Chapter 5

Measure and integration theory

Remark 5.1. 1. Idea of the Riemann Integral: Approximate f by "stair functions", i.e. decompose the domain into intervals (rectangles, cubes, ...) and use

$$g(x) = \sum_{i=1}^{n} \alpha_i \chi_{[a_i, a_{i+1}]}(x)$$

where for $A \subset \mathbb{R}$ the *characteristic function* of A is defined:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

The integral of a stair function is:

$$\int g(x)dx = \sum_{i=1}^{n} \alpha_i (a_{i+1} - a_i)$$

2. Idea of the Lebesgue integral: Decompose the range of the function into intervals $[\alpha_i, \alpha_{i+1})$ and approximate by "simple functions"

$$g(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(x)$$

e.g. $A_i = f^{-1}([\alpha_i, \alpha_{i+1}))$ (not interval in general).
The integral of a simple function is given by:

$$\int g(x)\, dx = \sum_{i=1}^{n} \alpha \lambda(A_i)$$

where $\lambda(A_i)$ is the "length" of A_i (area, volume, measure).

Example 5.2. $f(x) = \chi_{\mathbb{Q} \cap [0,1]}(x)$ is not Rieman integrable, but it is Lebesgue integrable:

$$\int_0^1 f(x)dx = 1 \cdot \lambda(\mathbb{Q} \cap [0,1]) + 0 \cdot \lambda([0,1] \backslash \mathbb{Q}) = 0$$

Remark 5.3. Two advantages of the Lebesgue integral:

1. There are more integrable functions, meaning spaces of Lebesgue integrable functions are complete.
2. The Lebesgue integral can be defined on all spaces where one can define a measure λ (not only on $\mathbb{R}$ or $\mathbb{R}^n$).

5.1 Basic notions of measure theory

In 1924 Banach and Tarski managed to prove that there exists no volume map $\mathrm{vol} : \mathcal{P}(\mathbb{R}^3) \to [0, \infty)$ such that

1. $\mathrm{vol}(\emptyset) = 0$, $\mathrm{vol}([0,1]^3) = 1$
2. $X_1, ..., X_k \in \mathcal{P}(\mathbb{R}^3)$ pairwise disjoint, then
$$\mathrm{vol}\Big(\bigcup_{i=1}^{k} X_i) = \sum_{i=1}^{k} \mathrm{vol}(X_i)$$
3. Invariant under transformations. Let $v \in \mathbb{R}^3$, $A \in O(3)$, $X \in \mathbb{R}^3$, then
$$\mathrm{vol}(\{Ax + v \ : \ x \in X\}) =: \mathrm{vol}(A \cdot X + v) = \mathrm{vol}(X)$$

To circumvent this problem σ-algebras and measure theory was created.

Definition 5.4 (σ-algebra)**.** A family $\mathcal{A} \subset \mathcal{P}(X)$ of subsets of a set X is called *σ-algebra*, if

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \quad \Rightarrow \quad A^C \in \mathcal{A}$
3. $A_k \in A$ for $k \in \mathbb{N} \quad \Rightarrow \quad \bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{A}$

The elements of $\mathcal{A}$ are called the $\mathcal{A}$-*measurable sets*.

Proposition 5.5. *Let $\mathcal{A}$ be a σ-algebra on X. Then*

1. $X \in \mathcal{A}$
2. $A_k \in \mathcal{A}$ *for* $k \in \mathbb{N} \quad \Rightarrow \quad \bigcap\limits_{k=1}^{\infty} A_k \in \mathcal{A}$
3. $A, B \in \mathcal{A} \quad \Rightarrow \quad A \cup B \in \mathcal{A}$, $A \cap B \in \mathcal{A}$, *and* $A \backslash B \in \mathcal{A}$

Exercise 5.1. Proof Proposition 5.5.

Example 5.6. 1. $\mathcal{P}(X)$ and $\{\emptyset, X\}$ are σ-algebras on X

2. If $A_j, j \in I$ are, σ-algebras on X, so is $\bigcap\limits_{j \in I} A_j$

Theorem 5.7 (Generating system)**.** *Let $\mathcal{F} \subset \mathcal{P}(X)$. Then the σ-algebra generated by $\mathcal{F}$ is:*

$$\mathcal{A}_{\mathcal{F}} = \bigcap_{\substack{\mathcal{B} \text{ is } \sigma\text{-alg.} \\ \mathcal{F} \subset \mathcal{B}}} \mathcal{B}$$

Any $\mathcal{F} \subset \mathcal{P}(X)$ that generates $\mathcal{A}$ is called generating system for $\mathcal{A}$.

