Wave Functions for QED

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Problems with QED

- Those inherited from QM, in particular lack of clear ontology.
- UV divergence
- There are field operators, but no particle position operators and thus no Born rule for position.

Working hypothesis

Quantum field theory is essentially relativistic quantum mechanics with particle creation and annihilation.

I'd like to explore how far one can get with this hypothesis, both concerning the unitary time evolution in Hilbert space and a particle ontology. (Alternative: field representation and field ontology.)

Photon wave function

- The covariant wave equation for a spin-1 mass-0 particle is the source-free complex Maxwell equation

$$\partial^\mu F_{\mu\nu} = 0, \quad \partial_{[\lambda} F_{\mu\nu]} = 0\,,$$

where $[\cdots]$ means anti-symmetrization as in $S_{[\mu\nu]} = \frac{1}{2}(S_{\mu\nu} - S_{\nu\mu})$.
- (There is a canonical bijection between *real* $F_{\mu\nu}$ and complex $F_{\mu\nu}$ with only positive frequencies.)
- As in classical electrodynamics, one can write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu} A_{\nu]} = dA$$

with d = exterior derivative of differential forms. The Maxwell eq. then becomes

$$2\partial^\mu \partial_{[\mu} A_{\nu]} = 0\,.$$

Another working hypothesis

Wave functions are physically real as fields of N space-time points.

Questions: What kind of mathematical functions are they?
Which PDEs do they satisfy?

[Landau and Peierls 1930, Lienert and Tumulka 2024]:
Complex $A_\mu(y)$ as photon wave function, Dirac equation for free electrons. Consider m electrons $x_1, \ldots, x_m \in \mathbb{R}^4$ and n photons $y_1, \ldots, y_n \in \mathbb{R}^4$:

$$\Psi^{(m,n)}_{s_1\ldots s_m,\mu_1\ldots\mu_n}(x_1\ldots x_m, y_1\ldots y_n)$$

with $s_j \in \{1,2,3,4\}$ an index for Dirac spin space $\mathbb{C}^4$.

Evolution equations

$$(i\gamma_j^\mu \partial_{x_j,\mu} - m_x)\Psi^{(m,n)}(x_1...x_m, y_1...y_n) = e\sqrt{n+1}\,\gamma_j^\rho\ \Psi^{(m,n+1)}_{\mu_{n+1}=\rho}(x_1...x_m, y_1...y_n, x_j) \quad (1)$$

$$2\partial^\mu_{y_k}\partial_{y_k,[\mu}\Psi^{(m,n)}_{\mu_k=\nu]}(x_1...x_m, y_1...y_n) = \frac{e}{\sqrt{n}}\sum_{j=1}^m \delta^3_\mu(y_k - x_j)\,\gamma_j^\mu\gamma_{j\nu}\ \Psi^{(m,n-1)}_{\widehat{\mu_k}}(x_1...x_m, y_1...y_{k-1}, y_{k+1}...y_n) \quad (2)$$

- But they have too many solutions [Lukas Nullmeier 2024].
- We need to impose a *gauge condition*.
- How to do this while staying invariant under gauge transformations was a breakthrough in 2024.

Classical gauge transformations

$$\widetilde{A}_\mu(y) = A_\mu(y) - \frac{1}{e_x}\partial_\mu \theta(y)$$

$$\widetilde{\psi}_s(x) = e^{i\theta(x)}\psi_s(x)$$

with $\theta : \mathbb{R}^4 \to \mathbb{R}$. Then $\widetilde{F}_{\mu\nu} = F_{\mu\nu}$. For *infinitesimal* change of gauge, replace $\theta(x) \to \theta(x) ds$ with some infinitesimal ds, so

$$\widetilde{A}_\mu(y) = A_\mu(y) - \frac{1}{e_x}\partial_\mu \theta(y)\, ds$$

$$\widetilde{\psi}_s(x) = \psi_s(x) + i\theta(x)\psi_s(x)\, ds$$

Corresponding transformation of Ψ ("first kind"):

$$\breve{\Psi}^{(m,n)}(x_1 ... x_m, y_1 ... y_n) = \Psi^{(m,n)}(x_1 ... x_m, y_1 ... y_n)$$

$$- \frac{1}{e_x\sqrt{n}} \sum_{k=1}^{n} \partial_{\mu_k}\theta(y_k)\, \Psi^{(m,n-1)}_{\widehat{\mu_k}}(\widehat{y_k})\, ds$$

$$+ i \sum_{j=1}^{m} \theta(x_j)\, \Psi^{(m,n)}(x_1 ... x_m, y_1 ... y_n)\, ds\,.$$

We need a second kind of gauge transformation

A solution of the theory is really an equivalence class (to be defined) of pairs $(\Psi, \mathscr{A})$, where

Definition

A gauge condition is a set $\mathscr{A}$ of complex vector fields A_μ that contains exactly one field A_μ for every complex $F_{\mu\nu}$, i.e., such that $d : \mathscr{A} \to \mathscr{F}$ is bijective ($\mathscr{F}$ = space of complex $F_{\mu\nu}$). It is a linear gauge condition iff $\mathscr{A}$ is a $\mathbb{C}$-linear subspace of all complex vector fields.

<u>Ex:</u> Coulomb gauge condition $\partial^1 A_1 + \partial^2 A_2 + \partial^3 A_3 = 0$.

If we want to replace $\mathscr{A}$ by $\widetilde{\mathscr{A}}$, then $\widetilde{d}^{-1} d : \mathscr{A} \to \widetilde{\mathscr{A}}$ maps any A_μ to the $\widetilde{A}_\mu$ with the same $F_{\mu\nu}$. Thus, θ depends linearly on A_μ:

$$\theta(x)\, ds = (\Theta A)(x)\, ds$$

with operator Θ defined by

$$A_\mu - \tfrac{1}{e_x} \partial_\mu (\Theta A)\, ds = \widetilde{d}^{-1} dA \quad \forall A \in \mathscr{A} \,.$$

Gauge transformations of the second kind

$$\widetilde{\Psi}^{(m,n)}(x_1...x_m, y_1..., y_n) = \Psi^{(m,n)}(x_1...x_m, y_1...y_n)$$

$$-\frac{1}{e_x}\sum_{k=1}^{n}\Big[I^{\otimes(m+k-1)} \otimes \partial_{\mu_k}\Theta \otimes I^{\otimes(n-k)}\Big]\Psi^{(m,n)}(x_1...x_m, y_1...y_n)\ ds$$

$$+\, i\sqrt{n+1}\sum_{j=1}^{m}\Big(\Big[I^{\otimes(m+n)} \otimes \Theta\Big]\Psi^{(m,n+1)}\Big)(x_1...x_m, y_1...y_n, x_j)\ ds\,.$$

Proposition [Lienert and Tumulka 2024]

If Ψ satisfies (1), (2) and gauge condition $\mathscr{A}$, then $\widetilde{\Psi}$ satisfies (1), (2), and gauge condition $\widetilde{\mathscr{A}}$.

Proposition [Lienert and Tumulka 2024]

When expressing a finite change of gauge $\mathscr{A} \to \widetilde{\mathscr{A}}$ as a succession of infinitesimal changes of gauge, then the finite transformation depends only on $\mathscr{A}$ and $\widetilde{\mathscr{A}}$, not on the succession (path in the space of gauge conditions) in between.

Open problems

- Confirm agreement with standard QED calculations.
- Probability current for photons?
- Analog for non-Abelian gauge theories (Yang-Mills theories)?
- What about positrons, Dirac sea?
- What about curved space-time?

Thank you for your attention