Propagation of Chaos

Prof. Dr. P. Pickl

Sheet 4

Aufgabe 1: Let $\Omega = [0,1]$ and $\rho : \Omega \to \mathbb{R}_0^+$ be the constant density on Ω, i.e. $\rho(x) = 1$ or all $x \in \Omega$.

Let for any $X = (x_1, x_2, \dots, x_N) \subset \Omega^N$ the empirical density ρ_X^{emp} be given by $\rho_X^{\text{emp}}(x) = \frac{1}{N}\sum_{j=1}^N \delta(x - x_j)$.

Find a $X \in \Omega^N$ such that $d_{\text{BL}}(\rho, \rho_X^{\text{emp}})$ is minimal. What is $\inf_{X\in\Omega^N} d_{\text{BL}}(\rho, \rho_X^{\text{emp}})$?

Discuss the case of general $\rho : \Omega \to \mathbb{R}_0^+$ with $\|\rho\|_\infty < C$ for some $C \in \mathbb{R}^+$.

Aufgabe 2: Let $A, B \subset \mathbb{R}^{6N}$, $\rho_{A/B}^{\text{emp}} : \mathbb{R}^6 \to \mathbb{R}_0^+$ be given by

$$\rho_A^{\text{emp}}(a) := \sum_{j=1}^N \delta(a - a_j) \qquad \rho_B^{\text{emp}}(a) := \sum_{j=1}^N \delta(a - b_j)$$

Show that $d_{\text{BL}}(\rho_A^{\text{emp}}, \rho_B^{\text{emp}}) \leq \|A - B\|_1$.

Aufgabe 3: Let $A_t = (X_t, V_t)$ and $\overline{A}_t = (\overline{X}_t, \overline{V}_t)$ be given via

$$\begin{aligned} \dot{x}_t^j &= v_t^j & \dot{v}_t^j &= \frac{1}{N-1}\sum_{j\neq k} f(x_t^j - x_t^k) \\ \dot{\overline{x}}_t^j &= \overline{v}_t^j & \dot{\overline{v}}_t^j &= \overline{f}(\overline{x}_t^j) \end{aligned}$$

Assume that $\|f\|_L < \infty$ and $\|f\|_\infty < \infty$. Show that for any $t \geq 0$

$$\lim_{N\to\infty} \mathbb{E}\left(N^{-1}\left\|A_t - \overline{A}_t\right\|_1\right) = 0\ .$$

Hint: It suffices to use the Tschebychew–version of the Law of Large Numbers.