Definition 5.8 (Borel σ-algebra)**.** Let $(X, \mathcal{T})$ be a topological space. Then

$$\mathcal{A}_{\mathcal{T}} = \mathcal{B}$$

is called the *Borel σ-algebra* on X.

Definition 5.9 (Measure)**.** Let $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra. A map $\mu : \mathcal{A} \to [0, \infty]$ is called a *measure*, if

1. $\mu(\emptyset) = 0$
2. For pairwise disjoint sets $A_k \in \mathcal{A}$, $k \in \mathbb{N}$,

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k) \qquad (\sigma\text{-additivity})$$

We further call μ

1. a *finite* measure, $\mu(X) < \infty$,
2. a *σ-finite* measure, if there exists a decomposition $X = \bigcup\limits_{k=1}^{\infty} A_k$ such that $\mu(A_k) < \infty \ \forall k$.

The pair $(X, \mathcal{A})$ is called a *measurable space*, the triple $(X, \mathcal{A}, \mu)$ is called a *measure space*.

Example 5.10. Let X be a set and $x_0 \in X$. Then

$$v : \mathcal{P}(X) \to [0, \infty], \quad A \mapsto v(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases} \qquad \text{"counting measure"}$$

and

$$\delta_{x_0} : \mathcal{P}(X) \to [0, \infty], \quad A \mapsto \delta_{x_0}(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases} \qquad \text{"Dirac measure at } x_0\text{”}$$

are measures.

Proposition 5.11. *Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$. Then*

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$$

and if $A \subset B$

$$\mu(B) = \mu(A) + \mu(B \backslash A) \quad \Rightarrow \mu(A) \leq \mu(B). \qquad \textit{monotony}$$

For $A_j \in \mathcal{A}$, $j \in \mathbb{N}$,

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu(A_j) \qquad \textit{sub-additivity}$$

and if $A_j \subset A_{j+1}$ $\forall j$, then

$$\lim_{j\to\infty} \mu(A_j) = \mu\Big(\bigcup_{j=1}^{\infty} A_j\Big)$$

Definition 5.12 (Measurable function and the push-forward of a measure)**.** Let $(X, \mathcal{A})$ and $(Y, \mathcal{C})$ be measure spaces. A map $f : X \to Y$ is called *$\mathcal{A}$-$\mathcal{C}$-measurable*, if

$$C \in \mathcal{C} \quad \Rightarrow \quad f^{-1}(C) \in \mathcal{A}.$$

If μ is a measure on $(X, \mathcal{A})$ then

$$f^*\mu : \mathcal{C} \to [0, \infty], \quad C \mapsto f^*\mu(C) = \mu\left(f^{-1}(C)\right)$$

is called its *push-forward* under f.

Remark 5.13 (Terminology from probability theory). A measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) = 1$ is called a *probability space*. Then the elements $A \in \mathcal{A}$ are called *events* and $\mu(A)$ the probability of the event. Measurable functions $f : X \to Y$, $(Y, \mathcal{C})$ a measurable space, are called *random variables* and the probability measure $f^*\mu$ is called the *distribution* of f.

Theorem 5.14 (Lebesgue measure)**.** *There is a unique measure λ on $(\mathbb{R}^n, \mathcal{B})$ that is translation invariant (i.e. $\lambda(A + x) = \lambda(A)$, $\forall A \in \mathcal{B} \forall x \in \mathbb{R}^n$) and normalised to $\lambda((0, 1)^n) = 1$. It is called the Lebesgue-Borel measure and its completion is called the Lebesgue measure.*

Exercise 5.2. Show that $\lambda(\mathbb{Q}) = 0$.

