

Topological, metric, and normed spaces

Definition 2.1 (Sequence).

A sequence $(x_n) = (x_1, x_2, x_3, \dots)$ in a set X is a map:

$$x : \mathbb{N} \rightarrow X, \quad n \mapsto x_n$$

A sequence (x_n) in \mathbb{R} *converges* to $x \in \mathbb{R}$, if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon : |x_n - x| < \varepsilon$$

(if for every $\varepsilon > 0$ it holds 'eventually' that $|x_n - x| < \varepsilon$) and we write

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{or simply} \quad x_n \rightarrow x.$$

Similarly, a sequence (x_n) in \mathbb{R}^k converges to $x \in \mathbb{R}^k$, if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon : \|x_n - x\|_2 < \varepsilon$$

where $\|\cdot\|_2$ denotes the usual Euclidean distance (see Example 2.3). We observe that in both examples the notion of *norm* is the key tool in order to define convergence of sequences.

Definition 2.2 (Norm and normed space).

Let V be a vector space either over \mathbb{R} or \mathbb{C} . A *norm* $\|\cdot\|$ on V is a map:

$$\|\cdot\| : V \rightarrow [0, \infty), \quad x \mapsto \|x\|$$

with the properties:

1. $\|x\| = 0 \iff x = 0$
2. $\forall x \in V, \lambda \in \mathbb{K} : \|\lambda x\| = |\lambda| \cdot \|x\|$

$$3. \forall x, y \in V : \quad \|x + y\| \leq \|x\| + \|y\|$$

The pair $(V, \|\cdot\|)$ is called a *normed space*.

Example 2.3. 1. On $V = \mathbb{R}^n$ or \mathbb{C}^n the following maps are norms:

$$\begin{aligned} \|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2} && \text{euclidean norm} \\ \|x\|_\infty &= \max\{|x_1|, \dots, |x_n|\} && \text{maximum norm} \\ \|x\|_1 &= |x_1| + |x_2| + \dots + |x_n| && \text{1-norm} \end{aligned}$$

or, more generally, for $p \in [1, \infty)$, we obtain:

$$\|x\|_p = \left(\sum_{j=1}^n |x_j|^p \right)^{\frac{1}{p}} \quad \text{p-norm}$$

2. Let X a set, $(Y, \|\cdot\|_Y)$ a normed space, and

$$V := \{f : X \rightarrow Y \mid \sup_{x \in X} \|f(x)\|_Y < \infty\}.$$

Then $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_Y$ is a norm on V .

Definition 2.4 (Convergence in normed spaces).

A sequence (x_n) in a normed space $(V, \|\cdot\|)$ converges to $x \in V$ if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon : \quad \|x_n - x\| < \varepsilon.$$

A norm, however, can only be defined on a vector space and ideally we would like to forget about such structure. In view of the definition of convergence for normed spaces, a notion of "distance" between points should suffice.

Definition 2.5 (Metric and metric space).

Let X be a set. A *metric* d on X is a map:

$$d : X \times X \rightarrow [0, \infty)$$

with the following properties:

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. Symmetry: $\forall x, y \in X : \quad d(x, y) = d(y, x)$
3. Triangle inequality: $\forall x, y, z \in X : \quad d(x, z) \leq d(x, y) + d(y, z)$

The pair (X, d) is called a *metric space*

Example 2.6. 1. Let $(V, \|\cdot\|)$ be a normed space. Then $d : V \times V \rightarrow [0, \infty)$, $(x, y) \mapsto d(x, y) := \|x - y\|$ defines a metric on V .

2. Let X be as set. The discrete metric on X is defined by:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

3. The euclidean unit sphere $S^2 := \{x \in \mathbb{R}^3 \mid \|x\|_2 = 1\}$ with the metric:

$$d(x, y) := \arccos(\langle x, y \rangle)$$

is a metric space.

Definition 2.7 (Convergence in metric spaces).

A sequence (x_n) in a metric space (X, d) converges to $x \in X$ if

$$\forall \varepsilon > 0 \exists N_\varepsilon \in \mathbb{N} \forall n \geq N_\varepsilon : d(x_n, x) < \varepsilon.$$

Definition 2.8 (Open sets in a metric space).

Let (X, d) be a metric space.

1. For $x_0 \in X$ and $r > 0$ the set:

$$B_r(x_0) := \{x \in X \mid d(x, x_0) < r\}$$

is called the *open ball*, with the radius r and the centre x_0 .

2. A subset of $U \subset X$ is called a *neighbourhood* of $x_0 \in X$, if U contains an open ball around x_0 , i.e.

$$\exists r > 0 : B_r(x_0) \subset U.$$

Then x_0 is called an *interior point* of U .

