Chapter 3

Continuity, compactness and connectedness

Definition 3.1 (Continuity and sequential continuity)**.**
Let X, Y be topological spaces, $f : X \to Y$ a map, and $a \in X$

1. We say that f is *sequentially continuous* at a, if for a sequence (x_n), $\lim_{n\to\infty} x_n = a$ implies that
$$\lim_{n\to\infty} f(x_n) = f(a).$$

2. We say that f is *continuous* at a, if
$$\forall U \in \mathcal{U}(f(a))\, \exists V \in \mathcal{U}(a) : \quad f(V) \subset U.$$ [1]

If a function is (sequentially) continuous at all points $a \in X$, then we say that f is *(sequentially) continuous on X*.

Proposition 3.2. *If $f : X \to Y$ is continuous at $x \in X$, then f is also sequentially continuous at x.*

Proposition 3.3 (ε-δ-continuity in metric spaces)**.** *A function $f : X \to Y$ between metric spaces X, Y is continuous at $x \in X$, if and only if*
$$\forall \varepsilon > 0 \exists \delta > 0 : \quad f(B_\delta(a)) \subset B_\varepsilon(f(a))$$

Proposition 3.4. *A function $f : X \to Y$ between metric spaces X, Y is continuous at $a \in X$, if and only if it is sequentially continuous at a.*

Proof. $\Rightarrow$ Proposition 3.2

[1] $\mathcal{U}(x)$ is the set of all neighbourhoods of the point x.

$\Leftarrow$ (by contraposition $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$)
Assume that f is not continuous at a, i.e.

$$\exists \varepsilon > 0 \forall \delta > 0 : \quad f(B_\delta(a)) \not\subset B_\varepsilon(f(a)).$$

For $\delta = \frac{1}{n}$ choose $x_n \in B_\delta(a) \backslash f^{-1}(B_\varepsilon(f(a))) \neq \emptyset$. Then $\lim_{n\to\infty} x_n = a$, but $f(x_n) \notin B_\varepsilon(f(a)) \forall n \Rightarrow f$ is not sequentially continuous.

□

Theorem 3.5. *Let X, Y be topological spaces. A map $f : X \to Y$ is continuous (on X), if the preimage $f^{-1}(O) \subset X$ of any open set $O \subset Y$ is open.*

Example 3.6.

1. In a metric space (X, d) the distance function to a point $b \in X$,

$$d_b : X \to [0, \infty), \quad x \mapsto d_b(x) := d(x, b)$$

is continuous.[2]

2. In a normed space $(V, \|\cdot\|)$ the norm:

$$\|\cdot\| : V \to [0, \infty),$$

addition:

$$+ : V \times V \to V, \quad (x, y) \mapsto x + y,$$

and multiplication by scalars:

$$\cdot : \mathbb{K} \times V \to V, \quad (\lambda, v) \mapsto \lambda \cdot v$$

are all continuous.

3. The composition of continuous functions is continuous. If $f : X \to Y$ and $g : Y \to Z$ are continuous then also $g \circ f : X \to Z$ is continuous.

4. If X is equipped with the discrete topology, then every map $f : X \to Y$ is continuous. If X is equipped with the trivial topology, then every map $f : Y \to X$ is continuous.

Remark 3.7.

1. Let (X, d_X) and (Y, d_Y) be metric spaces. Then a metric on $X \times Y$ is for example

$$d((x_1, y_1), (x_2, y_2)) := (d_x(x_1, x_2)^p + d_y(y_1, y_2)^p)^{1/p} \quad 1 \leq p < \infty$$

2. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ topological space. Then the (product) topology on $X \times Y$ is generated by

$$\{O_1 \times O_2 \ : \ O_1 \in \mathcal{T}_X, O_2 \in \mathcal{T}_Y\}$$

also called *bose topology*.

[2] Also $d : X \times X \to [0, \infty)$ is continuous using a suitable metric on $X \times X$. For the definition of this metric, see Remark 3.7.

3. Let $(X_i, \mathcal{T}_i)$, $i \in I$, be topological spaces. Then the product topology on $\prod_{i\in I} X_i$ is generated by

$$\Big\{ \prod_{i\in I} O_i \;:\; O_i \in \mathcal{T}_i \text{ and } O_i \neq X_i \text{ only for finitely many } i \in I \Big\}.$$

Definition 3.8 (Lipschitz continuity)**.**
Let X, Y be metric spaces. A function $f : X \to Y$ is called *Lipschitz-continuous*, if there exists $0 \leq L \leq \infty$ such that

$$\forall x_1, x_2 \in X : \quad d_Y(f(x_1), f(x_2)) \leq L \cdot d_X(x_1, x_2).$$

Then L is called a *Lipschitz-constant* for f. If f has a Lipschitz-constant $L < 1$, then f is called *contraction*.

Example 3.9.

1. $f(x) = ax + b$ is Lipschitz continuous with $L = a$.
2. $f \in C^1(\mathbb{R})$ then $L = \sup_{x\in\mathbb{R}} |f'(x)|$.
3. $f(x) = x^2$ is continuous but not Lipschitz continuous in $\mathbb{R}$.
4. $f(x) = \sqrt{|x|}$ is continuous but not Lipschitz continuous in $\mathbb{R}$, as its derivative around 0 diverges.

