Typicality Reasoning in Quantum Statistical Mechanics

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PSA meeting New Orleans
Session on typicality chaired by Isaac Wilhelm
17 November 2024

Main reference: `http://arxiv.org/abs/2210.10018`

The general concept of typicality

What is a typicality statement?

A statement asserting that "most" things A have a property P.

Example

Most numbers in [0,10] have "random-looking" sequence of decimal digits. (Even if they are not random, for example π.)

Example

Most points X on the unit sphere in $\mathbb{R}^n$ with large n have components $(X_1, X_2, ..., X_n)$ with statistical distribution close to Gaussian distribution with mean 0 and width $1/\sqrt{n}$.

Applications in physics

A = phase point $X = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N, \boldsymbol{P}_1, \ldots, \boldsymbol{P}_N)$ in classical mechanics
A = wave function Ψ in quantum mechanics
A = configuration $Q = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$ in Bohmian mechanics
A = Hamiltonian H (random matrix theory) [von Neumann 1929, Wigner 1955]

What's the measure?

- intuitively: uniform, natural, invariant under symmetries or time evolution; "volume" or "size"
- Usually given by the mathematical concept of "measure" (a number ≥ 0 for every set, like a probability distribution), but maybe some axioms (additivity?) can be relaxed.
- Sometimes uniform over all possibilities, sometimes only over admitted possibilities.
  Example: in classical statistical mechanics, uniform over a small region Γ_ν in phase space, corresponding to macro state ν.

Why is typicality relevant in statistical mechanics?

1. Because that is what thermal equilibrium means [e.g., X or Ψ]
2. Because of the past hypothesis (or other laws of nature) [e.g., X or Ψ or Q]
3. Because the typical behavior needs no further principles for its explanation [e.g., H or Q]
4. Because the typical behavior is our first guess [e.g., H]

1) Thermal equilibrium in classical mechanics

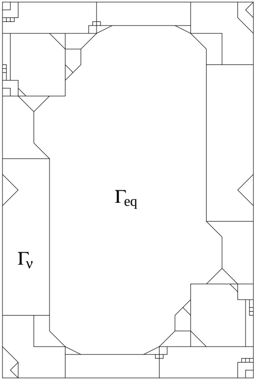
- energy shell in phase space Γ:
  $\Gamma_{\mathrm{mc}} = \{X \in \Gamma : E - \Delta E < H(X) \leq E\}$
- partition Γ_{mc} into "macro sets" Γ_ν corresponding to different macro states ν,

$$\Gamma_{\mathrm{mc}} = \bigcup_\nu \Gamma_\nu$$

- usually, one cell Γ_{eq} has most of the volume,

$$\frac{\mathrm{vol}\,\Gamma_{\mathrm{eq}}}{\mathrm{vol}\,\Gamma_{\mathrm{mc}}} \approx 1.$$

- Def: A system is in thermal equilibrium :⇔ its phase point lies in the set Γ_{eq}.



It is the nature of thermal equilibrium that most phase points in Γ_{mc} look macroscopically the same.

1) Thermal equilibrium in quantum mechanics

[Goldstein et al. Physical Review E and arxiv.org/abs/0911.1724]

- $H = \sum_\alpha E_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$
- energy shell $\mathscr{H}_{\mathrm{mc}} = \mathrm{span}\{\phi_\alpha : E - \Delta E < E_\alpha \leq E\}$
- orthogonal decomposition into subspaces $\mathscr{H} = \bigoplus_\nu \mathscr{H}_\nu$ ("macro spaces," each of high dimension) [von Neumann 1929, Lebowitz 1993]
- notation $P_\nu :=$ projection operator to $\mathscr{H}_\nu$
- usually, one of the $\mathscr{H}_\nu$ has most dimensions, "$\nu = \mathrm{eq}$":

$$\frac{\dim \mathscr{H}_{\mathrm{eq}}}{\dim \mathscr{H}_{\mathrm{mc}}} \approx 1$$

<u>Def:</u> $\psi \in$ macroscopic thermal equilibrium (MATE) $:\Leftrightarrow \quad \|P_{\mathrm{eq}}\psi\|^2 \approx 1$

Fact: Most ψ lie in MATE.

$u_{\mathrm{mc}}(\text{MATE}) \approx 1$
with u_{mc} the uniform normalized measure on the unit sphere $\mathbb{S}(\mathscr{H}_{\mathrm{mc}})$.

