Exercises

1. (Proposition 6.4) Let $\mathcal{A}$ be a σ-algebra on X. Show that

1. $X \in \mathcal{A}$
2. $A_k \in \mathcal{A}$ for $k \in \mathbb{N} \implies \bigcap_{k\in\mathbb{N}} A_k \in \mathcal{A}$
3. $A, B \in \mathcal{A} \implies A \cup B,\ A \cap B,\ A \setminus B \in \mathcal{A}$

2. (Proposition 6.10) Let μ be a measure on $(X, \mathcal{A})$ and $A, B \in \mathcal{A}$. Show that

(i) if $\mu(A \cup B) + \mu(A \cap B = \mu(A) + \mu(B)$.

(ii) if $A \subset B$, then $\mu(A) \leq \mu(B)$.

(iii) For $A_j \in \mathcal{A},\ j \in \mathbb{N}$,

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) \leq \sum_{j=1}^{\infty} \mu(A_j)\,.$$

(iv) If $A_j \subset A_{j+1}$ then

$$\lim_{j\to\infty} \mu(A_j) = \mu(\bigcup_{j=1}^{\infty} A_j)\,.$$

3. Find an example of

(a) a sequence in $L^p([0,1])$ that converges pointwise but not in L^p to a function in $L^p([0,1])$.

(b) a sequence in $L^p([0,1])$ that converges in L^p but not almost everywhere.