - ax$ ha no left-inverse. Then $R(1 - ax) \neq R$, so $(1 - ax)$ lies in a maximal left-ideal I. Since already $ax \in I$, we get $1 \in I$, a **contradiction**.

For the converse let x lie in the in the displayed set and let M be a maximal left-ideal. If $x \notin M$, then $M + Rx = R$, so there is an $a \in R$, such that $1 - ax \in M$, which is a contradiction.

Now let $\mathrm{Rad}'(R)$ be the intersection of all maximal right-ideals, then we see in the same manner, that

$$\mathrm{Rad}'(R) = \Big\{ x \in R : \forall_{a \in R}\ (1 - xa) \text{ has a right-inverse} \Big\}.$$

Let $x \in \mathrm{Rad}(R)$ and $a \in R$. Since $\mathrm{Rad}(R)$ is a right-ideal, it follows $xa \in \mathrm{Rad}(R)$ and so $(1 - xa)$ has a left-inverse. So there exists a $c \in R$ such that $(1 - c)(1 - xa) = 1$. Since $\mathrm{Rad}(R)$ is a two-sided ideal and $x \in \mathrm{Rad}(R)$, we infer that $(1 - xa)(1 - c) = 1 + xac - cxa \in 1 + \mathrm{Rad}(R)$. Therefore $1 + xac - cxa$ has a left-inverse and $(1 - c)$ has a left-inverse y. Multiplying the equation $(1 - c)(1 - xa) = 1$ from the left with y, we get $1 - xa = y$ and so $1 - xa$ is left-inverse to $1 - c$, hence $1 - xa$ also has a right-inverse $1 - c$ and x lies in $\mathrm{Rad}'(R)$. This measn we have $\mathrm{Rad}(R) \subset \mathrm{Rad}'(R)$ and be symmetry the other inclusion follows as well. □

Lemma 3.3.10 (Nakayama Lemma)**.** *Let M be a finitely-generated module of the rings R and assume that* $\mathrm{Rad}(R)M = M$. *Then $M = 0$.*

Proof. **Assume**, $M \neq 0$. Let $x_1, \ldots, x_m$ be generators, where m is chosen minimal. As $\mathrm{Rad}(R)M = M$, there are $r_1, \ldots, r_m \in \mathrm{Rad}(M)$, such that $x_m = r_1x_1 + \cdots + r_mx_m$, so

$$(1 - r_m)x_m = r_1x_1 + \cdots + r_{m-1}x_{m-1}.$$

By Proposition 3.3.9 the element $1 - r_m$ has a left-inverse, hence x_m is a linear combination of the other x_j and m is not minimal. **Contradiction!** □

Corollary 3.3.11. *If M is a finitely-generated module over a ring R and if $x_1 \ldots, x_m \in M$, then the x_j are generators iff their images in $M/\mathrm{Rad}(R)M$ generate the module $M/\mathrm{Rad}(R)M$.*

Proof. The direction "⇒" is trivial. So assume that $\bar{x}_1, \ldots, \bar{x}_m$ generate the module $M/\mathrm{Rad}(R)M$. Let $N = Rx_1 + \cdots + Rx_m$ and consider the module M/N. This satisfies the condition of the Nakayama-Lemma, hence is zero. □

Theorem 3.3.12. *Let R a local ring. Then every finitely-generated projective module is free with a uniquely determined rank. In particular, $K_0(R) \cong \mathbb{Z}$.*

Proof. Since R is local, the ring $D = R/\operatorname{Rad}(R)$ is a division ring. Let M be a finitely-generated projective R-module and N a module, such that $M \oplus N \cong R^k$. Then $M/\operatorname{Rad}(R)M$ and $N/\operatorname{Rad}(R)N$ are modules of D, hence free, of ranks, say, m and n with $m + n = k$. Choose bases and pull them back to elements in M and N. Write those as $x_1, \dots, x_m \in M$ and $x_{m+1}, \dots, x_{m+n} \in N$. By corollary 3.3.11 the x_j generate the module R^k. We need to show that the x_j are linearly independent. Then M is free of a unique rank

$$\operatorname{Rang}(M) = \dim_D(M/\operatorname{Rad}(R)M).$$

Let $e_1, \dots, e_k$ be the standard basis of R^k. Then there are elements a_{ij} and b_{ij} of R, making up matrices A and B, such that

$$e_i = \sum_{j=1}^{k} a_{ij} x_j, \quad \text{und} \quad x_i = \sum_{j=1}^{k} b_{ij} e_j.$$

Plugging in, we get $e_i = \sum_{j,l} a_{ij} b_{jl} e_l$, so

$$\sum_{j,l} (a_{ij} b_{jl} - \delta_{il}) e_l = 0.$$

Since the e_l are linearly independent, it follows that the coeffizients all are zero, i.e., $AB = I$. On the other hand the x_l are linearly independent modulo $\operatorname{Rad}(R)$, so we get $BA = I$ modulo $\operatorname{Rad}(R)$, or $BA - I \in \mathrm{M}_n(\operatorname{Rad}(R)) \subset \operatorname{Rad}(M_n(R))$. By Proposition 3.3.9 the matrix BA is invertible, say $CBA = I$. As already $AB = I$, the matrix A is invertible and by the same argument so is B and the two are inverses of each other, i.e., $BA = I$. Therefore $x_1, \dots x_k$ is a free basis of R^k and the Theorem is proven. □

* * *

3.4 Dedekind rings

Definition 3.4.1. A integral domain, which is not a field, is called a **Dedekind ring**, if every ideal I can be written as a product of prime ideals,

$$I = p_1 \cdots p_n,$$

where the p_j are uniquely determined up to order.

Examples 3.4.2. (a) Principal ideal domains are Dedekind rings.

(b) If R is a Dedekind ring with quotient field K, then every localisation $S^{-1}R$, which is not equal to K, a Dedekind ring.

