The Ghirardi-Rimini-Weber Theory of Wave Function Collapse: Principles and Recent Developments

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Plan

- Collapse in Bohmian mechanics
- GRW theory
    - Non-rel. GRW model (1986)
    - Non-rel. GRW model for identical particles (2005)
    - Rel. GRW model for non-interacting, non-identical particles (2004)
    - Rel. GRW model for interacting, non-identical particles (2020)
    - Rel. GRW model for interacting, identical particles (2024)

Collapse in Bohmian mechanics

Definition of Bohmian mechanics

Point particles with positions $\boldsymbol{X}_i(t) \in \mathbb{R}^3$ at time $t \in \mathbb{R}$ move in space according to Bohm's equation of motion

$$\frac{d\boldsymbol{X}_i}{dt} = \frac{\hbar}{m_i} \mathrm{Im} \frac{\nabla_i \Psi\big(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t\big)}{\Psi\big(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N, t\big)}\Bigg|_{\boldsymbol{x}_j = \boldsymbol{X}_j(t) \forall j}, \qquad (1)$$

which can be rewritten as

$$\frac{d\boldsymbol{X}_i}{dt} = \frac{\text{current}}{\text{density}} = \frac{\boldsymbol{j}_i(\boldsymbol{X}_1 \ldots \boldsymbol{X}_N)}{\rho(\boldsymbol{X}_1 \ldots \boldsymbol{X}_N)}$$

with prob. current $\boldsymbol{j}_i = \frac{\hbar}{m_i} \mathrm{Im}[\Psi^* \nabla_i \Psi]$ and prob. density $\rho = \Psi^* \Psi$. The wave function Ψ evolves according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\sum_{i=1}^{N} \frac{\hbar^2}{2m_i} \nabla_i^2 \Psi + V\Psi\,.$$

Initial conditions

We write $X(t) := (\boldsymbol{X}_1(t), \ldots, \boldsymbol{X}_N(t)) =:$ configuration at time t

At the initial time $t = 0$ of the universe, $X(0)$ is random with probability density $\rho(X(0) = x) = |\Psi(x, t = 0)|^2$. In short, $X(0) \sim |\Psi_0|^2$.

Equivariance theorem

If $X(t_0) \sim |\Psi_{t_0}|^2$ for one t_0, then $X(t) \sim |\Psi_t|^2$ for all t.

QM rules for making predictions

- Unitary evolution: The wave function ψ of an isolated system evolves according to the Schrödinger equation, $\psi_t = e^{-iHt/\hbar}\psi_0$.
- Born's rule: When an observer makes a "quantum measurement" of the observable $\mathscr{A}$ associated with the self-adjoint operator A with spectral decomposition $A = \sum_\alpha \alpha P_\alpha$ on a system with wave function ψ, the outcome is the eigenvalue α with probability $\|P_\alpha \psi\|^2 = \langle \psi | P_\alpha \psi \rangle$.
- Collapse rule: After a quantum measurement of $\mathscr{A}$ with outcome α, the wave function gets replaced by

$$\psi_{t+} = \frac{P_\alpha \psi_{t-}}{\|P_\alpha \psi_{t-}\|} \,.$$

Collapse of the wave function in Bohmian mechanics

The wave function Ψ of the universe does not collapse (but evolves according to the Schrödinger equation).
The wave function ψ of a system is the *conditional wave function*

$$\psi(x) = \mathcal{N}\ \Psi(x, Y)$$

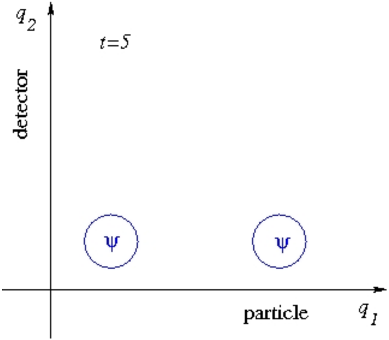
with $\mathcal{N}$ = normalizing constant, x = configuration variable of the system, Y = actual (Bohmian) configuration of the environment.

If x-system and y-system are disentangled, $\Psi(x, y) = \phi(x)\chi(y)$, and don't interact, then the conditional wave function ψ obeys its own Schrödinger eq., but in general it doesn't.

In BM, ψ collapses.

Here is why:

Evolution of Ψ in configuration space of particle + detector:



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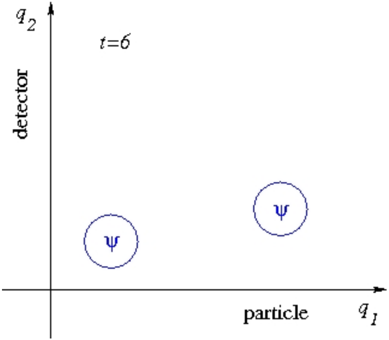

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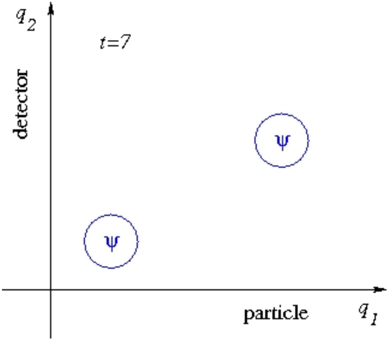
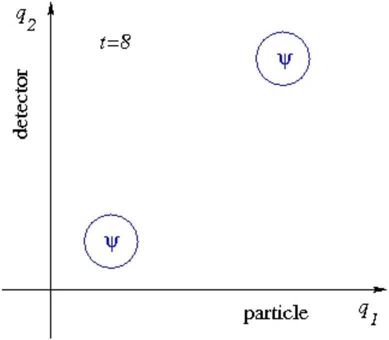


Evolution of Ψ in configuration space of particle + detector:



Collapse of ψ in Bohmian mechanics

- Since the configuration $Q = (X, Y) \sim |\Psi|^2$, Q lies in one of the packets; say, in the upper.
- Conditional on the configuration Y of the detector, $\psi(x)$ is a cross-section of the upper packet. That is, ψ has collapsed.
- Moreover, decoherence occurs: The two packets of Ψ do not overlap in configuration space and will not overlap any more in the future (for the next 10^{100} years). (As usual with macroscopically different packets.)
- As a consequence, $Q = (X, Y)$ will be guided only by the packet of Ψ containing Q (for the next 10^{100} years).
- Thus, ψ will follow the upper packet for the next 10^{100} years.

