Propagation of Chaos

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Sheet 3

Aufgabe 1: (Bounded Lipschitz distance) Let $\Omega := [0, 1]$, η, ρ be probability measures with sample space Ω (i.e. defined on the respective Borel set $\mathcal{B}(\Omega)$). Show that

(a)
$$d_{BL}(\eta, \rho) \leq \|\eta - \rho\|_1 \ .$$

(b)
$$d_{BL}(\eta, \rho) = \int_0^1 \left| \int_0^t \eta(s) - \rho(s) ds \right| dt$$

(c)
$$d_{BL}(\eta, \rho) = 0 \ \Leftrightarrow \ \eta = \rho \ .$$

Aufgabe 2: (Grönwall)

Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a continuous function which is differentiable from the left with $\frac{d^-}{dt} f(t) \leq C(f(t) + \varepsilon)$ for some $\varepsilon > 0$. Show that

$$f(t) \leq e^{Ct} f(0) + (e^{Ct} - 1)\varepsilon$$

for any $t \in \mathbb{R}_0^+$.

Aufgabe 3: (Differential inequality)

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function which is monotonously increasing and globally Lipschitz in the second argument. This means that there is a $L > 0$ such that for any $a \in \mathbb{R}$ and any $x, y \in \mathbb{R}$ with $x \leq y$ it holds that

$$0 \leq F(a, y) - F(a, x) \leq L(y - x)$$

Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a differentiable function which satisfies the following differential inequality:

$$\dot{f} \leq F(\cdot, f) \ .$$

Show that for all $t \in \mathbb{R}_0^+$ it holds that $f(t) \leq g(t)$ where g is a solution of the respective ODE: $\dot{g} = F(\cdot, g)$.

Does this ODE always have a solution for a given initial condition $g(0) = g_0$?