3. A subset $U \subset X$ is called *open*, if it contains only interior points, i.e.

$$\forall x \in U \exists r > 0 : B_r(x) \subset U.$$

Example 2.9. 1. Let (X, d) be a metric space. Then for any $x \in X$ and $r > 0$ the set $B_r(x)$ is open.

2. Let X be equipped with the discrete metric. Then any subset $U \subseteq X$ is open: $B_{\frac{1}{2}} = \{x\} \quad \forall x \in X$

Proposition 2.10. *Let (X, d) be a metric space. Then:*

1. \emptyset and X are open.
2. If $U, V \subset X$ are open, then also $U \cap V$ is open.

3. If $U_i \subset X$ is open for all $i \in I$, then also $\bigcup_{i \in I} U_i$ is open.

The second property implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Definition 2.11 (Closed set).

A subset $A \subset X$ of a metric space is *closed*, if its complement is open, i.e. $A^C = \{x \in X \mid x \notin A\}$ is open.

Example 2.12. 1. Let $X = \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$. Then $[a, b]$, $[a, \infty)$ are closed, but $[a, b)$ is neither open nor closed.

2. For any metric space (X, d) the sets \emptyset and X are open and closed.

A metric endows a set not only with a notion of convergence, but also with a geometry (distance, angles, etc.), but actually, open sets turn out to be enough in order to define convergence.

Definition 2.13 (Topology and topological spaces).

Let X be a set. A *topology* \mathcal{T} on X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X with the following properties:

1. $\emptyset, X \in \mathcal{T}$
2. $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$
3. $U_i \in \mathcal{T}$ for all $i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$

The sets $U \in \mathcal{T}$ are called the *open sets* and (X, \mathcal{T}) is called a *topological space*. $A \subset X$ is *closed*, if $A^C \in \mathcal{T}$. $U \subset X$ is called a *neighbourhood* of a $x_0 \in X$ (and x_0 is an *interior point* of U), if:

$$\exists O \in \mathcal{T} \quad \text{with} \quad x_0 \in O \subset U.$$

Remark 2.14. The property 2. implies that intersections of finitely many open sets are open. However, this does not hold for infinite intersections: Let $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}$, $n \in \mathbb{N}$. Then $\bigcap_{n \in \mathbb{N}} U_n = \{0\}$ is not open.

Example 2.15. 1. $\mathcal{T} = \{\emptyset, X\}$ forms the so-called *trivial topology* on X .

2. $\mathcal{T} = \mathcal{P}(X)$ forms the so called *discrete topology* on X .

3. According to Proposition 2.10, the open sets in a metric space form a topology.

Definition 2.16 (Relative topology).

(X, \mathcal{T}) a topological space and $Y \subset X$ a subset of X . Then $\mathcal{T}|_Y = \{O \cap Y \mid O \subset \mathcal{T}\}$ is a topology on Y , the *subspace* or *relative topology*. The elements $U \subset \mathcal{T}|_Y$ are called *relative open sets*.

Example 2.17. 1. $X = \mathbb{R}$, $Y = [0, 1]$. Then $Y \in \mathcal{T}|_Y$, i.e. Y is relatively open in itself. Also $[0, \frac{1}{2}) \subset Y$ is relatively open, since $[0, \frac{1}{2}) = (-\frac{1}{2}, \frac{1}{2}) \cap Y$

2. (X, d) a metric space and $Y \subset X$ a subset. Then $(Y, d|_Y)$ is a metric space.

$$d|_Y : Y \times Y \rightarrow [0, \infty], \quad (y_1, y_2) \mapsto d|_Y(y_1, y_2) = d(y_1, y_2)$$

3. $(V, \|\cdot\|)$ normed space and $U \subset V$ a vector subspace. Then $(U, \|\cdot\||_U)$ is a normed space.

Definition 2.18 (Interior, closure, boundary).

Let (X, \mathcal{T}) be a topological space and $Y \subset X$.

1. The set $\overset{\circ}{Y} = \bigcup_{\substack{U \in \mathcal{T} \\ U \subset Y}} U$ is called the *interior* of Y .

2. The set $\bar{Y} = \bigcap_{\substack{U \in \mathcal{T} \\ U \subset Y^C}} U^C$ is called the *closure* of Y .

3. The set $\partial Y = \bar{Y} \setminus \overset{\circ}{Y}$ is called the *boundary* of Y .

Proposition 2.19. 1. $\overset{\circ}{Y} \subset Y \subset \bar{Y}$.

2. $\overset{\circ}{Y}$ is the largest open set contained in Y .

3. Y is open $\Leftrightarrow Y = \overset{\circ}{Y}$.

4. \bar{Y} is the smallest closed set containing Y .