Measurement process more generally

- ideal quantum measurement of $A = \sum_\alpha \alpha P_\alpha$
- begins at t_0 and ends at t_1
- At t_0, the wave function of object and apparatus is

$$\Psi(t_0) = \psi(t_0) \otimes \phi$$

  with $\psi(t_0) =$ wave function of the object, $\phi =$ ready state of the apparatus.
- By the Schrödinger eq., Ψ evolves to

$$\Psi(t_1) = e^{-iH(t_1 - t_0)}\Psi(t_0) \,.$$

Measurement process, continued

We have that $\Psi(t_0) = \psi(t_0) \otimes \phi$ and $\Psi(t_1) = e^{-iH(t_1-t_0)}\Psi(t_0)$.

Suppose first that the object is in an eigenstate ψ_α of A. Then

$$\Psi_\alpha := \Psi(t_1) = e^{-iH(t_1-t_0)}[\psi_\alpha \otimes \phi]$$

should be a state in which the apparatus displays the value α (e.g., by the position of a needle).

Suppose next that $\psi(t_0) = \sum_\alpha c_\alpha \psi_\alpha$ is an arbitrary superposition. Then

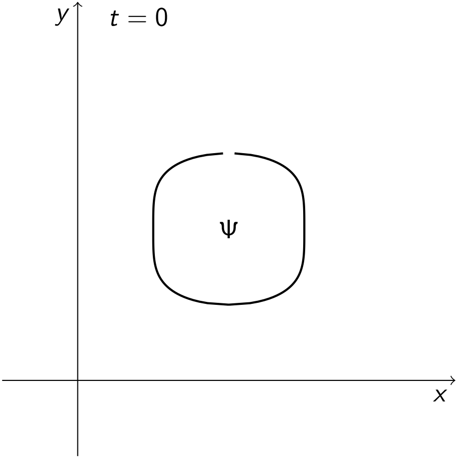
$$\Psi(t_0) = \sum_\alpha c_\alpha \, [\psi_\alpha \otimes \phi]$$

and, by linearity of the Schrödinger eq.,

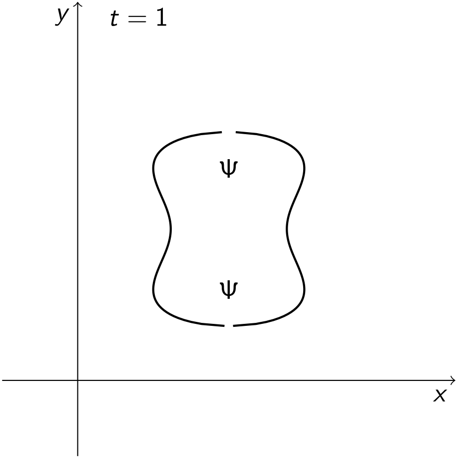
$$\Psi(t_1) = \sum_\alpha c_\alpha \Psi_\alpha \, ,$$

i.e., a superposition of wave functions of apparatuses displaying different outcomes.

Evolution of Ψ in configuration space of system x + apparatus y:



Evolution of Ψ in configuration space of system x + apparatus y:



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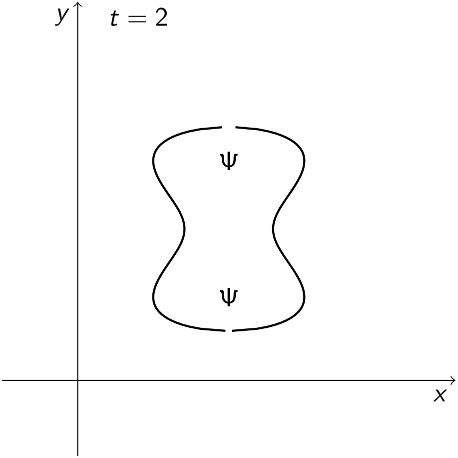
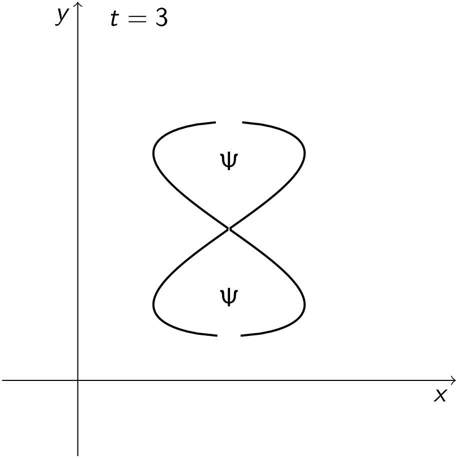


Evolution of Ψ in configuration space of system x + apparatus y:



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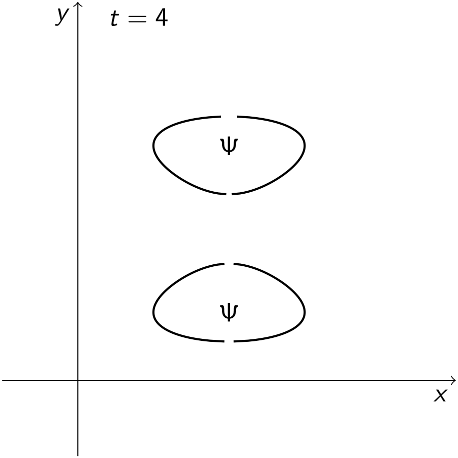

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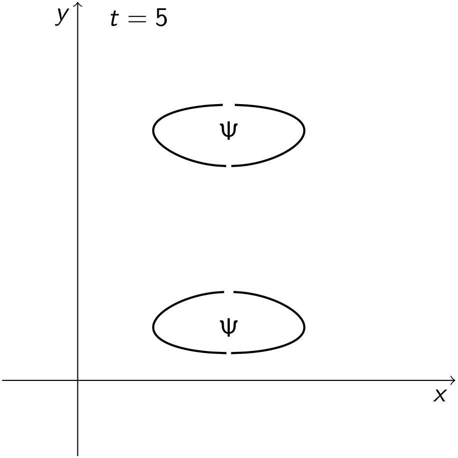

