Chapter 5

Implicit functions and ordinary differential equations

Implicit function theorem

Say we have a system of m algebraic equations on n variables

$$\begin{aligned} F_1(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ F_m(x_1,\ldots,x_n) &= 0 \end{aligned}$$

In the case of linear equations, if $n = m$, basic linear algebra tells us that the solvability depends on the degeneracy of the coefficient matrix, whereas if $n < m$, the degeneracy of a coefficient sub-matrix determines the parametrizability of the space solutions.
In the nonlinear case, one simply "linearizes" the problem around a point and obtains a similar statement locally. Consider a function

$$F : \underbrace{\mathbb{R}^n \times \mathbb{R}^m}_{\mathbb{R}^{n+m}} \to \mathbb{R}^m, \qquad (x,y) \mapsto F(x,y)$$

and think of the zero level set as the set of solutions to a system of algebraic equations, i.e.

$$F(x,y) = 0 \quad \iff \begin{cases} F_1(x_1,\ldots,x_n,y_1,\ldots,y_m) & = 0 \\ \quad \vdots \\ F_m(x_1,\ldots,x_n,y_1,\ldots,y_m) & = 0 \end{cases}$$

where we want to solve for the $(y_1,\ldots,y_m)$ variables in terms of the extra $(x_1,\ldots,x_n)$ parameters.

Theorem 5.1 (Implicite function theorem)**.** *Let $\Omega \subset \mathbb{R}^{n+m}$ be open, $F \in C^1(\Omega,\mathbb{R}^m)$ and*

$$N \doteq \{(x,y) \in \Omega \mid F(x,y) = 0\}.$$

If for $(a, b) \in N$ it holds that the matrix:

$$D_y F|_{(a,b)} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix} (a, b)$$

is invertible, then there exists open neighbourhoods $U_x \subset \mathbb{R}^n$ of a and $U_y \in \mathbb{R}^m$ of b with $U_x \times U_y \subset \Omega$ and a function $f \in C^1(U_x, U_y)$ such that

$$N \cap (U_x \times U_y) = \mathrm{graph}(f)\,,$$

i.e

$$\forall (x, y) \in U_x \times U_y : \; F(x, y) = 0 \quad \Leftrightarrow \quad f(x) = y\,.$$

In other words, one can solve $F(x, y) = 0$ locally for y. Moreover,

$$Df|_x = -\left(D_y F|_{(x,g(x))}\right)^{-1} \cdot D_x F|_{(x,f(x))}\,.$$

Definition 5.2.
Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open. A map $f \in C^1(\Omega, \Omega')$ is called a *diffeomorphism*, if it is bijective and also the inverse $f^{-1} \in C^1(\Omega', \Omega)$.

Theorem 5.3 (Inverse function theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1(\Omega, \mathbb{R}^n)$. If for $x \in \Omega$ it holds that $Df|_x$ is invertible then there exists an open neighbourhood U of x such that $f|_U : U \to f(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Definition 5.4 (Local extremum under constraint)**.**
Let $\Omega \subset \mathbb{R}^n$ be open and $f, h \in C^1(\Omega, \mathbb{R})$. Let $N \doteq \{x \in \Omega \mid h(x) = 0\}$ and $a \in N$. We say that f has a *local extremum* (maximum or minimum) *at the point a under the constraint $h = 0$* if $f|_N$ has a local extremum at a.

Theorem 5.5 (Necessary condition for local extremum under constraint)**.** *Let Ω, f, h, N as above. If $a \in N$ is a regular point of h (i.e. $Dh|_a \neq 0$) and a local extremum of f under the constraint $h = 0$, then there exists $\lambda \in \mathbb{R}$ such that:*

$$Df|_a = \lambda Dh|_a \tag{5.1}$$

with λ being the Lagrange parameter.

