Chapter 1

Linear Algebra

Definition 1.1 (Vector space).
A *vector space* (or *linear space*) over a field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a set V along with an *addition*

$$+ : V \times V \to V$$

and a *scalar multiplication*

$$\cdot : \mathbb{K} \times V \to V$$

satisfying

(i) additive associativity: $(u + v) + w = u + (v + w) \quad \forall\, u, v, w \in V$

(ii) additive identity: $\exists\, 0 \in V \,:\, v + 0 = v \quad \forall v \in V$

(iii) additive inverse: $\forall\, v \in V\, \exists (-v) \in V \,:\, v + (-v) = 0$

(iv) additive commutativity: $u + v = v + u \qquad \forall\, u, v \in V.$

(v) distributivity from the left: $\lambda \cdot (u+v) = \lambda \cdot u + \lambda \cdot v \quad \forall\, u, v \in V$ and $\lambda \in \mathbb{K}$

(vi) distibutivity from the right: $(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot v \quad \forall\, v \in V$ and $\forall\, \lambda, \mu \in \mathbb{K}$

(vii) multiplicative associativity: $(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v) \forall\, v \in V$ and $\forall\, \lambda, \mu \in \mathbb{K}$

(viii) multiplicative identity: $1 \in \mathbb{K} \,:\, 1 \cdot v = v \quad \forall v \in V$

If a set $W \subset V$ forms a vector space under the same operation, it is called a *linear subspace*.

Remark 1.2. Properties (i)-(iv) can be summarised as saying that V with its addition forms an Abelian group. Property (v) says that the map

$$\lambda \cdot : V \to V,\, v \mapsto \lambda \cdot v$$

is a group homomorphism for each $\lambda \in \mathbb{K}$. The remaining properties are equivalent to saying that the map

$$\mathbb{K} \to \mathrm{Hom}_{\mathrm{Grp}}(V, V),\, \lambda \mapsto \lambda \cdot$$

is a homomorphism of rings.

Definition 1.3 (Linear maps)**.**
A map $f : V \to W$ between vector spaces (over the same field) is said to be *linear* if for all $\lambda \in \mathbb{K}$ and $u, v \in V$ it holds that

$$f(u + v) = f(u) + f(v)$$
$$f(\lambda u) = \lambda f(u)\,.$$

A bijective linear map is called a *linear isomorphism*. We define the *kernel* and the *image* of the linear map by

$$\ker L = \{v \in V \ : \ L(v) = 0\}$$

and

$$\operatorname{Im} f = \{w \in W \ : \ \exists v \in V\,,\ w = L(v)\}$$

respectively. The set of all linear maps between V and W is denoted by $\mathcal{L}(V, W)$.

Definition 1.4.
Let V be a vector space and $B = (v_1, v_2, \ldots)$ a tuple of vectors in V. The tuple B is said to be *linearly independent* if for every tuple of scalars $(\lambda_1, \lambda_2, \ldots \lambda_n)$

$$\sum_{j=1}^{n} \lambda_j v_j = 0$$

implies $\lambda_j = 0$, $\forall j \in \{1, \ldots n\}$.

We say B *spans* V if any vector $v \in V$ can be written as a *linear combination* of elements of B, i.e. there exist scalars $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, such that

$$v = \sum_{j=1}^{n} \lambda_j v_j\,.$$

We call B a *basis* of V if it is both linearly independent and spans V.

Theorem 1.5 (Dimension theorem)**.** *Every vector space has a basis and all bases of a vector space have the same cardinality. This cardinality is called the dimension of the vector space.*

Two vector spaces are isomorphic if and only if they have the same dimension.

Proposition 1.6. *Every finite dimensional vector space over* $\mathbb{K}$ *is isomorphic to* $\mathbb{K}^n$ *for some* $n \in \mathbb{N}$. *Each choice of basis provides an isomorphism. A linear map* $f : V \to W$ *from an* n*-dimensional to an* m*-dimensional vector space can be represented by an* $m \times n$ *matrix.*

Remark 1.7. The definition of a basis presented here is also called Hamel-basis. In the context of infinite-dimensional vector spaces equipped with a topology one usually uses so called Schauder-bases instead.

Example 1.8.

1. $\mathbb{R}^3$ is a three dimensional vector space and the Cartesian coordinate vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$ form a basis.
2. $\mathbb{C}$ is a one dimensional vector space over $\mathbb{C}$ and a two dimensional vector space over $\mathbb{R}$.
3. The space $\mathbb{M}_{n,m}(\mathbb{K})$ of all $n \times m$ matrices is a vector space over $\mathbb{K}$ with component-wise operations.
4. For any two vector spaces V and W it holds that $\mathcal{L}(V, W)$ is a vector space under point-wise operations.
5. The Polynomial ring $\mathbb{K}[X]$ is an infinite dimensional vector space and $(1, X, X^2, X^3, \dots)$ is a basis.
6. The space $C(\mathbb{R}, \mathbb{R})$ of continuous real-valued functions on $\mathbb{R}$ is an infinite dimensional vector space with no countable basis.

Definition 1.9.
Let V be a vector space and W a subspace of V. The *quotient space* V/W is defined as the set of equivalence classes under the relation

$$u \sim v \iff u - v \in W$$

together with the natural addition and scalar multiplication.