5.2 Basic notions of integration theory

Definition 5.15 (Simple function)**.**
A function $g : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is called simple, if $g(X) = \{\alpha_1, \ldots, \alpha_k\}$ is finite, i.e.

$$g(x) = \sum_{j=1}^{k} \alpha_j \chi_{A_j}(x) \qquad \text{with } A_j \cap A_i = \emptyset \text{ for } i \neq j$$

Definition 5.16 (Integral of non-negative measurable functions)**.**
Let $(X, \mathcal{A}, \mu)$ be a measure space and $g : X \to [0, \infty]$ a simple and measurable, then

$$\int_X g \, d\mu = \sum_{j=1}^{k} \alpha_j \mu(A_j)$$

For a measurable function $f : X \to [0, \infty]$

$$\int_X f \, d\mu = \sup \left\{ \int_X g \, d\mu \;\middle|\; g : X \to [0, \infty] \text{ simple, measurable and } g \leq f \right\}$$

Definition 5.17 (Integral of measurable functions)**.**
A measurable function $f : X \to \overline{\mathbb{R}}$ is *integrable*, if for $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ it holds that

$$\int f_+ \, d\mu < \infty \qquad\qquad \int f_- \, d\mu < \infty.$$

Then

$$\int f \, d\mu = \int f_+ \, d\mu - \int f_- \, d\mu$$

Proposition 5.18. *Let $f, g : X \to \mathbb{R}$ be measurable and integrable and $\alpha \in \mathbb{R}$. Then*

1. $\int \alpha f \, d\mu = \alpha \int f \, d\mu$
2. $\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$
3. $f \leq g \quad \Rightarrow \int f \, d\mu \leq \int g \, d\mu$

Theorem 5.19 (Beppo-Levi, Monotone convergence)**.** *Let $f_n : X \to [0, \infty]$ measurable and $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$. Let $f := \lim_{n\to\infty} f_n$ (pointwise), then*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Corollary 5.20. *Let $f_n : X \to [0, \infty]$ be measurable. Then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu$$

Definition 5.21 (Almost everywhere)**.**
We say that a property of a point $x \in X$ holds *almost surely* or *almost everywhere* with respect to a measure μ on X, if it holds for $x \in A \subset X$ and

$$\mu(X \backslash A) = 0,$$

i.e. if it fails to hold a *null set* only.

Example 5.22. 1. A real number is almost surely irrational with respect to Lebesgue's measure on $\mathbb{R}$.

2. Let $f : X \to [0, \infty]$ be measurable. Then

$$\int_X f \, d\mu = 0 \quad \Leftrightarrow \quad f = 0 \text{ almost everywhere}$$

3. Changing an integrable function f on a null set does not change $\int f \, d\mu$.

4. For integrable functions we do not include $\pm\infty$ into the range anymore.

Remark 5.23. 1. Every Rieman integrable function $f : [a, b] \to \mathbb{R}$ is also Lebesgue integrable and the integrals coincide.

2. A function $f : X \to \mathbb{C}$ is integrable, if $|f|$ is integrable and

$$\int f \, d\mu = \int Re\, f \, d\mu + i \int Im\, f \, d\mu$$

3. Analogously for $f : X \to W$ (W-finite dimensional).

4. For $f : X \to W$, W a Banach space, the generalisation is called the Bochner integral.

Definition 5.24 (L^p-spaces)**.**
Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p < \infty$. Then

$$\mathcal{L}^p(X, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } |f|^p \text{ is integrable}\}$$

and for $f \in \mathcal{L}^p(X, \mu)$

$$\|f\|_{L^p} = \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} < \infty \,.$$

Moreover, $L^p(X, \mu) = \mathcal{L}^p(X, \mu) / \sim$ with respect to the equivalence relation

$$f \sim g \quad \Leftrightarrow \quad f = g \text{ almost everywhere}\,.$$

Theorem 5.25 (Completeness of L^p-spaces)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Then $(L^p(X, \mu), \|\cdot\|_{L_p})$ is a Banach space.*

Theorem 5.26 (Dominated convergence)**.** *Let $f_n : X \to \mathbb{C}$ be measurable, $n \in \mathbb{N}$, and assume that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for almost all $x \in X$. If for some $g \in L^p(X, \mu)$, $1 \leq p < \infty$ it holds that $|f_n| \leq |g|$ almost everywhere and for all $n \in \mathbb{N}$ then $f_n, f \in L^p(X, \mu)$ and*

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0$$

i.e. $f_n \to f$ in $L^p(X, \mu)$.

Definition 5.27 (L^∞ and the essential supremum)**.**
Let $(X, \mathcal{A}, \mu)$ be a measure space. For measurable $f : X \to \mathbb{C}$ ($|f| : X \to [0,\infty]$) we define

$$\|f\|_{L^\infty} = \inf\left\{0 \le \lambda \le \infty \mid \mu\left(|f|^{-1}((\lambda,\infty])\right) = 0\right\} = \operatorname{ess\,sup}|f|.$$

Using this definition one can define

$$\mathcal{L}^\infty(X) = \{f : X \to \mathbb{C} \mid f \text{ measurable and } \|f\|_{L^\infty} < \infty\}$$

and

$$L^\infty(X) = \mathcal{L}^\infty(X)/\sim$$

Example 5.28.