(c) If K is a **number field**, i.e., a finite extension of $\mathbb{Q}$, then the **ring of integers**

$$O_K = \left\{ \alpha \in K : \begin{array}{c} \alpha \text{ is zero of a polynomial } f(x) \in \mathbb{Z}[x] \\ \text{with leading coefficient 1} \end{array} \right\}$$

is a Dedekind ring.

Definition 3.4.3. A local principal ideal domain is also called a **discrete valuation ring**.

Example 3.4.4. Let A be a commutative ring and $\mathfrak{p} \subset A$ a maximal ideal. Then the localisation

$$A_\mathfrak{p} := S_\mathfrak{p}^{-1} A$$

is a local ring, where $S_\mathfrak{p} = A \smallsetminus \mathfrak{p}$. The only maximal ideal of $A_\mathfrak{p}$ is $S_\mathfrak{p}^{-1}\mathfrak{p}$. If A is a principal ideal domain, then the localisation $A_\mathfrak{p}$ is a discrete valuation ring.

Example 3.4.5. Any localisation $S^{-1}\mathbb{Z}$ of $\mathbb{Z}$ is a Dedekind ring and any prime localisation is a discrete valuation ring.

Definition 3.4.6. Let R be a Dedekind ring and let K be its quotient field. A **fractional ideal** is an R-sub-module $0 \neq M \subset K$, for which there exists an $\alpha \in K^\times$, such that $\alpha M \subset R$. If it happens to lie inside R, it is called an **integral ideal**

One can multiply fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ just like ideals of the ring R:

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{j=1}^{n} a_j b_j : a_j \in \mathfrak{a},\ b_j \in \mathfrak{b} \right\}.$$

Theorem 3.4.7. *If R is a Dedekind ring, then the fractional ideals form an abelian group under multiplication. The unit element is R and the inverse of a given $\mathfrak{a}$ is*

$$\mathfrak{a}^{-1} = \left\{ x \in K : x\mathfrak{a} \subset R \right\}.$$

This group is called the **ideal group** *of R and written as J_R.*

Proof. Algebraic Number Theory. □

Definition 3.4.8. The principal ideals $R\alpha$, $\alpha \in K^\times$ form a subgroup of the ideal group, which is isomorphic to $K^\times/R^\times$. Let $H_R \subset J_R$ be the sub-group of all principal ideals. We define the **ideal class group** as

$$\mathrm{Cl}(R) := J_R/H_R.$$

Lemma 3.4.9. *If R is a Dedekind ring, then every ideal is a finitely-generated projective module. Every finitely-generated projective module is isomorphic to a direct sum of ideals. In particular the group $K_0(R)$ is generated by ideals.*

Proof. Let I be a proper ideal. Since $I^{-1}I = R$, there are elements $x_1, \dots, x_n \in I^{-1}$ and $y_1, \dots, y_n \in I$ such that $1 = \sum_{j=1}^n x_j y_j$. If $b \in I$, then $b = \sum_j (bx_j)y_j$ with $bx_j \in I^{-1} = R$, hence $y_1, \dots, y_n$ generated the ideal I. The surjection $R^n \to I$, $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 y_1 + \cdots + \lambda_n y_n$ splits, a splitting is given by $s : I \to R^n$, $s(b) = (bx_1, \dots, bx_n)$, so I is a direct summand of R^n, i.e., finitely-generated projective.

For the second assertion let M be a finitely-generated projective module over R. We can assume, that M is a sub-module of R^n. We show by induction on n, that M is a direct sum of k ideals for some $k \leq n$. If $n = 1$, then M is an ideal itself. Assume the assertion has been proven for $n' < n$ and let $\pi : R^n \to R$ be the projection onto the last coordinate. Then $\pi(M)$ is an ideal. The ideal $I = \pi(M)$ is projective and so the sequence $0 \to \ker(\pi) \to M \to I \to 0$ splits and hence $M \cong \ker(\pi) \oplus I$. The kernel of π lies in R^{n-1}, hence is a direct sum of ideals by induction hypothesis. □

Lemma 3.4.10. (a) *Let R be a commutative ring with unit and let I_1, I_2 coprime ideals. Then $I_1 \cap I_2 = I_1 I_2$.*

(b) *Let R be a Dedekind ring, I a fractional ideal and J an ideal of R. Then there exists an $a \in I$, such that $I^{-1}a + J = R$.*

(c) *Let R be a Dedekind ring. Then every fractional ideal is generated by at most two elements.*

Proof. Algebraic Number Theory. □

Theorem 3.4.11. *Let R be a Dedekind ring. Then every finitely-generated projective module can be written in the form $R^k \oplus I$ for an ideal I, where the isomorphy class of I is uniquely determined. The map $R^k \oplus I \mapsto (k, [I])$ induces an isomorphism*

$$K_0(R) \xrightarrow{\cong} \mathbb{Z} \oplus \mathrm{Cl}(R).$$

Conversely, the map $[I] \mapsto [I] - [R]$ induces an isomorphism $\mathrm{Cl}(R) \xrightarrow{\cong} \tilde{K}_0(R)$.