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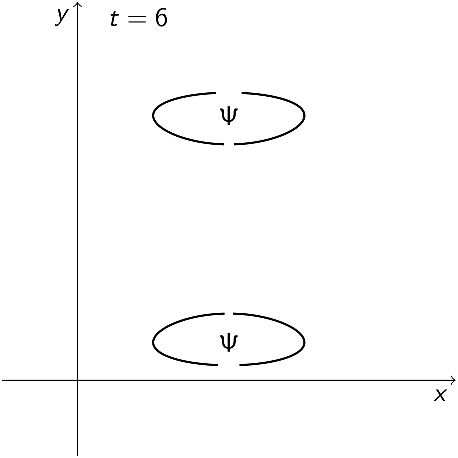


Measurement outcomes in BM

- Y provides the actual position of the needle, and thus the actual outcome $Z = f(Y)$.
- $\text{Prob}(Z = \alpha) = \|\Psi_\alpha\|^2 = |c_\alpha|^2$, in agreement with the rules of QM.
- If $\Psi_\alpha = \psi_\alpha \otimes \phi_\alpha$ for all α (i.e., if the measurement process doesn't change the state of the object), then the cond. wf is $\psi = \psi_\alpha|_{\alpha=Z}$ (collapse to eigenfunction), in agreement with the rules of QM.
- Moreover, by decoherence, also in Ψ the lower packet can henceforth be ignored.

Observers inhabiting a Bohmian universe (made out of Bohmian particles) observe random-looking outcomes of their experiments whose statistics agree with the rules of quantum mechanics for making predictions.

In short, Bohmian mechanics is empirically adequate.

The GRW theory of spontaneous collapse

Spontaneous collapse: GRW theory

Key idea:

The Schrödinger equation is only an approximation, valid for systems with few particles ($N < 10^4$) but not for macroscopic systems ($N > 10^{23}$). The true evolution law for the wave function is non-linear and stochastic (i.e., inherently random) and avoids superpositions (such as Schrödinger's cat) of macroscopically different contributions.

Put differently, regard the collapse of ψ as a physical process governed by mathematical laws.

GianCarlo Ghirardi (1935–2018)

Explicit equations by Ghirardi, Rimini, and Weber (1986), Bell (1987)

The predictions of the GRW theory deviate very very slightly from the quantum formalism. At present, no experimental test is possible.

GRW's stochastic evolution for ψ

- is designed for non-relativistic quantum mechanics of N particles
- meant to replace Schrödinger eq as a fundamental law of nature
- involves two new constants of nature:
  - $\lambda \approx 10^{-16}\,\text{sec}^{-1}$, called collapse rate per particle.
  - $\sigma \approx 10^{-7}$ m, called collapse width.
- Def: ψ evolves as if an observer outside the universe made, at random times with rate $N\lambda$, quantum measurements of the position observable of a randomly selected particle with inaccuracy σ.
- explicitly: Schrödinger evolution interrupted by jumps of the form

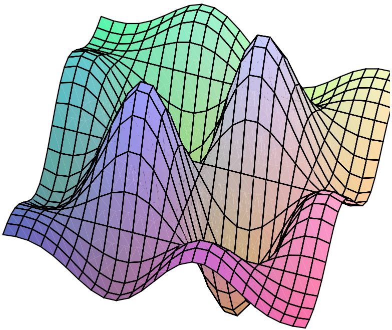
$$\psi_{T+} = \frac{K_i(\boldsymbol{X})\,\psi_{T-}}{\|K_i(\boldsymbol{X})\,\psi_{T-}\|} \quad \text{with} \quad K_i(\boldsymbol{X})\psi = g(\boldsymbol{x}_i - \boldsymbol{X})\,\psi\,,$$

where $g(\boldsymbol{x}) = (2\pi\sigma^2)^{-3/4}\exp(-\frac{\boldsymbol{x}^2}{4\sigma^2})$ is a Gaussian, $i \in \{1 \dots N\}$ is purely random, and $\boldsymbol{X}$ random with

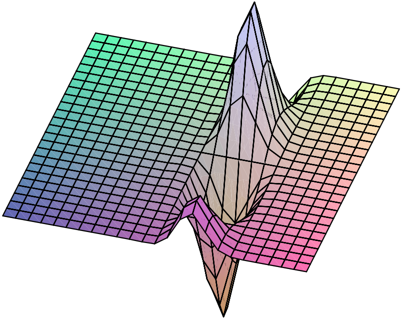
$$\rho(\boldsymbol{X} = \boldsymbol{x}|i) = \|K_i(\boldsymbol{x})\psi_{T-}\|^2 = |\psi_{T-}(\boldsymbol{x}_i = \boldsymbol{x})|^2 * g^2\ .$$

GRW's spontaneous collapse

before the spontaneous collapse:



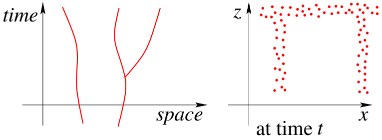
and after:



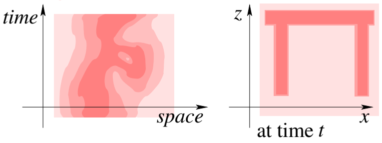
- In Hilbert space: piecewise deterministic stochastic jump process. ψ_t jumps at random times to random destinations.
- For a single particle, one collapse every 100 million years.
- For 10^4 particles, one collapse every 10,000 years.
- For 10^{23} particles, one collapse every 10^{-7} seconds.
- No-signaling theorem
- Measurement situation: As soon as a collapse occurs for one particle in the apparatus, the superposition in the test particle is gone as well.
- It would collapse, up to tails of the Gaussian, to one of the macroscopically distinct wave packets ψ_α.
- The probability that ψ collapses to ψ_α is, up to Gaussian tails, given by $\|\psi_\alpha\|^2$.

Ontology in 3-space

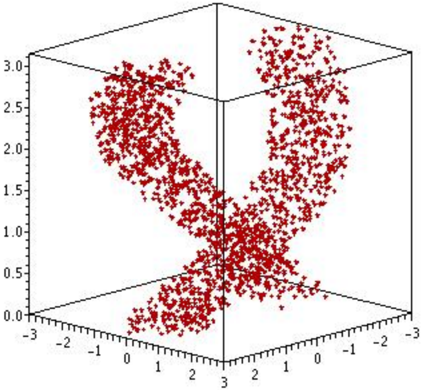
- Suppose a theory T talks a about particles in the literal sense, having world lines in space-time.



- Then we say that T has a particle ontology.
- Examples: Classical mechanics, Bohmian mechanics.
- Now suppose that a theory T' says that matter is continuously distributed in 4D space-time, with density function $m(t, \boldsymbol{x})$ [or $m_\mu(t, \boldsymbol{x})$ or $m_{\mu\nu}(t, \boldsymbol{x})$].