Theorem 5.6 (Sufficient condition for local extremum under constraint)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, $f, h \in C^2(\Omega, \mathbb{R})$. Let for $a \in N$ the necessary condition Eq. (5.1) be satisfied, i.e. there exists $\lambda \in \mathbb{R}$ such that $DF|_a \doteq D(f - \lambda h)|_a = 0$, then:*

1. *If $D^2F|_a(v, v) > 0$ for all $v \in \mathbb{R}^n \backslash \{0\}$ such that $Dh|_a v = 0$, then f has a strict local minimum at a under the constraint $h = 0$.*

2. *If $D^2F|_a(v, v) < 0$ for all $v \in \mathbb{R}^n \backslash \{0\}$ such that $Dh|_a(v) = 0$, then f has a strict local maximum at a under the constraint $h = 0$.*

3. *If $D^2F|_a$ is indefinite in the subspace spanned by vectors satisfying $Dh|_a(v) = 0$, then f has no local extremum at a.*

Remark 5.7. If $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$, then $N = \{h = 0\}$ is a $n-k$-dimensional submanifold. In this case, the necessary condition for extremum under constraint N becomes

$$\begin{aligned} & Df|_a \in \operatorname{span}\{Dh_1|_a, Dh_2|_a, \ldots, Dh_k|_a\} \\ \Leftrightarrow \quad & \exists \lambda \in \mathbb{R}^k : \quad D(f - \lambda \cdot h)|_a = 0 \quad (\text{i.e.} \quad Df|_a = \lambda_1 Dh_1|_a + \ldots + \lambda_k Dh_k|_a) \ . \end{aligned}$$

Ordinary differential equations

Definition 5.8 (Ordinary differential equation).
Let $I \subset \mathbb{R}$ be an open interval containing 0 and let $m \in \mathbb{N}$. An expression of the form

$$F\left(t, \gamma(t), \gamma'(t), \gamma''(t), \ldots, \gamma^m(t)\right) = 0$$

is called an *ODE of order m*, where

$$F : I \times \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{m\text{-times}} \to \mathbb{R}$$

is given and $\gamma \in C^m(I, \mathbb{R})$ is the unknown.

1. If F does not depend on t, the ODE is called *autonomous*.

2. If the expression is written like

$$\gamma^{(m)}(t) = f\left(t, \gamma(t), \gamma''(t), \ldots, \gamma^{(m-1)}(t)\right)$$

   it is called an *explicit* ODE.

3. If the expression can be writte like

$$\gamma^{(m)}(t) = \sum_{i=1}^{m-1} a_i(t)\gamma^{(i)}(t) + r(t)$$

   it is called *linear*, and $r(t)$ is called the *source term*. If the source term is equal to zero we call it *homogeneous*.

Definition 5.9 (System of ODEs).
Let $I \subset \mathbb{R}$ be an open interval containing 0, let $\Omega \subset \mathbb{R}^n$ open and let $m \in \mathbb{N}$. An expression of the form

$$F\left(t, \gamma(t), \gamma'(t), \gamma''(t), \ldots, \gamma^m(t)\right) = 0$$

is called an *system of ODEs of order m and dimension n*, where

$$F : I \times \Omega \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{m\text{-times}} \to \mathbb{R}^n$$

is given and $\gamma \in C^m(I, \mathbb{R}^n)$ is the unknown. All the nomenclature above translates easily to systems of ODEs.

Remark 5.10. Non-autonomous first-order and autonomous ODEs of any order all reduce to autonomous first-order ODEs.

Definition 5.11 (Integral curves).
Let $\Omega \subset \mathbb{R}^n$ open, $v \in C(\Omega, \mathbb{R}^n)$ a vector field and $I \subset \mathbb{R}$ an open interval containing 0. A solution $\gamma \in C^1(I, \Omega)$ to the initial value problem

$$\begin{cases} \gamma'(t) & = v(\gamma(t)) \\ \gamma(0) & = x_0 \end{cases}$$

is called an *integral curve* of v through $x_0 \in \Omega$.

Definition 5.12 (Local and global Lipschitz condition).
Let $U \subset \mathbb{R} \times \mathbb{R}^n$ and $v \in C(U, \mathbb{R}^n)$ be a time-dependent vector field.