Proof. We show the following principle:

- Let R be a Dedekind ring and I_1, I_2 fractional ideals. Then there is an R-module isomorphism
$$I_1 \oplus I_2 \cong R \oplus I_1 I_2.$$

Proof. Let $a_1 \in I_1$ and consider the integral ideal $J = a_1 I^{-1}$. By Lemma 3.4.10 with $I = I_2$ there is $a_2 \in I_2$ such that $I_2^{-1} a_2 + a_1 I_1^{-1} = R$. Choose $b_2 \in I_2^{-1}$ and $b_1 \in I_1^{-1}$ with $a_1 b_1 + a_2 b_2 = 1$. Then

$$\begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ -b_2 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that both matrices are invertible and

$$(x_1, x_2) \mapsto (x_1, x_2) \begin{pmatrix} b_1 & -a_2 \\ b_2 & a_1 \end{pmatrix}$$

is the desired isomorphism. □

By Lemma 3.4.9 every finitely-generated projective module P over R is a direct sum of ideals. Using the above isomorphism repeatedly, one can transform such a sum into the form $P = R^k \oplus I$. Assume $I \neq 0$. If $\mathfrak{m}$ is a maximal ideal of R, then the dimension of the $R/\mathfrak{m}$-vector space $P/\mathfrak{m}P$ equals $k + 1$. This means that k is determined by the isomorphy class of P. Finally let

$$\alpha : R^k \oplus I \xrightarrow{\cong} R^k \oplus J$$

be a module isomorphism, where $I, J \subset R$ are ideals. We need to show that $I \cong J$. Let $\beta = \alpha^{-1}$. Let K denote the quotient field of R, then $\left(R^k \oplus I\right) \otimes_R K \cong K^{k+1}$ as K-vector spaces. The homomorphisms α and β extend to K-linear maps, hence are given by invertible matrices A and B in $\mathrm{M}_{k+1}(K)$. Let $x \in I$ and let X be the diagonal matrix with diagonal $(1,\dots,1,x)$. Then X maps the module R^{k+1} to $R^k \oplus I$, so AX maps the module R^{k+1} to $R^k \oplus J$. The columns of AX are the images of the standard basis and they lie in $R^k \oplus J$, so in every column the last entry lies in J. The Laplace expansion along the last row shows that $x\det(A) = \det(AX) \in J$. This means that the module homomorphism $x \mapsto x\det(A)$ maps the ideal I to the ideal J. Similarly, $y \mapsto \det(B)y$ maps the ideal J to the ideal I and the two are inverse to each other. Therefore we have constructed an isomorphism $I \xrightarrow{\cong} J$.

One gets a bijective map $K_0(R) \to \mathbb{Z} \oplus \mathrm{Cl}(R)$. This also is a homomorphism by the above principle. □

* * *

3.5 $K_1(R)$

Definition 3.5.1. Let R be ring. A matrix in $\mathrm{GL}_n(R)$ is called **elementary**, if A has ones on the diagonal and has at most one further entry which is $\neq 0$.

We embed $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ via $A \mapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}$ and form the group $\mathrm{GL}(R) = \bigcup_n \mathrm{GL}_n(R)$. Let $E_n(R)$ the subgroup generated by all elementary matrices and let $E(R) \subset \mathrm{GL}(R)$ be the union of all $E_n(R)$.

If the ring R is commutative, then the determinant $\det : \mathrm{GL}(R) \to R^\times$ is a group homomorphism. Let $\mathrm{SL}(R)$ be the kernel of this homomorphism.

Theorem 3.5.2 (Whitehead Lemma)**.** (a) *For every $A \in \mathrm{GL}_n(R)$ the matrix $\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix}$ lies in $E(R)$.*

(b) *The group $E(R)$ is the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ of $\mathrm{GL}(R)$.*

Proof. We have

$$\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ -A^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

and

$$\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix}.$$

Part (a) follows.

(b) For $a \in R$ let $e_{ij}(a)$ be the element of $E(R)$ with a at the position (i, j), where $i \neq j$. One computes: $[e_{ik}(a), e_{kj}(1)] = e_{ij}(a)$, if i, j, k are distinct. So every generator of $E(R)$ is a commutator of two other generators and so $[E(R), E(R)] = E(R)$. It remains to show, that $[\mathrm{GL}(R), \mathrm{GL}(R)] \subset E(R)$. For $A, B \in \mathrm{GL}_n(R)$ we compute in $\mathrm{GL}_{2n}(R)$

$$\begin{pmatrix} ABA^{-1}B^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} AB & \\ & (AB)^{-1} \end{pmatrix}\begin{pmatrix} A^{-1} & \\ & A \end{pmatrix}\begin{pmatrix} B^{-1} & \\ & B \end{pmatrix}$$

and the right hand side lies in $E(R)$ by part (a). □

Definition 3.5.3. Let R be a ring. Define

$$K_1(R) := \mathrm{GL}(R)/E(R) = \mathrm{GL}(R)^{ab},$$

the **abelianisation** of $\mathrm{GL}(R)$.

Proposition 3.5.4. *If the ring R is commutative, then the determinant* $\det : \mathrm{GL}(R) \to R^\times$ *yields a split exact sequence*

$$1 \to SK_1(R) \to K_1(R) \xrightarrow{\det} R^\times \to 1.$$

Here we have written $SK_1(R)$ for the kernel of the determinant. One has

$$SK_1(R) \cong \mathrm{SL}(R)/E(R).$$

Proof. The generators of $E(R)$ are of determinant 1 and so the determinant map factors through $K_1(R)$. The sequence splits, since the map $R^\times \to \mathrm{GL}(R)$, $a \mapsto \left(\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right)$ defines a section. The last equation is clear, as $E(R) \subset \mathrm{SL}(R)$. □

Proposition 3.5.5. *If $R = K$ is a field, then $SK_1(K)$ is trivial, i.e., the determinant induces an isomorphism*

$$K_1(K) \xrightarrow{\cong} K^\times.$$

Proof. We need to show that $\mathrm{SL}(K) = E(K)$. Let $A \in \mathrm{SL}_n(K)$. By row transformations, which are given by left-multiplication with matrices of $E(R)$, one can transform A to upper triangular form then into diagonal form. Since $\det(A) = 1$, multiplication with diagonal matrices brings A of the diagonal $(1, \dots, 1, a, a^{-1}, 1, \dots, 1)$ transforms A to the unit matrix. These diagonal matrices are in $E(K)$ by the whitehead lemma. □

xxx

Proposition 3.5.6. *Let R a local ring. The inclusion* $R^\times = \mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}(R)$ *induces a Surjektion*

$$(R^\times)_{ab} \twoheadrightarrow K_1(R).$$

Proof. Der gleiche Beweis wie im kommutativen Fall zeigt, that every matrix in $\mathrm{GL}_n(R)$ on the Diagonalform $(a, 1, \dots)$ gebracht werden kann. Hierbei is the entscheidende point, that for $A \in \mathrm{GL}_n(R)$ the erste Spalte a element aus $R^\times$ enthalten muss, denn andernif waeren alle Eintraege im maximal ideal M, such that the erste Spalte modulo M gleich Null waere, was a Widerspruch zur Invertierbarkeit of A darstellt.