- Then we say that T' has a matter density ontology.



Instead of particle world lines, there are world points in space-time, called "flashes." A macroscopic object consists of a galaxy of flashes.

Laws for the primitive ontology

Def: GRWf [Bell 1987]

If ψ collapses at time T with center $\boldsymbol{X}$ then put a flash at $(T, \boldsymbol{X})$.

Def: GRWm [Diósi 1989; Ghirardi, Grassi, Benatti 1995; Goldstein 1998]

matter is continuously distributed with density given by

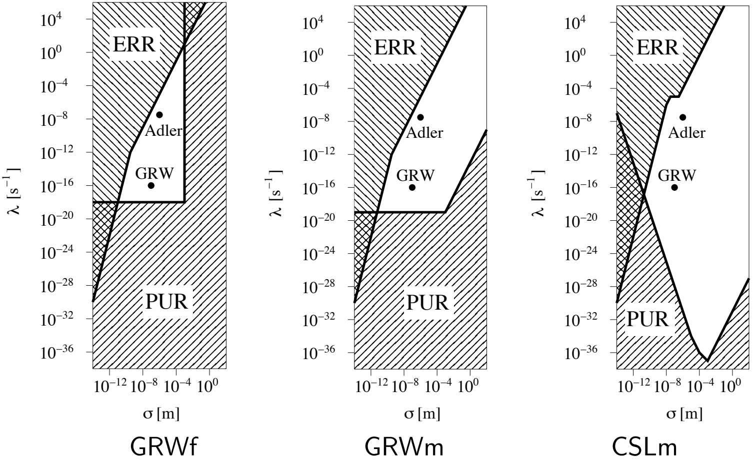
$$m(t,\boldsymbol{x}) = \sum_{i=1}^{N} m_i \int \delta^3(\boldsymbol{x}-\boldsymbol{x}_i)\, |\psi_t(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)|^2 \, d^3\boldsymbol{x}_1 \cdots d^3\boldsymbol{x}_N$$
$$= \langle \psi_t | \mathcal{M}(\boldsymbol{x}) | \psi_t \rangle$$

with $\mathcal{M}(\boldsymbol{x}) = \sum_{i=1}^{N} m_i\, \delta^3(\boldsymbol{x} - \hat{\boldsymbol{X}}_i)$ the mass density operators.

GRWf and GRWm are empirically equivalent.

GRW theories are empirically adequate

Their predictions deviate very very slightly from the quantum formalism.



Parameter diagrams (log-log scale). ERR = empirically refuted region, PUR = philosophically unsatisfactory region [Feldmann, Tumulka 1109.6579]

- But in principle, the predictions of GRW theory can differ from standard QM.
- For example, in a double slit experiment in which it takes the particle 300 million years to travel from the double slit to the screen, the interference pattern would disappear.
- It is not easy to test GRW against standard QM.
- Dramatic energy increase for much smaller σ values than 10^{-7} m
- Slight energy increase for $\sigma = 10^{-7}$ m

Non-relativistic GRW model for identical particles (2005)

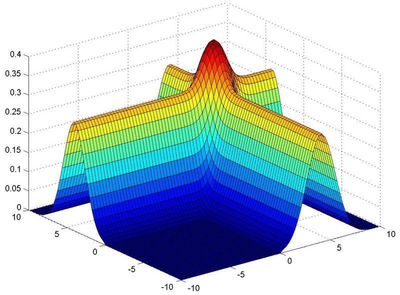
[Tumulka quant-ph/0508230]

GRW collapse for identical particles

$$\psi_{T+} = \frac{K(\boldsymbol{X})\psi_{T-}}{\|K(\boldsymbol{X})\psi_{T-}\|}$$

$$K(\boldsymbol{X})\psi(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N) = \left(\sum_{i=1}^{N} g^2(\boldsymbol{x}_i - \boldsymbol{X})\right)^{1/2} \psi(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N)$$

$$\rho(\boldsymbol{X} = \boldsymbol{x}) = \|K(\boldsymbol{x})\psi_{T-}\|^2$$



Relativistic GRW model for non-interacting particles (rGRW, 2004)

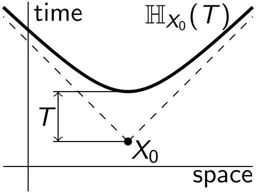
[Tumulka quant-ph/0406094, quant-ph/0602208, 0711.0035]

- works also in curved space-time, described here in Minkowski space-time $\mathbb{M} = \mathbb{R}^4$
- Bohmian mechanics in $\mathbb{M}$ needs preferred foliation of $\mathbb{M}$ into spacelike surfaces, rGRW doesn't.
- Now, rGRW for fixed number N of distinguishable particles
- works also with matter density ontology [Bedingham et al. 1111.1425], described here with flash ontology
- unitary part of evolution: e.g., free Dirac [arising from $L^2(\mathbb{R}^3, \mathbb{C}^4)$]
- with every Cauchy surface Σ there is associated a Hilbert space $\mathscr{H}_\Sigma$
- easier without interaction b/c
  - $U_\Sigma^{\Sigma'} = \otimes_i U_{i\Sigma}^{\Sigma'}$,
  - so propagators for different particles commute,
  - and we can evolve different particles to different surfaces
- need Σ = Cauchy surface or hyperboloid
  ($U_\Sigma^{\Sigma'}$ exists for free Dirac with $m > 0$ [Dürr, Pickl math-ph/0207010])

The rGRW process for $N = 1$

Given: initial wave fct ψ_0 on some 3-surface Σ_0, seed flash $X_0 \in \mathbb{M}$

Randomly select next flash $X \in \mathbb{M}$:

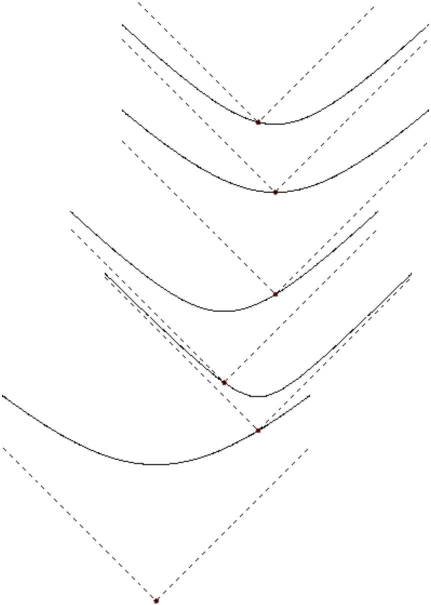


- Randomly select waiting time $T \sim \text{Exp}(\lambda)$, T = proper time between X_0 and X, i.e., $X \in \mathbb{H}_{X_0}(T)$
- Evolve $\psi_0 \to \psi_\Sigma$ from Σ_0 to $\Sigma = \mathbb{H}_{X_0}(T)$.
- Randomly select $X \in \Sigma$ with probability density $|\psi_\Sigma|^2 * g^2$, where $*$ = convolution and g the Gaussian on Σ

$$g(z) = \mathcal{N} \exp\Big(-\frac{\text{dist}_\Sigma(x,z)^2}{4\sigma^2}\Big),$$

$\text{dist}_\Sigma(x,z)$ = spacelike dist. from x to z along Σ, normalization $\int_\Sigma d^3x\, g_x^2(z) = 1$.

The rGRW process for $N = 1$



Repeat with

ψ_0 replaced by $\dfrac{g_X \psi_\Sigma}{\|g_X \psi_\Sigma\|}$

and X_0 by X.

The rGRW process for $N = 1$

It follows from the definition that the joint distribution of the first n flashes is of the form

$$\mathbb{P}\Big((X_1, \ldots, X_n) \in B\Big) = \langle\psi_0|G(B)|\psi_0\rangle, \qquad B \subseteq (\mathbb{R}^4)^n$$

where $\psi_0 \in L^2(\Sigma_0)$, and G is a positive-operator-valued measure (POVM).

The rGRW process for $N > 1$

Let the joint probability distribution of the first n_1 centers for particle 1, ..., the first n_N flashes for particle N be

$$\mathbb{P}\Big((X_{11}, \ldots, X_{n_N,N}) \in B\Big) = \langle\psi_0|G^{(N)}(B)|\psi_0\rangle, \quad B \subseteq (\mathbb{R}^4)^{n_1+\ldots+n_N}$$

where $\psi_0 \in L^2(\Sigma_0)^{\otimes N}$, and $G^{(N)}$ is the product POVM defined by

$$G^{(N)}(B_1 \times \cdots \times B_N) = G(B_1) \otimes \cdots \otimes G(B_N).$$

Explicit form of distribution

- $X_{ik} \in \mathbb{M}$ is the k-th flash of i-th particle
- $\mathbb{H}_{ik} := \mathbb{H}_{X_{ik-1}}(|X_{ik} - X_{ik-1}|)$
- consider n_i flashes for i-th particle
- $\underline{X} = (X_{ik} : 1 \leq i \leq N, 1 \leq k \leq n_i), \qquad d\underline{x} = \prod_{i=1}^{N} \prod_{k=1}^{n_i} d^4 x_{ik}$

$$\mathbb{P}(\underline{X} \in d\underline{x}) = \langle \psi_0 | D(\underline{x}) | \psi_0 \rangle \, d\underline{x} \quad \text{with}$$

$$D(\underline{x}) := \left(\prod_{i=1}^{N} \prod_{k=1}^{n_i} 1_{x_{ik} \in \text{future}(x_{ik-1})} \lambda e^{-\lambda |x_{ik} - x_{ik-1}|} \right) L(\underline{x})^\dagger \, L(\underline{x})$$

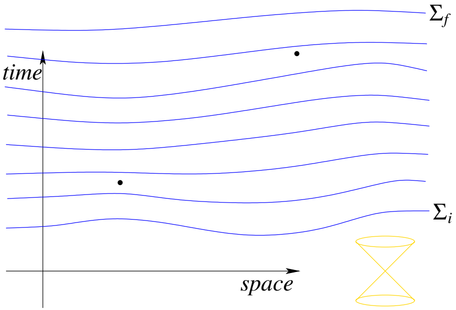
$$L(\underline{x}) := \bigotimes_{i=1}^{N} \prod_{k=1}^{n_i} K(x_{ik}), \qquad K(x_{ik}) := U^{0}_{i\mathbb{H}_{ik}} \, P(g_{x_{ik-1} x_{ik}}) \, U^{\mathbb{H}_{ik}}_{i0}$$

P = multiplication operator, g_{yx} = Gaussian centered at $x \in \mathbb{H}_y(s)$

- Key fact: $\int_{\mathbb{M}^\nu} d\underline{x} \, D(\underline{x}) = I$

ψ_Σ

- We have defined the joint distribution of the flashes.
- random wave function ψ_Σ:
- If the flashes X_{ik} up to Σ are given, ψ_Σ is determined by the initial $\psi_0 \in \mathscr{H}_{\Sigma_0}$: Roughly speaking, collapse ψ at every flash and evolve ψ unitarily in-between.



Relativistic GRW model for interacting particles (2020)

[Tumulka 2002.00482]

Interacting rGRW model

- fixed number N of distinguishable particles
- works also in curved space-time, described here in Minkowski space-time $\mathbb{M} = \mathbb{R}^4$
- works also with matter density ontology [Bedingham et al. 1111.1425], described here with flash ontology
- still has the form $\mathbb{P}(\underline{X} \in d\underline{x}) = \langle\psi_0|D(\underline{x})|\psi_0\rangle\, d\underline{x}$
- still need to make sure that $\displaystyle\int_{\mathbb{M}^\nu} d\underline{x}\, D(\underline{x}) = I$
- non-relativistic limit = known GRW with interaction
- non-interacting case ≈ known 2004 model
- regard the unitary part $U_\Sigma^{\Sigma'}$ as given and including the interaction
- still of the form

$$D(\underline{x}) = \Bigl(\prod_{i=1}^{N}\prod_{k=1}^{n_i} 1_{x_{ik}\in\text{future}(x_{ik-1})}\lambda e^{-\lambda|x_{ik}-x_{ik-1}|}\Bigr) L(\underline{x})^\dagger\, L(\underline{x})$$