Theorem 1.10 (Isomorphism Theorem)**.**
Let $f\colon V \to W$ be a linear map. The quotient space $V/\ker(f)$ is isomorphic to $\mathrm{Im}(f)$

Proposition 1.11.
Let V be a finite dimensional vector space and W a subspace of V. It holds that

$$\dim(V/W) = \dim(V) - \dim(W)\,.$$

Theorem 1.12 (The Rank nullity theorem)**.**
Let $f : V \to W$ be a linear map and suppose that V is finite dimensional. Then,

$$\dim(V) = \dim(\ker f) + \dim(\mathrm{Im}\, f)\,.$$

Definition 1.13 (Eigenvalues and eigenvectors)**.**
Let $f \in \mathcal{L}(V, V)$ be a linear map. We say that a scalar λ is an *eigenvalue* of f with *eigenvector* $v \in V$ whenever

$$f(v) = \lambda v$$

holds. The linear subspace $\ker(f - \lambda \mathrm{id}_V)$ is called the *eigenspace* of λ. The set of all eigenvaules $\sigma(f)$ is called the *spectrum* of f.

Definition 1.14 (Inner product)**.**
An *inner product* on a vector space V over $\mathbb{K}$ is a map $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ satisfying

1. $\langle v, w\rangle = \overline{\langle w, v\rangle}, \quad \forall v, w \in V.$
2. $\langle \lambda v + w, u\rangle = \lambda\langle v, u\rangle + \langle w, u\rangle, \quad \forall v, w, u \in V.$
3. $\langle v, v\rangle > 0, \quad \forall v \in V \setminus \{0\}.$

A vector space together with an inner product is called an *inner product space*.

Definition 1.15. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space and $f \in \mathcal{L}(V, V)$. We denote by f^* the unique linear map that satisfies

$$\langle f(v), w\rangle = \langle v, f^*(w)\rangle, \qquad \forall v, w \in V$$

it is called the adjoint of f. We call f *normal* if $f \circ f^* = f^* \circ f$ and *self-adjoint* if $f = f^*$.

Remark 1.16. In the finite dimensional case, all the above concepts have their matrix counterpart: once we fix a basis on each vector space, vectors and linear maps are uniquely represented by their component matrices.

Theorem 1.17 (Finite dimensional Spectral theorem)**.** *Let V be a finite dimensional complex (real) inner product space and consider a linear map $f \in \mathcal{L}(V, V)$. If f is normal, then there exists a basis of V consisting of eigenvectors of f.*

Exercises

1. Let $L \in \mathcal{L}(V, W)$ be a linear map between vector spaces. Show the following:

(i) $\ker L$ and $\operatorname{Im} L$ are linear subspaces of V and W respectively.

(ii) L is injective if and only if $\ker L = \{0\}$.

(iii) If $\dim(V) = \dim(W)$, then L is injective if and only if it is surjective.

2. Let V, W be finite dimensional real vector spaces. Prove the following isomorphisms:

(i) $\mathbb{C} \cong \mathbb{R}^2$ ($\mathbb{C}$ as a real vector space).

(ii) $\mathcal{L}(\mathbb{R}, \mathbb{R}^n) \cong \mathbb{R}^n$.

(iii) $\mathrm{V} \cong \mathbb{R}^n$ for some $n \in \mathbb{N}$.

(iv) $\mathcal{L}(V, W) \cong \mathbb{R}^{dim(V) \times \dim(W)}$.

3. Consider three maps $f, g, h : \mathbb{R}^2 \to \mathbb{R}^2$ acting as shown in the image below. Select whether the following statements are true or false.

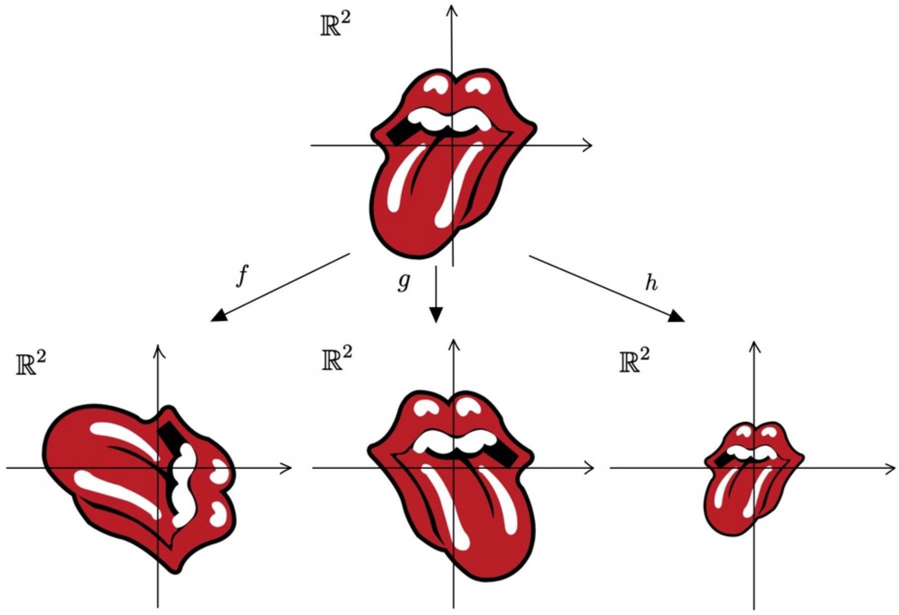
(i) f and h are linear but g is not.

(ii) f and g are linear but h is not.

(iii) f has a positive real eigenvalue.

(iv) g has a unique real eigenvalue.

(v) Any vector of $\mathbb{R}^2$ is an eigenvector of h.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