Since $K_1(R)$ abelsch ist, faktorisiert the entstehende Surjektion $R^\times \to K_1(R)$ over $(R^\times)_{ab}$. □

Theorem 3.5.7. *Let R a local ring (nicht notwendig kommutativ). Then induces the inclusion* $R^\times = \mathrm{GL}_1(R) \to \mathrm{GL}(R)$ *a isomorphism*

$$(R^\times)_{ab} \xrightarrow{\sim} K_1(R).$$

Proof. We brauchen a Determinante, the es im nichtkommutativen Fall aber nicht gibt. Ersatz schafft:

Lemma 3.5.8. *Let R a local ring. Then there exists exactly one map*

$$\det : \mathrm{GL}(R) \to R^\times_{ab}$$

with folgenden Eigenschaften:

(a) *the Determinante is invariant unter elementaren Zeilentransformationen, i.e., is A' aus A entstanden durch addieren des (links) λ-fachen einer Zeile zu einer anderen, then is* $\det(A) = \det(A')$,

(b) $\det(I) = 1$,

(c) *entsteht A' aus A durch Multiplikation einer Zeile of links with $\mu \in R^\times$, then is* $\det(A') = \overline{\mu}\det(A)$, *wober $\overline{\mu}$ the Bild of μ in $R^\times_{ab}$ ist.*

Zusaetzlich gilt: det *is a groupnhomomorphism. Entsteht A' aus A durch Vertauschen zweier Zeilen, then is* $\det(A') = \overline{(-1)}\det(A)$. *Ferner one has* $\det(A^t) = \det(A)$.

Proof. We zeigen zunaechst Eindeutigkeit and that the Zusatzeigenschaften aus (a)-(c) folgen.

The Eindeutigkeit is klar, weil man every invertierbare matrix durch Zeilentransformationen on Diagonalgestalt with Diagonale $a, 1, \dots$ bringen kann. Dass the Determinante a groupnhomomorphism ist, folgert man wie im Koerperfall: The Zeilentransformationen werden durch Linksmultiplikation with erweiterten elementarmatrizen given, where zu den obigen noch every Diagonalmatrix with Diagonale $1, \dots, 1, a, 1, \dots$ with $a \in R^\times$ hinzugenommen wird. Es folgt $\det(SA) = \det(S)\det(A)$, wenn S a erweiterte elementarmatrix ist. Sind Matrizen A and B given, so can man $A = S_1 \cdots S_k$ and $B = T_1 \cdots T_l$ als Produkte of elementarmatrizen schreiben and erhaelt

$$\begin{aligned}\det(AB) &= \det(S_1 \cdots S_k T_1 \cdots T_l)\\ &= \det(S_1)\cdots\det(S_k)\det(T_1)\cdots\det(T_l)\\ &= \det(A)\det(B).\end{aligned}$$

Das Zeilenvertauschen wird durch the matrix $\left(\begin{smallmatrix} & 1\\ 1 & \end{smallmatrix}\right) = \left(\begin{smallmatrix} -1 & \\ & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right)$ bewerkstelligt. The matrix $\left(\begin{smallmatrix} & -1\\ 1 & \end{smallmatrix}\right)$ is aber nach dem Beweis des Whitehead Lemmas in $E(R)$, hat also Determinante 1. The Transponierte matrix erhaelt man als Produkt the transponierten elementarmatrizen, also braucht man the Invarianz nur for elementarmatrizen zu checken, wo es aber klar ist.

Bleibt the Existenz zu zeigen. We define $\det_n : \mathrm{GL}_n(R) \to R^\times_{ab}$ induktiv nach n, such that the Eigenschaften (a)-(c) erfuellt are. For $n = 1$ setze $\det_1(x) = \overline{x}$, the Klasse of x in $R^\times_{ab}$ and the Eigenschaften are klar, the Induktionsanfang is geschafft.

Let also $\det_k$ for $k < n$ konstruiert. Wegen the Eindeutigkeit are the verschiedenen $\det_k$ untereinander kompatibel, d.h. $\det_{k+1}\left(\begin{smallmatrix} A & \\ & 1\end{smallmatrix}\right) = \det_k(A)$. Let also $A \in \mathrm{GL}_n(R)$ with den Zeilen $Z_1, \dots, Z_n$. Seien $b_1, \dots, b_n$ the Eintraege the ersten Zeile of A^{-1}. Wegen $A^{-1}A = 1$

folgt

$$b_1 Z_1 + \cdots + b_n Z_n = (1, 0, \ldots, 0).$$

Schreiben we also $Z_j = (z_j, B_j)$, then folgt $\sum_j b_j B_j = 0$. Since nicht alle b_j im maximal ideal liegen koennen, muss eines invertierbar sein, sagen we $b_i \in R^\times$. Es folgt

$$b_i^{-1} b_1 B_1 + \cdots + B_i + \cdots + b_i^{-1} b_n B_n = 0.$$

Indem man also Vielfache anderer Zeilen zur i-ten addiert, reduziert man A on the form

$$\begin{pmatrix} z_1 & B_1 \\ \vdots & \vdots \\ z_{i-1} & B_{i-1} \\ b_i^{-1} & 0 \\ z_{i+1} & B_{i+1} \\ \vdots & \vdots \\ z_n & B_n \end{pmatrix}$$

