Propagation of Chaos

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Sheet 10

Exercise 1: Let $\Psi \in L^2\left(\mathbb{R}^{3N}\right)$ and μ^Ψ be the respective reduced one-particle density matrix. Show that μ^Ψ is a positive, self-adjoint operator of trace one.

Exercise 2: Let A be an operator on a given Hilbert space $\mathcal{H}$ of dimension $d < \infty$.

Show that the trace norm can be alternatively defined via

$$\|A\|_{tr} = \operatorname{tr}\left(\sqrt{AA^\dagger}\right) \ .$$

Here $A^\dagger$ stands for the adjoint of A.

Exercise 3: Let $\phi \in L^2(\mathbb{R}^3)$ with $\|\phi\| = 1$. Show that for any $j \in 1, \ldots, N$ the operator $p_j^\phi := 1 \otimes 1 \otimes \ldots \otimes \overset{j^{th}\text{comp.}}{|\phi\rangle\langle\phi|} \otimes 1 \otimes \ldots \otimes 1$ is a projector on $L^2(\mathbb{R}^{3N})$.

Show that for any $\Psi \in L^2(\mathbb{R}^{3N})$

$$\operatorname{tr}(p_1^\phi \mu^\Psi) = \langle \Psi, p_1^\phi \Psi \rangle$$

Exercise 4: Let $\mathcal{V}$ be a normed vector space. On the set of operators on $\mathcal{V}$ the operator-norm is defined via

$$\|T\|_{op} := \sup_{\Psi \in \mathcal{V} \setminus 0} \frac{\|T\Psi\|}{\|\Psi\|}$$

Assuming that $\mathcal{V}$ is a Hilbert space we can define for any given orthonormal basis $\mathcal{B} := \{b_1, b_2, \ldots\}$ the norm $\| \cdot \|_\mathcal{B}$ via

$$\|T\|_\mathcal{B} = \sup_{1 \leq i,j \leq d} |\langle b_i, T b_j \rangle| \ .$$

Here $d \in \mathcal{N} \cup \{\infty\}$ is the dimension of the vector space.

Show that $\| \cdot \|_\mathcal{B}$ is in fact a norm and that in the case of finite dimension

$$\| \cdot \|_\mathcal{B} \leq \| \cdot \|_{op} \leq d \| \cdot \|_\mathcal{B} \ .$$

Show that for $d = \infty$ the operator norm is neither equivalent to $\| \cdot \|_\mathcal{B}$ nor to $\| \cdot \|_{tr}$.