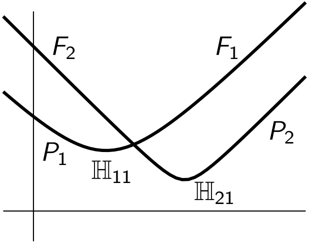
Difficulty

- We want something like

$$\text{“}\quad L(\underline{x}) = \prod_{i=1}^{N} \prod_{k=1}^{n_i} U^0_{\mathbb{H}_{ik}} \, P_{\mathbb{H}_{ik}}(g_{ik}) \, U^{\mathbb{H}_{ik}}_0 \quad\text{”} \tag{2}$$

  with $g_{ik}(z_1 ... z_N) := g_{x_{ik-1} x_{ik}}(z_i)$ on $\mathbb{H}^N_{ik}$.
- But now $P(g_{ik})$ don't commute for different i. Problem of operator ordering.
- Rough idea:
  - when $x_{j\ell} \in \text{future}(x_{ik})$, put $P(g_{j\ell})$ left of $P(g_{ik})$
  - when $x_{j\ell}$ spacelike from x_{ik}, maybe $P(g_{j\ell})$ commutes with $P(g_{ik})$?
- Don't actually commute because even if $x_{j\ell}$ spacelike from x_{ik}, support$(g_{j\ell})$ is not spacelike from support(g_{ik}).
- Idea: cut off g_{ik} to get better control of support.

Cut off Gaussians



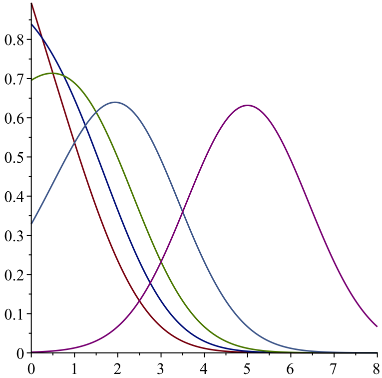
- Previously, $\int_{\mathbb{H}} d^3x\, g_{yx}^2(z) = 1$.
- Now subdivide $\mathbb{H}_{ik}$ in pieces $=$ past/future of $\mathbb{H}_{j\ell}$.
- For each piece $A \subset \mathbb{H}_{ik}$, define cut-off Gaussian g_A so that $\int_A d^3x\, g_{yAx}(z)^2 = 1_{z\in A}$.

Refined way of cutting off the Gaussian:

$$g_{yAx}(z) := 1_{z\in A}\, 1_{x\in A}\, \big\|\text{Gaussian}_{yz} 1_A\big\|^{-1}\, \text{Gaussian}_{yx}(z) \qquad (3)$$

Cut off Gaussians

$$g_{yAx}(z) := 1_{z\in A}\, 1_{x\in A} \left\| \mathrm{Gaussian}_{yz} 1_A \right\|^{-1} \mathrm{Gaussian}_{yx}(z) \qquad (3)$$

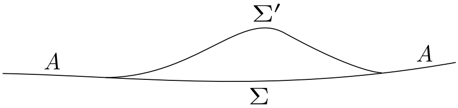


Deviation from Gaussian shape, here on $\mathbb{R}$ instead of $\mathbb{H}$ with $A = [0, \infty)$.

Assumptions

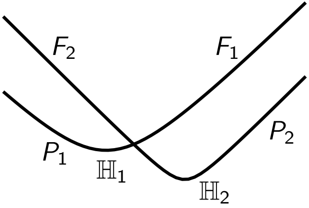
- $U_\Sigma^{\Sigma'} : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$
- P_Σ PVM on Σ^N acting on $\mathscr{H}_\Sigma$
- **Interaction locality** (IL): No interaction at spacelike separation. More precisely [Lienert, Tumulka 1706.07074], we need this slightly weaker statement:
  For any set $A \subseteq \Sigma \cap \Sigma'$ in the overlap and any $i \in \{1, \ldots, N\}$,

$$P_{\Sigma'}\Big((\Sigma')^{i-1} \times A \times (\Sigma')^{N-i-1}\Big) = U_\Sigma^{\Sigma'} P_\Sigma\Big(\Sigma^{i-1} \times A \times \Sigma^{N-i-1}\Big) U_{\Sigma'}^{\Sigma}$$



The case of two flashes

$$L(x_1,x_2) := \begin{cases} U^0_{\mathbb{H}_2}\, P_{\mathbb{H}_2}(g_{y_2 P_2 x_2 2})\, U^{\mathbb{H}_2}_{\mathbb{H}_1}\, P_{\mathbb{H}_1}(g_{y_1 P_1 x_1 1})\, U^{\mathbb{H}_1}_0 & \text{if } x_1 \in P_1, x_2 \in P_2 \\ U^0_{\mathbb{H}_2}\, P_{\mathbb{H}_2}(g_{y_2 F_2 x_2 2})\, U^{\mathbb{H}_2}_{\mathbb{H}_1}\, P_{\mathbb{H}_1}(g_{y_1 P_1 x_1 1})\, U^{\mathbb{H}_1}_0 & \text{if } x_1 \in P_1, x_2 \in F_2 \\ U^0_{\mathbb{H}_1}\, P_{\mathbb{H}_1}(g_{y_1 F_1 x_1 1})\, U^{\mathbb{H}_1}_{\mathbb{H}_2}\, P_{\mathbb{H}_2}(g_{y_2 P_2 x_2 2})\, U^{\mathbb{H}_2}_0 & \text{if } x_1 \in F_1, x_2 \in P_2 \\ U^0_{\mathbb{H}_1}\, P_{\mathbb{H}_1}(g_{y_1 F_1 x_1 1})\, U^{\mathbb{H}_1}_{\mathbb{H}_2}\, P_{\mathbb{H}_2}(g_{y_2 F_2 x_2 2})\, U^{\mathbb{H}_2}_0 & \text{if } x_1 \in F_1, x_2 \in F_2. \end{cases}$$

