Propagation of Chaos

Prof. Dr. P. Pickl

Blatt 1

Aufgabe 1: Let Ω be a set, I be some index set. Let for any $i \in I$ $\mathcal{A}_i$ be a σ-algebra with respect to Ω. Show that

$$\bigcap_{i \in I} \mathcal{A}_i$$

is also a σ-algebra. Find an example such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra.

Aufgabe 2: Let Ω be a set $\mathcal{E} \subset \mathcal{P}(\Omega)$ (a subset of the power set of Ω). Let $\sigma(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E}$, i.e.

$$\sigma(\mathcal{E}) = \bigcap_{\mathcal{A} \text{ is } \sigma\text{-alg}}^{\mathcal{E} \subset \mathcal{A}} \mathcal{A} \, .$$

Show that for any $(A_i)_{i \in \mathbb{N}} \subset \mathcal{E}$ it follows that $\bigcap_{i=1}^{\infty} A_i \in \sigma(\mathcal{E})$.

Aufgabe 3: Show that the Borel σ-algebra can alternatively be defined as the σ-algebra generated by all open intervals $]a, b[$ for $a, b \in \mathbb{R}$ (instead of the σ-algebra generated by all open subsets of $\mathbb{R}$).

Show that all closed intervals $[a, b]$ for $a, b \in \mathbb{R} \cup \{\pm\infty\}$ are elements of the Borel σ-algebra.

Aufgabe 4: Let $(X_j)_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables with probability measure $\rho \in L^\infty$, i.e. $\mathbb{P}(X_j \in B) = \int_B \rho(x) d^3x$ for any element B of the Borel set. Let for some $a > 0$ $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x) = |x|^{-2}$ if $|x| \geq N^{-a}$ and $f(x) = 0$ else. Define $Y_i^N = f(X_i)$.

Use the strong version of the Law of Large Numbers we found in class to estimate

$$\mathbb{P}\left(|\overline{Y}_N^N - \mathbb{E}\left(Y_1^N\right)| \geq \varepsilon \right) \, .$$

For which range of ε does the estimate hold? For which values of ε do you get a non-trivial estimate?