1. We say that v satisfies a *Lipschitz condition*, if there exists $L \geq 0$ such that

$$\forall (t, x), (t, y) \in U : \quad \|v(t, x) - v(t, y)\| \leq L\|x - y\|$$

2. We say that v satisfies a *local Lipschitz condition*, if every $(t, x) \in U$ admits a neighbourhood $V \subset U$ such that $v|_V$ satisfies a Lipschitz condition.

Theorem 5.13 (Picard-Lindelöf). *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be a domain and let $v \in C(U, \mathbb{R}^n)$ satisfy a local Lipschitz condition.*

1. *Local existence: For any $(t_0, x_0) \in U$ there exists $\delta > 0$ and a curve $\gamma \in C^1((t_0 - \delta, t_0 + \delta), \mathbb{R}^n)$ that is a solution of $\gamma' = v(t, \gamma)$ with initial datum $\gamma(t_0) = x_0$.*

2. *Uniqueness: If $J \subset \mathbb{R}$ is an interval with $t_0 \in J$ and $\tilde{\gamma} : J \to \mathbb{R}^n$ solves $\gamma' = v(t, \gamma)$ with $\tilde{\gamma}(t_0) = x_0$, then*

$$\tilde{\gamma}(t) = \gamma(t) \qquad \forall t \in J \cap (t_0 - \delta, t_0 + \delta)\,.$$

Definition 5.14 (Maximal solution).
Let $v \in C(J \times \Omega, \mathbb{R}^n)$ satisfy a local Lipschitz condition. A solution $\gamma : I \to \Omega$ of $\gamma' = v(t, \gamma)$ is called maximal solution, if the following holds: If $I \subset \tilde{I} \subset J$ and $\tilde{\gamma} : \tilde{I} \to \Omega$ is a solution of $\gamma' = v(t, x)$ with $\tilde{\gamma}|_I = \gamma$, then $\tilde{I} = I$.

Corollary 5.15. *Under the conditions of the Picard-Lindelöf-theorem, there exists for any initial value a unique maximal solution.*

Theorem 5.16. *Let $J = (j_-, j_+) \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ a domain, and $v \in C(J \times \Omega, \mathbb{R}^n)$ satisfy a local Lipschitz condition. Let $\gamma : (t_-(t_0, x_0), t_+(t_0, x_0)) \to \Omega$ be the unique maximal solution of $\gamma' = v(t, x)$ for the initial value $(t_0, x_0) \in J \times \Omega$. If $t_+(t_0, x_0) < j_+$, then for any compact $K \subset \Omega$ there exists $0 < \tau_K < t_+(t_0, x_0)$ such that*

$$\gamma(t) \notin K \qquad \forall t \in (\tau_K, t_+(t_0, x_0))\,.$$

Definition 5.17.
A locally Lipschitz vector field $v \in C(\Omega, \mathbb{R}^n)$ is *complete*, if there exists a global solution $\gamma_{x_0} \in C^1(\mathbb{R}, \Omega)$ of $\gamma' = v(\gamma)$ with $\gamma_{x_0}(0) = x_0$ for any initial value $x_0 \in \Omega$. The associated *flow* is:

$$\Phi : \mathbb{R} \times \Omega \to \Omega, \qquad (t, x) \mapsto \Phi(t, x) = \gamma_x(t)$$

and

$$\Phi_t : \Omega \to \Omega, \qquad x \mapsto \Phi_t(x) = \Phi(t, x)$$

is called the *flow map* at time t. It satisfies

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \qquad \forall t, s \in \mathbb{R}$$

i.e.

$$\mathbb{R} \to \mathrm{Bij}(\Omega \to \Omega), \quad t \mapsto \Phi_t$$

is a groups action of $(\mathbb{R}, +)$ on the set Ω.

Theorem 5.18. *If v satisfies a local Lipschitz condition and is complete, then the corresponding flow maps $\Phi_t : \Omega \to \Omega$ are continuous. If $v \in C^1$, then the flow maps $\Phi_t : \Omega \to \Omega$ are also C^1.*

Linear ordinary differential equations

Definition 5.19 (Non-autonomous homogeneous linear system)**.**
Let $J \subset \mathbb{R}$ be open interval, $A : J \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ continuous and $\gamma : J \to \mathbb{R}^n$.