Dawith the Eigenschaften gelten, muessen we also setzen

$$\det_n(A) = \overline{(-1)^{i} b_i^{-1}} \det_{n-1} \begin{pmatrix} B_1 \\ \vdots \\ \widehat{B_i} \\ \vdots \\ B_n \end{pmatrix}.$$

We muessen zeigen, that diese Definition nicht of the Wahl of i abhaengt. Nenne the matrix rechts C_i, so is also folgendes zu zeigen: Sind $b_i, b_j \in R^\times$ with $i < j$, so ist

$$\overline{(-1)^{i} b_i^{-1}} \det_{n-1}(C_i) = \overline{(-1)^{j} b_j^{-1}} \det_{n-1}(C_j).$$

Man erhaelt C_j aus C_i dadurch, that man erst the Zeilen permutiert zu

$$C = \begin{pmatrix} B_1 \\ \vdots \\ B_{i-1} \\ B_j \\ B_{i+1} \\ \vdots \\ \widehat{B_j} \\ \vdots \\ B_n \end{pmatrix}$$

and then B_j durch B_i ersetzt. Man gelangt of C_i nach C durch zyklisches permutieren the Zeilen $B_{i+1}, \ldots, B_j$, such that $\det_{n-1}(C) = (-1)^{j-i-1} \det_{n-1}(C_i)$. Schliesslich is $B_i = -b_i^{-1} b_j B_j +$ (a Linearkombination the anderen Zeilen), such that $\det_n$ wohldefined ist.

Nun bleibt zu zeigen, that $\det_n$ the Eigenschaften (a)-(c) hat. (b) and (c) are sofort klar. Man verifiziert (a), indem man annimmt that A' aus A durch a Zeilentransformation entsteht and checkt the Auswirkung dieser. Dies Let dem Leser zur Uebung gelassen. □

Der Satz is with dem Lemma klar. □

* * *

3.6 Relative K_0 and K_1

Definition 3.6.1. If $f : R \to S$ a ringhomomorphism, then macht f den ring S insbesondere zu einem R-Rechtsmodul. If then P a R-Linksmodul, so wird the Tensorprodukt

$$f_* P := S \otimes_R P$$

zu einem S-Linksmodul. The map f_* bildet projective modules on projective modules ab and respektiert direct sumn, such that sie a groupnhomomorphism

$$f_* : K_0(R) \to K_0(S)$$

induces.

Definition 3.6.2. Let R a ring and I a zweiseitiges ideal. Setze

$$D(R,I) = \Big\{(x,y) \in R^2 : x - y \in I\Big\}.$$

Then is $D(R,I)$ a Unterring of R^2 and the projection on the erste Koordinate liefert a exact sequence

$$0 \to I \to D(R,I) \xrightarrow{p_1} R \to 0$$

von ringen (ohne Eins, Since I ka hat), or auch of R-modules.

Definition 3.6.3. The **relative** K_0 group is defined als the Kern

$$K_0(R,I) := \ker\Big[(p_1)_* : K_0(D(R,I)) \to K_0(R)\Big].$$

Lemma 3.6.4. *Let R a ring and $I \subset R$ a ideal. is $A \in \mathrm{GL}_n(R/I)$, then liftet the matrix $\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix}$ zu einer invertierbaren matrix in $\mathrm{GL}_{2n}(R)$.*

Proof. One has

$$\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ -A^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & A \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Choose $X, Y \in \mathrm{M}_n(R)$ with $X \cong A \mod I$ and $Y \cong -A^{-1} \mod I$, then ist

$$\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ Y & 1 \end{pmatrix}\begin{pmatrix} 1 & X \\ & 1 \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

a matrix in $\mathrm{GL}_{2n}(R)$, the modulo I on $\begin{pmatrix} A & \\ & A^{-1} \end{pmatrix}$ reduziert. □

Theorem 3.6.5. *Let R a ring and I a zweiseitiges ideal. Then there is a natuerliche exact sequence*

$$K_0(R,I) \xrightarrow{p_{2*}} K_0(R) \xrightarrow{q_*} K_0(R/I).$$

The map q_* is induces durch the Quotientenabbildung $q : R \to R/I$ and the map $K_0(R,I) \to K_0(R)$ is induces durch the projection p_2 on the zweite Variable.

Proof. Let $[e]-[f] \in K_0(R,I)$, where $e=(e_1,e_2)$ and $f=(f_1,f_2)$ Idempotente in $\mathrm{M}(D(R,I))$ are. Das Bild of $[e]-[f]$ in $K_0(R\times R)=K_0(R)\times K_0(R)$ is $([e_1]-[f_1],[e_2]-[f_2])$. Also ist

$$q_* \circ (p_2)_*([e]-[f]) = q_*([e_2]-[f_2]) = [\bar{e}_2]-[\overline{f}_2],$$

where $\overline{f}$ the Bild in R/I of $f\in R$ ist. Ferner is $[e_1]-[f_1]=0$, Since $[e]-[f]\in\ker((p_1)_*)$. Since aber $e,f\in D(R,I)$, folgt $\bar{e}_1=\bar{e}_2$ and $\overline{f}_1=\overline{f}_2$, such that $[\bar{e}_2]-[\overline{f}_2]=[\bar{e}_1]-[\overline{f}_1]=0$. Dawith is the Bild the ersten map im Kern the zweiten.