Definition 3.10 (Homeomorphic functions, isometries and isometric isomorphisms)**.**

1. Two topological spaces X, Y are *homeomorphic* if there exists a bicontinuous bijection
$$f : X \to Y \quad \text{a } \textit{homeomorphism}$$
2. A map $f : X \to Y$ between metric spaces is an *isometry*, if
$$\forall x_1, x_2 \in X : \quad d_Y(f(x_1), f(x_2)) = d_x(x_1, x_2)\,.$$
X and Y are *isometric*, if there exists a bijective isometry $f : X \to Y$.
3. Two normed spaces V and W are *isometrically isomorphic*, if there exists a linear bijection (isomorphism) $A : V \to W$ such that
$$\forall v \in V : \quad \|Av\|_W = \|v\|_V\,.$$

Example 3.11.

1. The interval $(a, b) \subset \mathbb{R}$ is homeomorphic, but not isometric to $\mathbb{R}$. The map
$$f : (a, b) \to \mathbb{R}, \quad x \mapsto f(x) = \frac{1}{a - x} + \frac{1}{b - x}$$
is an example of a homeomorphism.

2. The isometries of Euclidean space $(\mathbb{R}^n, d_2)$ are translations, rotations and reflections and compositions thereof (euclidean group).

3. $\mathbb{R}^2$ and $\mathbb{C}$ with the standard norms are isometrically isomorphic.

Definition 3.12 (Pointwise and uniform convergence)**.**
Let X be a set, Y a metric space and

$$f_n : X \to Y,\ n \in \mathbb{N} \quad \text{and} \quad f : X \to Y$$

both functions.

1. We say that f_n *converges pointwise* to f, if

$$\forall x \in X : \quad \lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0. \quad \Leftrightarrow \quad \lim_{n\to\infty} f_n(x) = f(x)$$

2. We say that f_n *converges uniformly* to f, if

$$\lim_{n\to\infty} \sup_{x\in X} d_Y(f_n(x), f(x)) = 0$$

If $(Y, \|\cdot\|)$ is a normed space, then $f_n \to f$ uniformly, if and only if

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

Example 3.13. $f_n : [0,1] \to [0,1]$, $x \mapsto f_n(x) = x^n$, then pointwise

$$f_n(x) \overset{n\to\infty}{\longrightarrow} f(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \end{cases}.$$

However, (f_n) does not converge uniformly to f since $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1$. To see this consider $x = 1 - \delta$ for arbitrarily small $\delta > 0$. Then, $f_n(x) = (1-\delta)^n = 1 - n\delta + O(\delta^2)$, whereas $f(x) = 0$, so after sending $\delta \to 0$ we get $\sup_{x\in[0,1]} |f_n(x) - f(x)| \geq 1$.

Proposition 3.14 (Uniform limits of continuous functions are continuous)**.**
Let $(X, \mathcal{T})$ a topological and (Y, d) a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions and let $f_n \to f$ uniformly. Then f is continuous.

Corollary 3.15. *Let X be a topological space, $(Y, \|\cdot\|_Y)$ a complete normed space and $C_b(X, Y)$ the space of continuous bounded functions, i.e.*

$$C_b(X,Y) = \{f : X \to Y \text{ continuous} \mid \sup_{x\in X} \|f(x)\|_Y < \infty\}.$$

Then the normed space $(C_b(X,Y), \|\cdot\|_\infty)$ is complete.

Definition 3.16 (Open cover and finite subcover)**.**
Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. A family $(U_i)_{i \in I}$ of open sets, $U_i \in \mathcal{T}\ \forall i \in I$, is called an *open cover* of Y, if

$$Y \subset \bigcup_{i \in I} U_i$$

A set $K \subset X$ is called *compact*, if any open cover $(U_i)_{i \in I}$ of K admits a finite subcover, i.e. there exists $i_1, \dots, i_n \in I$ such that:

$$K \subset \bigcup_{i = i_1, \dots, i_n} U_i$$

Example 3.17.

1. Every finite subset $K = \{x_1, \dots, x_n\}$ of a topological space is compact.
2. $(0, 1] \subset \mathbb{R}$ is not a compact set. The open cover $(0, 1] \subset \bigcup_{n=2}^{\infty} (\frac{1}{n}, 2)$ admits no finite subcover.

Theorem 3.18 (Bolzano-Weierstraß)**.** *Let $K \subset X$ be compact. Then any sequence in K has a cluster point in K.*

Remark 3.19. In metric spaces also the converse is true, namely, that if every sequence in a subset has a cluster point, then it is compact.

Proposition 3.20. *Let $f : X \to Y$ be a continuous function and $K \subset X$ a compact set. Then also $f(K) \subset Y$ is compact.*

Proposition 3.21.

