Propagation of Chaos

Prof. Dr. P. Pickl

Sheet 5

Aufgabe 1: Let $A_t = (X_t, V_t)$ and $\overline{A}_t = (\overline{X}_t, \overline{V}_t)$ be given via

$$\dot{x}_t^j = v_t^j \qquad \dot{v}_t^j = f^{ext}(x_j) + \frac{1}{N-1}\sum_{j\neq k} f(x_t^j - x_t^k)$$
$$\dot{\overline{x}}_t^j = \overline{v}_t^j \qquad \dot{\overline{v}}_t^j = f^{ext}(\overline{x}_j) + \overline{f}(\overline{x}_t^j)$$

Assume that $\|f^{ext}\|_L < \infty$,$\|f\|_L < \infty$ and $\|f\|_\infty < \infty$. Show that for any $t \geq 0$

$$\lim_{N\to\infty} \mathbb{E}\left(\left\|A_t - \overline{A}_t\right\|_\infty\right) = 0 \ .$$

Aufgabe 2: Consider the system of exercise 1. Find a function $g(t)$ with

$$\lim_{N\to\infty} g(t) = 0 \ \forall t \in \mathbb{R}_0^+$$

such that

$$\mathbb{E}\left(\left[\left\|A_t - \overline{A}_t\right\|_\infty - g(t)\right]_+\right)$$

is, for sufficiently large N, smaller than any polynomial in N.

Here $[h]^+$ stands for the positive part of the function, i.e. $[h]^+ = 0$ if $h \leq 0$, $[h]^+ = h$ if $h > 0$.

Aufgabe 3: Let N be even. In this exercise we consider the solution of N Newtonian particles in two spacial dimensions, i.e. the one-body phase space has dimension 4.

Assume that the interaction potential is repulsive Coulomb with $\frac{1}{N}$-coupling, i.e. the force for each particle is given by

$$\dot{x}_t^j := \frac{1}{N}\sum_{k\neq j} \frac{x_j - x_k}{|x_j - x_k|^3} \ .$$

Assume that at time $t = 0$ the velocities of all particles are zero and that they are distributed on a uni-circle of radius $N^{-1/2}$, centered at the origin. That means, in polar coordinates $x_j = (N^{-1/2}, \frac{2\pi j}{N})$.

Show that $\lim_{N\to\infty} |x_t^j| = \infty$ for any $t > 0$ and any $j \in \{1, \ldots, N\}$.