Seien nun e,f Idempotente in $\mathrm{M}(R)$ such that $q_*([e]-[f])=0$. Das bedeutet, that es $r\in\mathbb{N}$ gibt such that $\bar{e}\oplus I_r$ and $\overline{f}\oplus I_r$ bezueglich $\mathrm{GL}(R/I)$ konjugiert are. Ersetzen we e durch $e\oplus I_r$ and f durch $f\oplus I_r$, koennen we annehmen, that $\overline{f}=g\bar{e}g^{-1}$ for a $g\in\mathrm{GL}(R/I)$. Es gibt a $h\in\mathrm{GL}(R)$ such that $\overline{h}=g\oplus g^{-1}$ and diese matrix konjugiert $e\oplus 0$ zu $f\oplus 0$, such that we f durch $h(f\oplus 0)h^{-1}$ and e durch $e\oplus 0$ ersetzen koennen and annehmen, that $\bar{e}=\overline{f}$. Das bedeutet gerade, that (e,f) a Idempotent in $\mathrm{M}(D(R,I))$ ist, also is $[(e,e)]-[(e,f)]$ a Klasse in $K_0(D(R,I))$, the unter p_1 on Null and unter p_2 on $[e]-[f]$ abgebildet wird. □

Definition 3.6.6. Let R a ring and I a ideal and definiere

$$K_1(R,I) := \ker\Big[(p_1)_* : K_1(D(R,I)) \to K_1(R)\Big].$$

Definition 3.6.7. Let R a ring with Eins and I a zweiseitiges ideal. Let $\mathrm{GL}(R,I)$ the Kern the map $\mathrm{GL}(R)\to\mathrm{GL}(R/I)$.

Let $E(R,I)\subset E(R)$ the kleinste normale Untergruppe, the alle elementarmatrizen in $\mathrm{GL}(R,I)$ enthaelt. Also insbesondere $E(R,I)\subset\big(E(R)\cap\mathrm{GL}(R,I)\big)$, where aber ka Gleichheit herrschen muss.

Theorem 3.6.8 (Relatives Whitehead-Lemma)**.** *Let R a ring with Eins and I a zweiseitiges ideal. Then is $E(R,I)$ normal in $\mathrm{GL}(R,I)$ and in $\mathrm{GL}(R)$. One has*

$$\mathrm{GL}(R,I)/E(R,I)\cong K_1(R,I)$$

und $\mathrm{GL}(R,I)/E(R,I)$ is the Zentrum of $\mathrm{GL}(R)/E(R,I)$. Ferner one has $E(R,I)=[E(R),E(R,I)]=[\mathrm{GL}(R),E(R,I)]$.

Proof. Der Beweis wird ausgelassen. Nicht weil er zu schwierig, sondern weil er zu lang ist. □

Theorem 3.6.9. *Let R a ring and I a ideal. Then there is a exact sequence*

$$K_1(R,I) \xrightarrow{p_{2*}} K_1(R) \xrightarrow{q_*} K_1(R/I) \xrightarrow{\partial} K_0(R,I) \xrightarrow{p_{2*}} K_0(R) \xrightarrow{q_*} K_0(R/I).$$

Proof. Ausgelassen. □

* * *

3.7 $K_2(R)$

Definition 3.7.1. Let R a ring with Eins and let $n \geq 3$ in $\mathbb{N}$. The **Steinberg-group** $\mathrm{St}_n(R)$ is the group generated of allen Symbolen $x_{i,j}(r)$, $1 \leq i, j \leq n$, $i \neq j$, $r \in R$ with den Relationen

(a) $x_{i,j}(r)x_{i,j}(s) = x_{i,j}(r+s)$,

(b) $\left[x_{i,j}(r), x_{k,l}(s)\right] = \begin{cases} 1 & j \neq k, i \neq l, \\ x_{i,l}(rs) & j = k, i \neq l, \\ x_{k,j}(-sr) & j \neq k,\ i = l. \end{cases}$

Remark 3.7.2.

- The elementarmatrizen $e_{i,j}(r)$ erfuellen the o.g. Relationen. Man erhaelt also a groupnhomomorphism

$$\mathrm{St}_n(R) \to E_n(R).$$

- The Steinberggruppe is funktoriell, i.e., for every ringhomomorphism $f : R \to S$ induces $x_{i,j}(r) \mapsto x_{i,j}(f(r))$ a groupn homomorphism $f_* : \mathrm{St}_n(R) \to \mathrm{St}_n(S)$.
- with the offensichtlichen map $\mathrm{St}_n(R) \to \mathrm{St}_{n+1}(R)$ defined man $\mathrm{St}(R) = \lim_n \mathrm{St}_n(R)$ and erhalt a surjektiven groupnhomomorphism $\mathrm{St}(R) \to E(R)$.

Definition 3.7.3. For a ring R sei

$$K_2(R) = \ker\left(\phi : \mathrm{St}(R) \to E(R)\right).$$

Lemma 3.7.4. *$K_2(R)$ is funktoriell in R and es gibt a funktorielle exact sequence*

$$0 \to K_2(R) \to \mathrm{St}(R) \to \mathrm{GL}(R) \to K_1(R) \to 0.$$

Proof. $K_2(R)$ is funktoriell, Since $E(R)$ and $\mathrm{St}(R)$ es are. Der erste Teil the sequence is exakt nach Definiton of $K_2(R)$. Der zweite Teil is exakt nach the Definition of $K_1(R)$ and the Surjektivitaet of $\mathrm{St}(R) \to E(R)$. □

Lemma 3.7.5 (Steinberg)**.** *The group $K_2(R)$ is the Zentrum of* $\mathrm{St}(R)$*. Insbesondere is $K_2(R)$ abelsch.*

Proof. Let z im Zentrum of $\mathrm{St}(R)$. Then commutes z with allen elementsn of $\mathrm{St}(R)$ and $\phi(z)$ commutes with allen elementsn of $E(R)$. Das Zentrum of $\mathrm{GL}(R)$ and the of $E(R)$ are beide trivial, Since beide nur aus Vielfachen the Eins bestehen koennen, the aber nicht in $\mathrm{GL}(R) = \lim_n \mathrm{GL}_n(R)$ liegen. Dawith folgt $z \in \ker(\phi) = K_2(R)$.