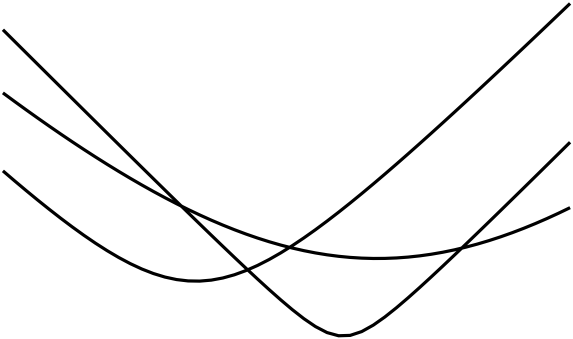


Key theorem

$$\text{(IL)} \Rightarrow \int d^4x_1 \int d^4x_2\, D(x_1,x_2) = I$$

The general case

Division into 4-cells and 3-cells



Admissible sequences

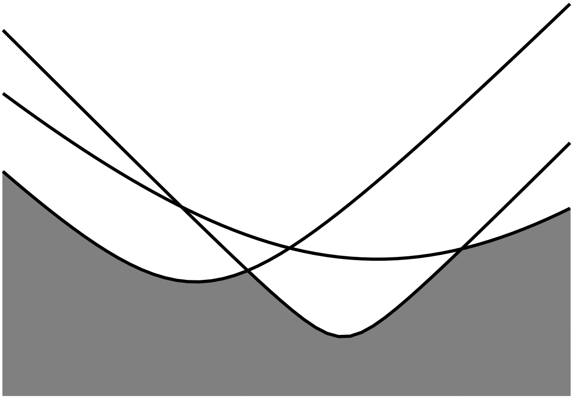
Def: $S \subseteq \mathbb{M}$ is *past complete* $\Leftrightarrow$ past$(S) \subseteq S$

Fact: $S \neq \mathbb{M}$ past complete iff $S =$ past(∂S)

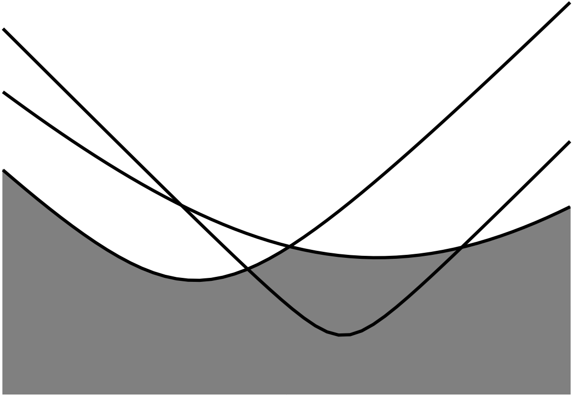
Def: *admissible sequence*: $\big({}^4C_1, \ldots, {}^4C_r\big)$ such that ${}^4C_1 \cup \ldots \cup {}^4C_r = \mathbb{M}$, no repetitions, and for every $n = 1, \ldots, r$, ${}^4C_1 \cup \ldots \cup {}^4C_n$ is past complete.

Proposition: There exist admissible sequences.

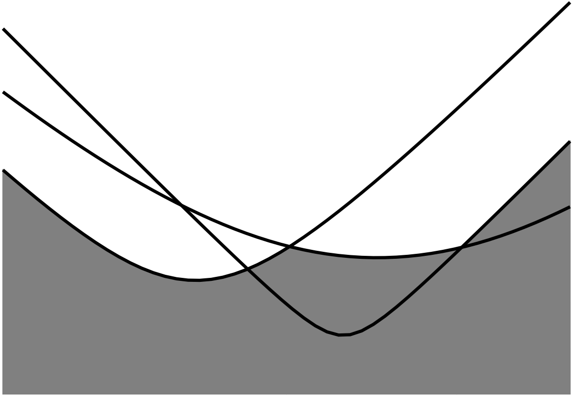
Example of an admissible sequence



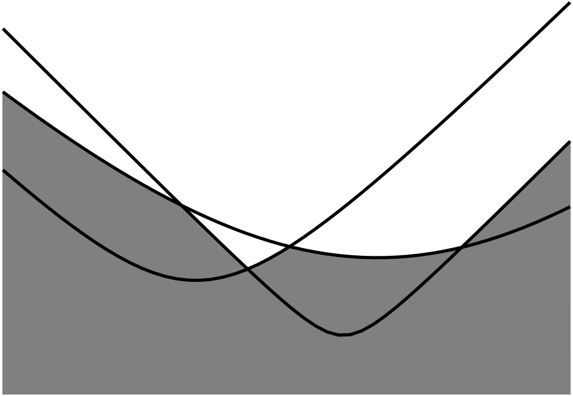
Example of an admissible sequence



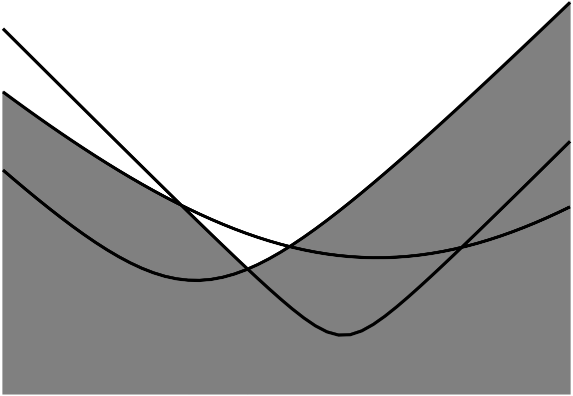
Example of an admissible sequence



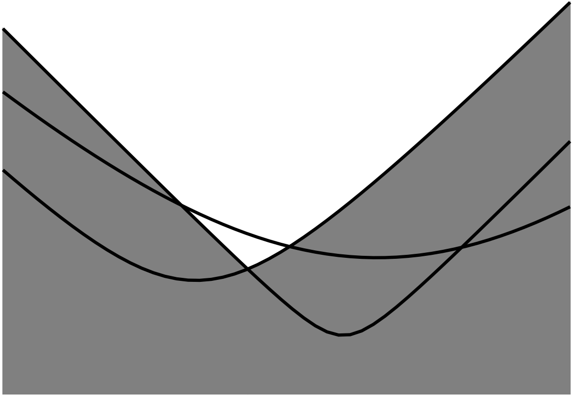
Example of an admissible sequence



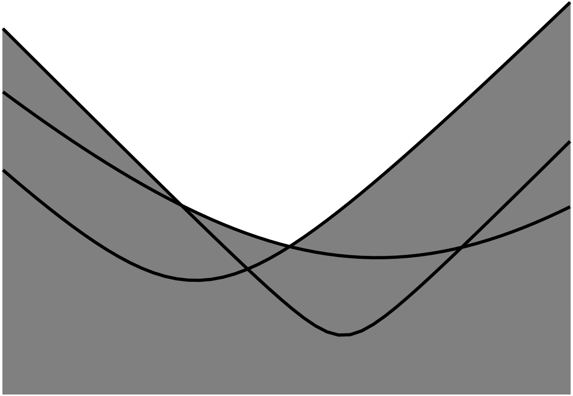
Example of an admissible sequence



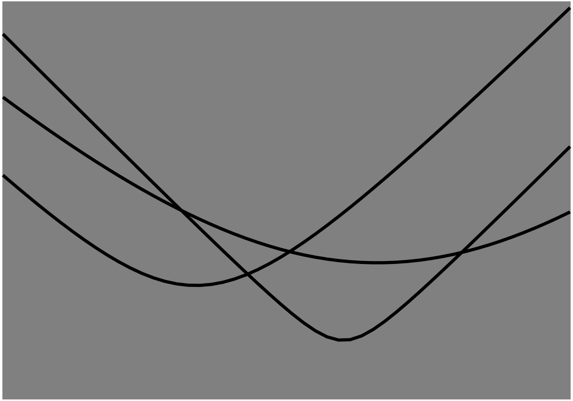
Example of an admissible sequence



Example of an admissible sequence



Example of an admissible sequence



Proposition: Every admissible sequence crosses every 3-cell exactly once.