1. If $\mu(X) < \infty$ and $f \in L^\infty(X)$, then

$$\int_X |f|\,d\mu \le \int_X \underbrace{\|f\|_{L^\infty}}_{=\|f\|_\infty}\,d\mu = \|f\|_{L^\infty}\cdot\mu(X)$$

   In particular $L^\infty(X) \subset L^1(X)$ in this case. Actually, $L^p(X) \subset L^q(X)$ if $p > q$ and $\mu(X) < \infty$.

2. $X = \mathbb{R}^n$, $\mu = \lambda^n$, then $\mu(X) = \infty$ and Item 1 does not apply. We prove this by the following: Let $\alpha \in \mathbb{R}$

$$f_\alpha : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}, \quad x \mapsto f_\alpha(x) = \frac{1}{\|x\|^\alpha}$$

   (a) $A = B_1(0) \subset \mathbb{R}^n$. Then for $\alpha > 0$,

$$\int_{B_1(0)} |f_\alpha|^p\,d\lambda = \int_{B_1(0)} \frac{1}{\|x\|^{\alpha+p}}\,d\lambda = C_n\int_0^1 \frac{1}{r^{\alpha p}}r^{n-1}\,dr$$
$$= C_n\int_0^1 \frac{1}{r^{\alpha p+1-n}}\,dr = \begin{cases} <\infty & \text{if } \alpha < \frac{n}{p} \\ =\infty & \text{if } \alpha \ge \frac{n}{p}\end{cases}$$

   We also could have used $L^p(B_1(0)) \subsetneq L^q(B_1(0))$ for $p \ge q$.

   (b) $A = \mathbb{R}^n\setminus B_1(0)$. Then

$$\int_A |f_\alpha|^p\,d\lambda = \int_A \frac{1}{\|x\|^{\alpha+p}}\,d\lambda = C_n\int_1^\infty \frac{1}{r^{\alpha p}}r^{n-1}\,dr$$
$$= C_n\int_1^\infty \frac{1}{r^{\alpha p+1-n}}\,dr = \begin{cases} <\infty & \text{if } \alpha > \frac{n}{p} \\ =\infty & \text{if } \alpha \le \frac{n}{p}\end{cases}$$

Putting both together we can conclude that neither $L^p(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ nor $L^q(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $p \neq q$.

3. At last we want to show that pointwise convergence does not imply convergence in the L^p-norm. Let $f_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto \chi_{[n,n+1]}(x)$. Then for all $p \geq 1$ $f_n \in L^p$ with $\|f_n\|_p = 1$ and $f_n \xrightarrow{\text{p.w.}} f = 0$, but
$$\|f_n - f\|_{L^p} = \|f_n\|_{L^p} = 1\,.$$
This is because there exists no dominating function.

Theorem 5.29 (Hölder inequality)**.** *Let $f, g : X \to \mathbb{C}$ be measurable and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (conjugated exponents) where $\frac{1}{\infty} = 0$. Then*
$$\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

Remark 5.30. For $p = q = 2$ this is the *Cauchy-Schwarz inequality* on the Hilbert space L^2. Hence for $f, g \in L^2 \Rightarrow \overline{f}g \in L^1$, since
$$\underbrace{\left| \int \overline{f} g \, d\mu \right|}_{=|\langle f,g|f,g\rangle_{L^2}|} \leq \int |\overline{f}g| \, d\mu \leq \|f\|_{L^2} \cdot \|g\|_{L^2}\,.$$

Theorem 5.31 (Minkowski inequality)**.** *Let $f, g : X \to \mathbb{C}$ be measurable and $1 \leq p \leq \infty$. Then*
$$\|f + g\|_p \leq \|f\|_p + \|g\|_p\,.$$

5.3 Product measures and Fubini's theorem

Definition 5.32 (Product σ-algebras)**.**
Let $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ be measurable spaces. Then $\mathcal{A}_1 \otimes \mathcal{A}_2$ denotes the product σ-algebra on $X_1 \times X_2$ generated by sets of the form $A_1 \times A_2 \subset X_1 \times X_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, the so called *product σ-algebra*.