Let umgekehrt $y \in K_2(R)$, also $\phi(y) = 1$. Choose n so gross, that y a Darstellung als Produkt of $x_{i,j}(r)$ with $i, j < n$ hat.

We zeigen: y commutes with everym $x_{k,l}(s)$ with $k, l < n$ and $s \in R$.
Let $p = x_{k,n}(s)$ with $k < n$ and $s \in R$. Byden Steinberg-Relationen is $[y, p] \in P_n(R)$, the of den $x_{k,n}(s)$ with $k < n$ and $s \in R$ generatedn Untergruppe. Man macht sich klar, that $\phi|_{P_n(R)}$ injektiv ist. Daher also $[y, p] = 1$, also commutes y with allen $x_{k,n}(s)$ with $k < n$. Analog folgt, that y with allen $x_{n,k}(s)$, $k < n$ commutes. For $k, l < n$ is aber $x_{k,l}(s) = [x_{k,n}(s), x_{n,l}(1)]$. Also commutes y with everym solchen $x_{k,l}(s)$, liegt also im Zentrum of $\mathrm{St}(R)$. □

Remark 3.7.6. Man can show, that

$$K_2(R) \cong H_2(E(R), \mathbb{Z})$$

gilt.

Wieder defined man the relative Theorie als

$$K_2(R, I) = \ker(K_2(D(R, I)) \to K_2(R))$$

und erhaelt a exact sequence

$$\begin{aligned} K_2(R,I) &\xrightarrow{p_{2*}} K_2(R) \xrightarrow{q_*} K_2(R/I) \\ \xrightarrow{\partial} K_1(R,I) &\xrightarrow{p_{2*}} K_1(R) \xrightarrow{q_*} K_1(R/I) \\ \xrightarrow{\partial} K_0(R,I) &\xrightarrow{p_{2*}} K_0(R) \xrightarrow{q_*} K_0(R/I). \end{aligned}$$

* * *

3.8 The +-Konstruktion

Erinnerung. We machen Anleihen in the topology:

- Der **Satz of Hurewicz** besagt, that for a CW-Komplex T the kanonische map $\pi_i(T) \to H_i(T,\mathbb{Z})$ a isomorphism $\pi_1(T)_{ab} \cong H_1(T,\mathbb{Z})$ induces. If ferner $H_1(T,\mathbb{Z}) = 0$, induces sie auch a isomorphism $\pi_2(T) \cong H_2(T,\mathbb{Z})$.
- Zu every group Γ there is a **Klassifizierenden Raum** $B\Gamma$, dieser is bis on Homotopie-Aequivalenz eindeutig bestimmt durch the Eigenschaften:
  (a) $\pi_1(B\Gamma) = \Gamma$,
  (b) the universelle Ueberlagerung $\tilde{B\Gamma}$ is zusammenziehbar.

  Der space $B\Gamma$ can als CW-Komplex vorausgesetzt werden.

a group G is called **perfekt**, if the Abelisierung trivial ist, also wenn $G = [G,G]$ gilt.

Theorem 3.8.1 (Quillen). *Let X a zusammenhaengender CW-Komplex with Basispunkt x_0, the im Nullskelett liegt. Let π a perfekte normle Untergruppe the Fundamentalgruppe $\pi_1(X,x_0)$. Then can man durch hinzufuegen of 2 and 3-Zellen a CW Komplex X^+_π konstruieren so that*

(a) *The map $X \hookrightarrow X^+_\pi$ induces a exact sequence of groupn*

$$1 \to \pi \to \pi_1(X,x_0) \to \pi_1(X^+_\pi,x_0) \to 1.$$

(b) *The inducesen maps the Homologiegruppen*

$$H_i(X, \mathbb{Z}) \to H_i(X^+, \mathbb{Z})$$

is a isomorphism for every $i \geq 0$*.*

Proof. If S a Erzeuger set of π, then liefert every $s \in S$ a Homotopieklasse of maps $\phi : S^1 \to X$. We kleben nun entlang ϕ a 2-Zelle an and tun thatelbe for every $s \in S$. Der so entstehende CW-Komplex Y hat Eigenschaft (a) and hat π_1/π als Fundamentalgruppe. Let $\tilde{X}$ the universelle covering of X and let $\tilde{X} \to Z \to X$ the Zwischenueberlagerung $Z = \tilde{X}/\pi$. Then hat Z the Fundamentalgruppe π, also ist, Since π perfekt ist, $H_1(Z, \mathbb{Z}) = \pi_{ab} = 1$. We erhalten a diagram

$$\begin{array}{ccc} \tilde{X} & & \tilde{Y} \\ \downarrow \pi & \nearrow & \\ Z & & \downarrow \pi_1/\pi \\ \downarrow \pi_1/\pi & & \\ X & \hookrightarrow & Y \end{array}$$

Hier are $\tilde{X}$ and $\tilde{Y}$ the universellen Ueberlagerungen of X and Y. Since $Z \to X$ dieselbe Ueberlagerungsgruppe wie the Fundamentalgruppe of Y hat, liftet the Injektion $X \hookrightarrow Y$ zu einer Injektion $Z \hookrightarrow \tilde{Y}$. Wegen the langen exactn sequence of Raumpaaren reicht es, aus Y durch ankleben of 3-Zellen a space X^+_π zu konstruieren, the $H_i(X^+_\pi, X, \mathbb{Z}) = 0$ erfuellt. Nun is $H_i(Y, X, \mathbb{Z}) = 0$ ausser for $i = 2$, Since Y aus X durch Ankleben of 2-Zellen entsteht. Ausserdem is $H_2(Y, X, \mathbb{Z})$ the freie abelian group generated of den angeklebten 2-Zellen $[e_i^2]$. Nun entsteht aber $\tilde{Y}$ aus Z ebenif durch Ankleben of 2-Zellen, also is $H_2(\tilde{Y}, Z, \mathbb{Z})$ the freie $\mathbb{Z}[\pi_1/\pi]$-module generated of den $[e_i^2]$. Since the Verbindungshomomorphism

$$\partial : H_2(\tilde{Y}, Z, \mathbb{Z}) \to H_1(Z, \mathbb{Z}) = \pi_{ab} = 0$$

trivial ist, unterscheidet sich $H_2(\tilde{Y}, \mathbb{Z})$ of $H_2(Z, \mathbb{Z})$ also durch a direct sum $\bigoplus_{i \in I} \mathbb{Z}[\pi_1/\pi][e_i^2]$.