1. *Let X be a topological space and $K \subset X$ compact. Then any close subset $A \subset K$ is also compact.*
2. *If X is a Hausdorff space and K compact, then K is closed.*

Definition 3.22 (Sequential compactness)**.**
Let X be a topological space. Then, $K \subset X$ is called *sequentially compact* if every sequence in K has a convergent subsequence with limit in K.

Proposition 3.23. *A subset $K \subset (X, d)$ of a metric space is compact if and only if it is sequentially compact.*

Definition 3.24 (Bounded sets and the diameter of a set)**.**
Let X be a metric space.

1. A subset $B \subset X$ is *bounded*, if

$$\exists C \in \mathbb{R} \forall x, y \in B : \quad d(x, y) \leq C$$

2. The *diameter* of the set $Y \subset X$ is

$$\operatorname{diam}(Y) = \sup\{d(x, y) \mid x, y \in Y\} \in [0, \infty) \cup \{\infty\}$$

Theorem 3.25.
Let X be a metric space and $K \subset X$ compact. Then K is bounded and closed.

Theorem 3.26 (Heine-Borel)**.** *A subset K of a finite-dimensional normed space is compact if it is bounded and closed.*

Theorem 3.27 (Weierstraß)**.** *Let $f : K \to \mathbb{R}$ be a continuous function and K compact. Then f is bounded ($f(K) \subset \mathbb{R}$ is bounded) and attains its maximum and its minimum.*

Definition 3.28 (Equicontinuity)**.**
Let X, Y be metric spaces and $A \subset C(X, Y)$. Then the set A is called *equicontinuous* at $x \in X$, if

$$\forall \varepsilon > 0 \exists \delta > 0 \, \forall f \in A : \quad f(B_\delta(x)) \subset B_\varepsilon(f(x)) \,.$$

Theorem 3.29 (Arzela-Ascoli)**.** *Let X be a compact metric space and consider $C(X, \mathbb{C})$ equipped with the $\| \cdot \|_\infty$-norm. A subset $K \subset C(X, \mathbb{C})$ is compact, if and only if it is closed, bounded pointwise (i.e. $\forall x \in X$:*

$$\sup_{f \in K} |f(x)| < \infty)$$

and equicontinuous.

Definition 3.30 (Connected, disconnected and path connected spaces)**.**
Let X be a topological space. If X is the union of two disjoint, open, non-empty sets, then X is *disconnected*, otherwise *connected*.
X is *path-connected*, if any two points $x_0, x_1 \in X$ can be connected by a continuous path, i.e. there exists

$$\gamma : [0, 1] \to X$$

continuous, with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Proposition 3.31. *If X is path-connected then X is connected.*

Proposition 3.32. *Let O be an open subset of a normed space. Then O is connected, if and only if it is path connected.*

Proposition 3.33. *Let $f : X \to Y$ be continuous and $A \subset X$ (path) connected. Then also $f(A) \subset Y$ is (path) connected.*

Definition 3.34 (Bounded functions)**.**
A function $f : X \to Y$ with X a set and (Y, d) a metric space, is called bounded, if and only if $f(X) \subset Y$ is bounded.

Definition 3.35 (Bounded linear maps and their norms)**.**
A linear map $A : V \to W$ between normed spaces is called bounded, if $A(B_1(0))$ is *bounded*, i.e.

$$\exists C \in \mathbb{R} \forall x \in V : \quad \|Ax\|_W \leq C \|x\|_V \,.$$

The smallest such constant C is called the *operator norm* of A, i.e.

$$\|A\|_{op} := \sup\{\|Ax\|_W \mid x \in \overline{B_1(0)}\}$$

The space of bounded linear maps $V \to W$ is denoted by

$$\mathcal{L}(V, W) \text{ or } \mathcal{B}(V, W)$$

and $\|\cdot\|_{op}$ is a norm on $\mathcal{L}(V, W)$.

Remark 3.36.

1. If $A \in \mathcal{L}(V, W)$ we have for all $x \in V$

$$\|Ax\|_W \leq \|A\|_{op} \cdot \|x\|_V$$

2. $A \in \mathcal{L}(V, W)$ is bounded if and only if it is continuous.
3. If $\dim V < \infty$, then all linear maps $V \to W$ are bounded.
4. If $(W, \|\cdot\|_W)$ is a Banach space, then $(\mathcal{L}(V, W), \|\cdot\|_{op})$ is also complete.

Exercises

1. **(Proposition 3.2)** Let $f : X \to Y$ be a map between topological spaces and assume that it is continuous at $x \in X$. Prove that it is also sequentially continuous at x.

2. **(Proposition 3.3)** Show that a map $f : X \to Y$ between metric spaces X, Y is continuous at $x \in X$, if and only if

$$\forall \varepsilon > 0 \; \exists \delta > 0 : \quad f(B_\delta(a)) \subset B_\varepsilon(f(a)) \, .$$

3. **(Theorem 3.18)** Let $K \subset X$ be a compact subset of a topological space. Show that any sequence in K has a cluster point in K.

4. **(Proposition 3.21)** Show that any compact subset of a Hausdorff space is closed.

5. Find an example of

(i) a sequence of maps that converges pointwise but not uniformly.

(ii) a connected but not path connected topological space.