Example 5.33. Let $\mathcal{B}^n \subset \mathcal{P}(\mathbb{R}^n)$ be the Borel-σ-algebra. Then $\mathcal{B}^n = \mathcal{B}^1 \otimes \ldots \otimes \mathcal{B}^1$.

Theorem 5.34. *Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. There exists a unique measure μ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that for all $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$*
$$\mu(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2),$$
called the product measure and denoted by $\mu = \mu_1 \otimes \mu_2$.

Example 5.35. The Lebesgue-Borel measure
$$\lambda^n = \lambda^1 \otimes \ldots \otimes \lambda^1.$$

Theorem 5.36 (Tonelli)**.** *Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces. Let $f : X_1 \times X_2 \to [0, \infty]$ be $\mathcal{A}_1 \otimes \mathcal{A}_2$-mensurable. Then*

$$\int\limits_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2) = \int\limits_{X_1} \Big(\underbrace{\int\limits_{X_2} f_{x_1} \, d\mu_2}_{fct.\ of\ x_1} \Big) d\mu_1 = \int\limits_{X_2} \Big(\int\limits_{X_1} f_{x_2} \, d\mu_1 \Big) d\mu_2$$

where $f_{x_1} : X_2 \to \overline{\mathbb{R}}, \quad x_2 \mapsto f(x_1, x_2)$.

Example 5.37. $X_1 = X_2 = [0, 1], \mu_1 = \lambda^1, \mu_2 = v$ the counting measure. Hence (X_2, μ_2) is not σ-finite.

$$f : X_1 \times X_2 \to [0, \infty]$$
$$(x_1, x_2) \mapsto \delta_{x_1, x_2} := \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

Now the results for the integrals. For the first integral, we calculate

$$\int \underbrace{f(x_1, x_2)}_{= f_{x_2}(x_1)} \, d\mu_1 = 0$$

and obtain

$$\int \Big(\int f_{x_2}(x_1) \, d\mu_1 \Big) d\mu_2 = 0$$

For the second integral, we calculate

$$\int \underbrace{f(x_1, x_2)}_{f_{x_1}(x_2)} \, d\mu_2 = 0 \cdot \mu_2(\chi_{\{f=0\}}) + 1 \cdot \mu_2(\chi_{\{f=1\}}) = 1$$

hence obtain

$$\int \Big(\int f_{x_1}(x_2) \, d\mu_2 \Big) \, d\mu_1 = 1.$$

I.e. the two integral do not agree.

Theorem 5.38 (Fubini)**.** *Let $(X_1, \mathcal{A}_1, \mu_1)$ and $(X_2, \mathcal{A}_2, \mu_2)$ be σ-finite measure spaces and $f : X_1 \times X_2 \to \mathbb{C}$ measurable. Then the following two statements hold.*

1. *We find the following equivalence:*

$$\int\limits_{X_1} \Big(\int\limits_{X_2} |f_{x_1}| \, d\mu_2 \Big) d\mu_1 < \infty \quad \text{or} \quad \int\limits_{X_2} \Big(\int\limits_{X_1} |f_{x_2}| \, d\mu_1 \Big) d\mu_2 < \infty$$

if and only if

$$f \in L^1(X_1 \times X_2, \mu_1 \otimes \mu_2) \, .$$

2. If $f \in L^1(X_1 \times X_2, \mu_1 \otimes \mu_2)$*, then*

$$\int_{X_1 \times X_2} f \, d(\mu_1 \otimes \mu_2) = \int_{X_1} \Big(\int_{X_2} f_{x_1} d\mu_2 \Big) d\mu_1 = \int_{X_2} \Big(\int_{X_1} f_{x_2} d\mu_1 \Big) d\mu_2$$

Example 5.39. $X = \mathbb{N}, \mathcal{A} = \mathcal{P}(\mathbb{N}), \mu = v$ counting measure. $f : X \to \mathbb{R}$, $n \mapsto f(n)$ (real sequences). The integral is then defined as:

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} f(n)$$

Now we consider $g : X \times X \to \mathbb{R}$, $(n, m) \mapsto g(n, m)$. There appears the question of whether it is possible to change the order of summation. Fubini's theorem allows us to answer yes to that question in the case of absolute convergence. Hence if $\sum\limits_m \sum\limits_n |g(n, m)| < \infty$, or equivalently with the summation order changed, we can change the summation.