Since $\tilde{Y}$ einfach zusammenhaengend ist, is the Huerwicz-map a isomorphism $\pi_2(\tilde{Y}) \to H_2(\tilde{Y})$. Choose a $h_i \in H_2$, the on e_i^2 abbildet. Pushe h_i on Y runter and benutze

$h_i : S^2 \to Y$, um a 3-Zelle anzukleben. Das machen we with everym i and erhalten so den space X_π^+, the the verlangten Eigenschaften hat. □

Theorem 3.8.2. *One has*

$$K_1(R) \cong \pi_1\Big(B\,\mathrm{GL}(R)^+_{E(R)}\Big).$$

Proof. Klar nach Hurewicz. □

Definition 3.8.3. (Quillen K-groupn) Let R a ring with Eins. For $i \geq 1$ define wir

$$K_i(R) := \pi_i\Big(B\,\mathrm{GL}(R)^+_{E(R)}\Big).$$

Ergebnisse:

(a) A ringhomomorphism $f : R \to S$ induces Homomorphismen on den K-groupn.

$$f_* : K_j(R) \to K_j(S).$$

(b) For a ring R and a zweiseitiges ideal I defined man

$$K_j(R, I) = \ker(K_j(C(R, I)) \to K_j(R)).$$

Man erhaelt a lange exact sequence

$$\cdots \to K_j(R, I) \to K_j(R) \to K_j(R/I) \to K_{j-1}(R, I) \to \ldots$$

Anwendung: Nur a Beispiel, stellvertretend for viele. If M a kompakte manifold and $f : M \to M$ a glatte function, the die Diagonale nur Transversal schneidet, was soviel bedeutet, that $Df : T_pM \to T_pM$ for every Fixpunkt of f nicht den Eigenwert 1 hat. The Zahl

$$\mathrm{ind}_p(f) = \frac{\det(f - 1 : T_p)}{|\det(f - 1 : T_p)|}$$

misst, in wievielen Richtungen f the Diagonale aufsteigend or absteigend durchquert. The **Lefschetz-formel** besagt:

$$\sum_p \mathrm{ind}_p(f) = \sum_{k=0}^{\dim(M)} (-1)^k \operatorname{tr}\left(H^k(f)\right),$$

where $H^k(f)$ the of f inducese map on the Kohomologie ist. The Zahlen $\mathrm{ind}_p(f)$ deutet man als **sectionzahlen** zwischen the Diagonale Δ and dem Graphen of f. Diese sectionzahlen lassen sich verallgemeinern zu sectionzahlen zwischen zwei Untermannigfaltigkeiten, or in the algebraischen Geometrie zweier Untervarietaeten einer glatten Varietaet X.

Definition 3.8.4. Let X a glatte Varietaet. Let $Z(X)$ the freie abelian group generated of allen Untervarietaeten. Jede Untervarietaet $Y \subset X$ and every rationale function f on Y defined a element in $Z(X)$ durch

$$\text{Nullstellen} - \text{Pole}$$

Let $Z_0(X)$ The Untergruppe generated of all diesen for alle Paare (Y, f).

Der **Chow-ring** $\mathrm{Ch}(X)$ is the group

$$\mathrm{Ch}(X) = Z(X)/Z_0(X),$$

where the ringmultiplikation durch sectionzahlen defined wird. Genauer is called das

$$Y * Z = \sum_{\substack{U \subset Y \cap Z \\ \text{irreduzibel}}} \sigma(Y, Z, U)\, U.$$

Let $\mathrm{Ch}^k(X)$ the Untergruppe generated of allen Untervarietaeten the Codimension k, then liefert the sectionprodukt a map

$$\mathrm{Ch}^k \times \mathrm{Ch}^l \to \mathrm{Ch}^{k+l}.$$

Das bedeutet, that $\mathrm{Ch}(X)$ a graduierter ring ist.

The additive group $\mathrm{Ch}_0(X)$ wird of the Klasse $[X]$ generated and dies is the Eins des rings. $\mathrm{Ch}^1(X)$ besteht aus den Aequivalenzklassen of Divisoren D. If L a Geradenbundle on X and is s a nichttrivialer section, then liefert the Nullstellendivisor of s a element of $\mathrm{Ch}^1(X)$ and dies defined a bijection zwischen $\mathrm{Ch}^1(X)$ and the set the

Isomorphieklassen of Geradenbundlen Pic(X) auch the **Picard-group** genannt. The groupnverknuepfung is hier the Tensorprodukt of Geradenbundlen. Nun is aber

$$\mathrm{Ch}^1(X) \cong \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X),$$

where $\mathcal{O}_X$ the Strukturgarbe ist. Lange Zeit war es unklar, ob the hoeheren Chow-groupn ebenif a kohomologische Interpretation zulassen. Dies Problem wurde of Quillen with the hoeheren K-Theorie geloest. Er zeigte

$$\mathrm{Ch}^k(X) \cong H^k(X, \mathcal{K}_k),$$

where $\mathcal{K}_k$ the Garbe on X ist, the durch Garbifizierung von

$$U \mapsto K_k(\mathcal{O}(U))$$

entsteht.

* * *