5. Y is closed $\Leftrightarrow Y = \bar{Y}$.

6. $(\overset{\circ}{Y})^C = \overline{(Y^C)}$ and $(Y^C)^\circ = (\bar{Y})^C$.¹

Proposition 2.20. Let (X, \mathcal{T}) be a topological space and $Y \subset X$. Then:

1. $\overset{\circ}{Y}$ is the set of interior points of Y .

¹This can be proven using de Morgan's laws: For a family of sets $(A_i)_{i \in I}$ it holds

$$(a) \quad \left(\bigcap_{i \in I} A_i \right)^C = \bigcup_{i \in I} A_i^C,$$

$$(b) \quad \left(\bigcup_{i \in I} A_i \right)^C = \bigcap_{i \in I} A_i^C.$$

2. $x \in \partial Y \Leftrightarrow$ for any neighbourhood U of x $U \cap Y \neq \emptyset$ and $U \cap Y^C \neq \emptyset$.
3. $\overset{\circ}{Y} = Y \setminus \partial Y$ and $\bar{Y} = Y \cup \partial Y$

Example 2.21. 1. For $Y = [a, b) \subset \mathbb{R}$ we have $\overset{\circ}{Y} = (a, b)$, $\bar{Y} = [a, b]$, $\partial Y = \{a, b\}$.

2. For $\mathbb{Q} \subset \mathbb{R}$ we have $\overset{\circ}{\mathbb{Q}} = \emptyset$, $\bar{\mathbb{Q}} = \mathbb{R}$, $\partial \mathbb{Q} = \mathbb{R}$.

Definition 2.22 (Convergence in topological spaces).

Let X be a topological space. A sequence (x_n) in X *converges* to $a \in X$ and we write:

$$\lim_{n \rightarrow \infty} x_n = a,$$

if for any neighbourhood U of the point a there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Remark 2.23. In general convergence points are not unique: On any set X with the trivial topology any sequence converges to every point in X !

Definition 2.24 (Hausdorff spaces).

A topological space (X, \mathcal{T}) is *Hausdorff*, if:

$$\forall x, y \in X, x \neq y, \exists U, V \in \mathcal{T} : x \in U, y \in V, U \cap V = \emptyset.$$

Proposition 2.25. 1. In Hausdorff spaces, sequences have at most one limit.

2. All metric spaces are Hausdorff.

Definition 2.26 (Cluster point).

A point $a \in X$ is called a *cluster point* of a sequence (x_n) if any neighbourhood U of a contains infinitely many elements of (x_n) .

Thus far, we have encountered spaces with different level of abstraction,

$$\text{normed spaces} \subset \text{metric spaces} \subset \text{Hausdorff spaces},$$

in which we call always define a notion of convergence and they indeed coincide. Both in physics and mathematics, the analysis is often carried out in so-called *Banach spaces*.

Definition 2.27 (Cauchy sequence).

A sequence (x_n) in a metric space X is called *Cauchy sequence*, if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N : d(x_n, x_m) < \varepsilon$$

Proposition 2.28. Every convergent sequence in a metric space is also a Cauchy sequence.

Definition 2.29 (Complete metric space, Banach space).

1. A metric space X is called *complete* if every Cauchy sequence in X converges.
2. A complete normed space (the norm induces the metric) is called a *Banach space*.

Example 2.30. 1. $\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$ (with e.g. the Euclidean norm) are Banach spaces.

2. $(\mathbb{Q}, |\cdot|)$ is not complete, as there exists a Cauchy sequence $(x_n) \subset \mathbb{Q}$ with $\lim_{n \rightarrow \infty} x_n = \sqrt{2}$ but $\sqrt{2} \notin \mathbb{Q}$.

Exercises

1. Let X be a set and $(Y, \|\cdot\|_Y)$ a normed space. Consider the vector space

$$V \doteq \{f : X \rightarrow Y \mid \sup_{x \in X} \|f(x)\|_Y\}.$$

Prove that $\|f\|_\infty \doteq \sup_{x \in X} \|f(x)\|_Y$ is a norm on V .

2. Let (X, d) be a metric space.

- (i) Prove that $B_r(x)$ is open for any $r > 0$ and $x \in X$.
- (ii) Prove that \emptyset and X are open.
- (iii) Prove that if $Y, Z \subset X$ are open, then $Y \cap Z$ and $Y \cup Z$ are open.
- (iv) Consider now the metric

$$d(x, y) \doteq \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

Prove that any subset $Y \subset X$ is open.

3. Prove that

- (i) all sequences in a Hausdorff space have at most one limit.
- (ii) all metric spaces are Hausdorff.

4. Prove that every convergent sequence in a metric space is a Cauchy sequence.

5. Find two examples of

- (i) metric spaces that are not normed spaces.
- (ii) normed spaces that are not Banach.
- (iii) metric spaces that are not complete.
- (iv) topological spaces that are not Hausdorff.