MATHEMATISCH-NATURWISSENSCHAFTLICHE FAKULTÄT WiSe 2024/25
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Propagation of Chaos

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Sheet 11

Exercise 1:

(a) Let $\|\cdot\|_{p\wedge q}$ be given by

$$\|f\|_{p\wedge q} = \inf_{f_1+f_2=f} \left\{ \|f_1\|_p + \|f_2\|_q \right) \} .$$

Show that $\|\cdot\|_{p\wedge q}$ defines for any $1 \leq p < q \leq \infty$ a norm on the respective space of functions where $\|\cdot\|_{p\wedge q}$ is finite (called $L^{p\wedge q}$).

(b) Let $\phi \in L^2 \cap L^\infty$ and $V \in L^{2\wedge\infty}$. Show that $\|V \star |\phi|^2\|_\infty$ and $\|V^2 \star |\phi|^2\|_\infty$ are bounded.

Exercise 2: Let $\Psi \in L^2(\mathbb{R}^{3N})$ with $\|\Psi\| = 1$ and μ^Ψ be the respective one-particle reduced density matrix.

Show that there is a sequence of vectors $\Psi_n \in L^2(\mathbb{R}^{3N})$ such that $\lim_{n\to\infty} \|\mu^\Psi - \mu^{\Psi_n}\|_{tr} = 0$ and such that μ^{Ψ_n} is diagonalizable for any $n \in \mathbb{N}$.

Exercise 3: Let ϕ_t be a solution of the Hartree equation with $\phi_t \in L^2 \cap L^\infty$ for any $t \geq 0$ for a potential $V \in L^{2\wedge\infty}$. Let Ψ_t be a solution of the respective Schrödinger equation, i.e.

$$i\frac{d}{dt}\phi_t = (-\Delta + V \star |\phi_t|^2)\phi_t$$

$$i\frac{d}{dt}\Psi_t = (-\sum_{j=1}^{N} \Delta_j + \frac{1}{N-1} V(x_j - x_k))\Psi_t$$

Assume that $\Psi_0 = \prod_{j=1}^N \phi_0(x_j)$. Show that there is a $C < \infty$ such that

$$\langle \Psi_t, q_1^{\phi_t} q_2^{\phi_t} \Psi_t \rangle \leq C N^{-2}(e^{Ct} - 1)$$