Since $x_{ik} \in \mathbb{H}_{ik}$, it lies in some 3-cell ${}^3C(x_{ik})$. Set

$$K(x_{ik}) := U^0_{\mathbb{H}_{ik}} \, P_{\mathbb{H}_{ik}}\big(g_{x_{ik-1},{}^3C(x_{ik}),x_{ik}}(z_i)\big) \, U_0^{\mathbb{H}_{ik}}.$$

Given an admissible sequence AS, define

$$L(\underline{x}) = \prod_{ik} K(x_{ik})$$

in the order from right to left in which the 3-cells are crossed in AS.

Proposition: When two 3-cells are crossed in the same step, and if (IL) holds, then their K operators commute. Thus, AS unambiguously defines the product $L(\underline{x})$.

Proposition: If (IL) holds, then any two admissible sequences lead to the same operator $L(\underline{x})$. Thus, $L(\underline{x})$ is unambiguously defined.

Key theorem

(IL) $\Rightarrow \int_{\mathbb{M}^\nu} d\underline{x}\, D(\underline{x}) = I$

Properties

- Non-local
- Size of 3-cells: back-of-envelope estimate 10^{-3} m (no problem)
- Stochastic evolution of ψ_Σ: similarly as before
- Non-interacting special case $\approx$ 2004 model (exact if we replace cut-off Gaussians by Gaussians; tiny change if size of 3-cell $= 10^4\,\sigma$)
- Microscopic parameter independence (i.e., joint distribution of flashes before Σ is independent of external fields after Σ): holds approximately.
- No superluminal signaling (follows from microscopic parameter independence): holds approximately
- Non-relativistic limit ("$c \to \infty$") $=$ GRW 1986
  hyperboloid $\to$ horizontal 3-plane
  3-cell $\to$ horizontal 3-plane
  4-cell $\to$ layer between horizontal 3-planes
  cut-off Gaussian $\to$ Gaussian
  there is only 1 admissible sequence

Rel. GRW model for interacting, identical particles

[Kaimal and Tumulka 2024]

Number function on Σ^N: $n_{A,\Sigma}(q) = \#(q \cap A)$.
Symmetrized cut-off Gaussian

$$f_{yAx}(z_1, \ldots, z_N) = \Big(\sum_{i=1}^{N} g^2_{yAx}(z_i)\Big)^{1/2}$$

has the property $\int_A d^3x\, f^2_{yAx}(q) = n_{A,\Sigma}(q)$ for $A \subseteq \Sigma$. Set

$$K\left(x_{ik}\right) := U^0_{\mathbb{H}_{ik}}\, P_{\mathbb{H}_{ik}}\left(f_{x_{ik-1},{}^3C(x_{ik}),x_{ik}}\right) U^{\mathbb{H}_{ik}}_0$$

$$L(\underline{x}) := N^{-(n_1+\ldots+n_N)/2} \prod_{i=1}^{N} \prod_{k=1}^{n_i} K(x_{ik})$$

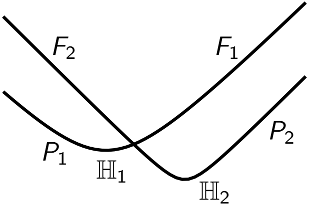
in the order (right to left) of an admissible sequence.

Proposition: Assume (IL). When two 3-cells are crossed in the same step, then their K operators commute. Any two admissible sequences lead to the same operator $L(\underline{x})$. Thus, $L(\underline{x})$ is unambiguously defined.

Key theorem: (IL) $\Rightarrow$ $\int_{\mathbb{M}^\nu} d\underline{x}\, D(\underline{x}) = I$

The case of two flashes

$$L(x_1,x_2) := \begin{cases} \frac{1}{2}\, U^0_{\mathbb{H}_2} P_{\mathbb{H}_2}(f_{y_2 P_2 x_2}) U^{\mathbb{H}_2}_{\mathbb{H}_1} P_{\mathbb{H}_1}(f_{y_1 P_1 x_1}) U^{\mathbb{H}_1}_0 & \text{if } x_1 \in P_1, x_2 \in P_2 \\ \frac{1}{2}\, U^0_{\mathbb{H}_2} P_{\mathbb{H}_2}(f_{y_2 F_2 x_2}) U^{\mathbb{H}_2}_{\mathbb{H}_1} P_{\mathbb{H}_1}(f_{y_1 P_1 x_1}) U^{\mathbb{H}_1}_0 & \text{if } x_1 \in P_1, x_2 \in F_2 \\ \frac{1}{2}\, U^0_{\mathbb{H}_1} P_{\mathbb{H}_1}(f_{y_1 F_1 x_1}) U^{\mathbb{H}_1}_{\mathbb{H}_2} P_{\mathbb{H}_2}(f_{y_2 P_2 x_2}) U^{\mathbb{H}_2}_0 & \text{if } x_1 \in F_1, x_2 \in P_2 \\ \frac{1}{2}\, U^0_{\mathbb{H}_1} P_{\mathbb{H}_1}(f_{y_1 F_1 x_1}) U^{\mathbb{H}_1}_{\mathbb{H}_2} P_{\mathbb{H}_2}(f_{y_2 F_2 x_2}) U^{\mathbb{H}_2}_0 & \text{if } x_1 \in F_1, x_2 \in F_2. \end{cases}$$



Key theorem

$$(\text{IL}) \Rightarrow \int d^4x_1 \int d^4x_2\, D(x_1,x_2) = I$$

Open problem

- How to set up a GRW model for an H with particle creation

Thank you for your attention