1. The ODE
$$\gamma' = A(t) \cdot \gamma \qquad (v(\gamma) = A(t) \cdot \gamma)$$
is called a non-autonomous, homogeneous, linear system.

2. If $b : J \to \mathbb{R}^n$ is continuous, then
$$\gamma' = A(t) \cdot \gamma + b(t)$$
is called a non-autonomous, inhomogeneous, linear ODE.

Example 5.20. In the homogeneous autonomous case

$$\gamma' = A\gamma$$

the unique global solution with initial datum $x_0 \in \mathbb{R}^n$ is

$$\gamma(t) = e^{At}x_0$$

where $e^{At} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$.

Theorem 5.21. *$J \subset \mathbb{R}$ open, $A : J \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $b : J \to \mathbb{R}^n$ continuous. Then for very $t_0 \in J$ and $x_0 \in \mathbb{R}^n$ there exists a unique maximal solution $\gamma : J \to \mathbb{R}^n$ of the ODE*

$$\gamma' = A(t)\gamma + b(t), \qquad \textit{with} \qquad \gamma(t_0) = x_0 \,.$$

Lemma 5.22 (Grönwall)**.** *Let $a < b$ and $\gamma, A : [a, b] \to \mathbb{R}$ continuous. Assume that γ is differentiable in (a, b) and that*

$$\gamma'(t) \leq A(t)\gamma(t) \qquad \forall t \in (a, b) \,.$$

Then

$$u(t) \leq u(a) \exp\left(\int_a^t A(s)ds\right) \,.$$

Definition 5.23 (The propagator of a non-autonomous, homogeneous linear system)**.**
Let $J \subset \mathbb{R}$ open and $A : J \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ continuous. For fixed $t_0 \in J$ we define the maps

$$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n, \quad x_0 \mapsto \gamma_{x_0}(t) \quad \forall t \in J \tag{5.2}$$

for each $t \in J$, where $\gamma_{x_0} : J \to \mathbb{R}^n$ the solution to $\gamma' = A \cdot \gamma$ with initial data $\gamma_{x_0}(t_0) = x_0$ and call it the flow map or the *propagator*.

Theorem 5.24. *$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ from Eq. (5.2) is a linear isomorphism.*

We hence get that the solutions $\{\gamma \in C^1(J, \mathbb{R}^n) \mid \gamma' = A(t)\gamma\}$ form a n-dimensional subspace of $C^1(J, \mathbb{R}^n)$.

Theorem 5.25 (Variation of constants)**.** *Let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ be the propagator of a homogeneous linear system $\gamma' = A(t)\gamma$ and $b : J \to \mathbb{R}^n$ continuous. Then the solution of the inhomogeneous equation:*

$$\gamma' = A(t)\gamma + b(t) \qquad \textit{with} \qquad \gamma(t_0) = x_0$$

is

$$\gamma(t) = \Phi_t\Big(x_0 + \int_{t_0}^{t} \Phi_s^{-1} b(s)\, ds\Big) \,.$$

This approach is called the variation of constants.

Exercises

1. Show that if $f : \mathbb{R}^n \supset \Omega \to \Omega' \subset \mathbb{R}^n$ is a diffeomorphism, its differential $Df|_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for every $x \in \Omega$.

2. Find an example of a continuously differentiable bijection that is not a diffeomorphism.

3. Determine and draw some integral curves for the vector fields

$$\begin{aligned} v :&\mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto v(x,y) = \begin{pmatrix} -x \\ y \end{pmatrix}, \\ w :&\mathbb{R}^2 \to \mathbb{R}^2, \quad (x,y) \mapsto w(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

4. Show that every autonomous ODE of order m can be reduced to a system of m first order autonomous ODEs.

5. Let $u : \mathbb{R} \to \mathbb{R}$, $a, b \in \mathbb{R}$. Name the type of the following ODEs and find a general solution.

(i) $u'(t) = -b(u(t) - a)$

(ii) $u'(t) + au(t) = e^{-t}$