2) The past hypothesis as demanding typicality

[Albert 2000, Goldstein et al. 1903.11870]

Past hypothesis (maybe a law of nature?)

(classical) The initial phase point X_0 of the universe lies in a certain subset Γ_0 of the phase space Γ of the universe and is typical in Γ_0.

(quantum) The initial wave function Ψ_0 of the universe lies in a certain subspace $\mathscr{H}_0$ of the Hilbert space $\mathscr{H}$ of the universe and is typical in $\mathbb{S}(\mathscr{H}_0)$.

- Suggestion: $\Gamma_0 = \Gamma_{\nu_0}$ for a certain low-entropy macro state ν_0. Or [Penrose 1979] $\Gamma_0 = \{\text{states with zero Weyl curvature}\}$. Similarly $\mathscr{H}_0$.
- Boltzmann's insight: Most $X \in \Gamma_{\nu_0}$ evolve to higher and higher entropy $S(X_t) = k_B \log \text{vol}\, \Gamma_{\nu(X_t)}$. This explains the thermodynamic arrow of time (2nd law).
- "is typical" = looks as if random (for our purposes here: can be taken to be random)

2) The universe: Typicality vs probability

- The usual idea of “probability” involves the possibility to repeat.
- Thus, for things we see only once (e.g., the universe as a whole), “typicality” is the more fitting concept, also because the typical thing A need not be “truly random” (think of π).
- If we can repeat, then we may be able to observe whether the actual distribution is uniform.
  If we can't repeat, we may only have theoretical reasons for selecting a measure.

Three recent typicality theorems in quantum statistical mechanics

Result 1: Distribution typicality

- For any experiment with random outcome Z on a quantum system with wave function Ψ, $\mathbb{P}(Z=z) = \langle\Psi|E_z|\Psi\rangle$ for some POVM E_z.
- POVM = positive-operator-valued measure
  = "unsharp observable"
- E_z positive operators with $\sum_z E_z = I$
- special case "ideal observable": E_z = projection operator

Theorem [Reimann 0810.3092, Teufel & Tumulka & Vogel 2307.15624]

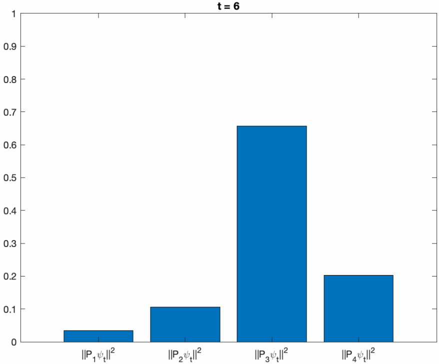
Suppose $\dim \mathscr{H}_0$ is large, and the number of possible z's is not too large. Then most $\Psi \in \mathbb{S}(\mathscr{H}_0)$ have nearly the same $\langle\Psi|E_z|\Psi\rangle$ (for all z).

Consequence for the past hypothesis (PH) [Chen & Tumulka 2410.16860]

PH $\Rightarrow$ empirical observations can't distinguish between different Ψ's. Empirical observations yield almost zero information about the actual Ψ.

Result 2: Macroscopic appearance

- Generic Ψ are superpositions of contributions $P_\nu \Psi$ from several macro spaces $\mathscr{H}_\nu$.
- To describe the macroscopic appearance of Ψ, we say how big the contribution from each $\mathscr{H}_\nu$ is, $\|P_\nu \Psi\|^2$.



- <u>Def:</u> macro history $(\nu, t) \mapsto \|P_\nu \Psi_t\|^2$

Result 2: Dynamical typicality

[Bartsch & Gemmer 0902.0927, Reimann 1805.07085]

- Most $\Psi_0 \in \mathbb{S}(\mathscr{H}_0)$ have nearly the same macro history for a long time.

