Propagation of Chaos

Prof. Dr. P. Pickl

Sheet 8

Aufgabe 1: (LLN and relative entropy)

Let $\phi : \mathbb{K} \to \mathbb{R}_0^+$ be probability densities on some one-particle configuration space $\mathbb{K}$ and $\chi_N : \mathbb{K}^N \to \mathbb{R}_0^+$ be a sequence of probability densities on the respective N-particle configuration space. Assume that there is a sequence $(\alpha_N)_{N\in\mathbb{N}}$ with $\lim_{N\to\infty} \alpha_N = 1$ such that for any $A \subset \mathbb{K}$ and any $x_1, x_2, \ldots, x_{N-1}$ the (conditional) probability to find $x_N \in A$ given $x_1, x_2, \ldots, x_{N-1}$ is bounded by

$$\mathbb{P}^{\chi_N}_{(x_1,\ldots,x_{N-1})}(x_N \in A) \leq \alpha_N \int_A \rho(x)dx \, .$$

(a) Show that $\lim_{N\to\infty} r(\chi_N, \phi^{\otimes N}) = 0$ where r is the relative entropy.

(b) Given a bounded function $X : \mathbb{K} \to \mathbb{R}$ such that $\int |X(x)|\rho(x)dx < \infty$. This defines a sequence $(X_N)_{N\in\mathbb{N}}$ of i.i.d. (w.r.t. the probability measure $\rho^{\otimes N}$) random variables via $X_N := X(x_N)$. Show that the Law of Large Number holds for this sequence, that means that $\overline{X}_N := \frac{1}{N}\sum_{j=1}^N X_j$ converges in probability to $\mu = \mathbb{E}(X)$.

Aufgabe 2: (Keller-Segel) Consider the two-dimensional Keller-Segel equation

$$\frac{\partial}{\partial t}\rho_t = \Delta\phi_t - \operatorname{div}(\phi_t(\phi_t \star f))$$

where the force f is the two dimensional Coulomb force, i.e. $f(x) = \mu\frac{x}{|x|^2}$.

Let ϕ_0 be a Gaussian, centered at $x = 0$ with width δ. Calculate $\Delta\phi_0$ and $\operatorname{div}(\phi_t\phi_t \star f)$.

Argue on a heuristic level that there is a critical value for μ separating the regimes where ρ_t is unstable respectively stable. Hint: What happens for very small δ for different values of μ?

Aufgabe 3: (Fokker Planck)

Give the PDE that effectively describes a gas of many particles subject to Newtonian motion with weak interaction, i.e. force $\frac{1}{N}\sum_{k\neq j}^N f(x_j - x_k)$ acting on the j^{th} particle, and Brownian motion influencing the velocities with $\sigma = 1$. Assume that initially all particles are i.i.d. with respect to ρ_0.

Show that the propagated probability density converges to the product ρ_t in relative entropy.