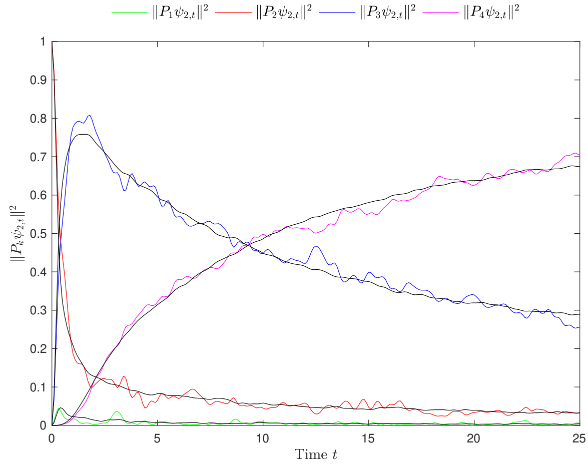
Theorem [Teufel & Tumulka & Vogel 2210.10018]

Fix $T > 0$, and let $\dim \mathscr{H}_0$ be sufficiently large. Then for every $t \in [0, T]$, most $\Psi_0 \in \mathbb{S}(\mathscr{H}_0)$, and all ν,

$$\|P_\nu \Psi_t\|^2 \approx \mathbb{E}_{\Psi_0} \|P_\nu \Psi_t\|^2 .$$

A kind of macroscopic determinism.

Result 2: Numerical example



Result 3: Fraction equilibrium = generalized normal equilibrium

- Most $\Psi \in \mathbb{S}(\mathscr{H})$ have $\|P_\nu \Psi\|^2 \approx d_\nu / d$ (the "normal histogram").
- Von Neumann 1929 proposed to take this as the definition of thermal equilibrium. But it is not really a *thermal* equilibrium, it is a different kind of equilibrium ("normal equilibrium").
- But it tends to occur in the long run:

Theorem on normal equilibrium [von Neumann 1929, Goldstein et al. 0907.0108]

For most orthonormal bases B, if H has B as its eigenbasis (and under some technical conditions), every $\Psi \in \mathbb{S}(\mathscr{H})$ satisfies for most $t \in [0, \infty)$ that

$$\|P_\nu \Psi_t\|^2 \approx \frac{d_\nu}{d} \,.$$

Result 3

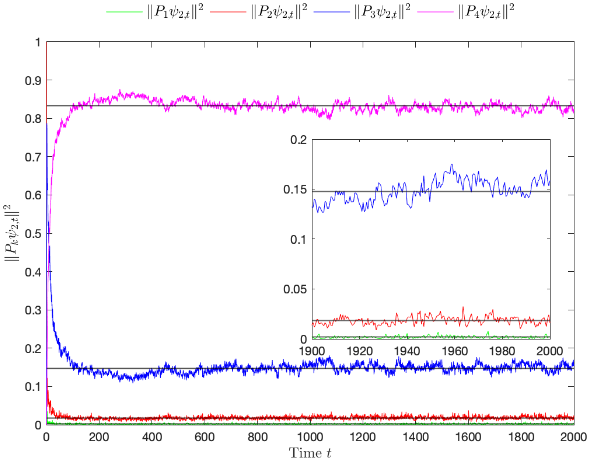
Let us move away from the unrealistic assumption that B is uniformly distributed: Every initial macro state ν_0 has a typical long-time histogram:

Theorem on fraction equilibrium [Teufel & Tumulka & Vogel 2210.10018]

Consider any fixed non-random H (under some technical assumptions) and suppose $\dim \mathscr{H}_0$ is large. Then most $\Psi_0 \in \mathbb{S}(\mathscr{H}_0)$ are such that for most $t \in [0, \infty)$

$$\|P_\nu \Psi_t\|^2 \approx \mathbb{E}_{\Psi_0} \mathbb{E}_t \|P_\nu \Psi_t\|^2 \,.$$

Result 3: Numerical example



Typical Ψ vs typical H

- Reasons 1) and 2) [thermal equilibrium and laws] vs reasons 3) and 4) [explanation and guessing]
- Typical $\Psi \in \mathbb{S}(\mathscr{H}_{\mathrm{mc}})$ is empirically wrong since the universe is not in thermal equilibrium. But typical $\Psi \in \mathbb{S}(\mathscr{H}_0)$ (PH) is right, as far as we can tell.
- H with typical eigenbasis B is empirically wrong:
    - ultrafast thermalization [Goldstein et al. 1307.0572]
    - violates local conservation of particle number/ kinetic energy/momentum/angular momentum
    - super-long distance interaction (realistic: $1/r$ potential)
    - super-many particle interaction (realistic: pair interaction)
- The actual H of the universe may have a simple formula (in terms of Lagrangian?) from a suitable perspective.
- But in practice, the effective H of a system may be very complicated. It may thus be appropriate to model H by a random matrix (typical H).
- Leads to the question: which measure for H?

Thank you for your attention