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Foundations of Quantum Mechanics: Assignment 11

Exercise 43: Essay question. (25 points)
Describe the Einstein–Podolsky–Rosen argument (either in terms of position and momentum or in terms of spin matrices).

Exercise 44: Positive operators (25 points)
An operator $S : \mathbb{C}^d \to \mathbb{C}^d$ is positive (= positive semi-definite) iff $\langle\psi|S\psi\rangle \geq 0$ for all ψ. Are the following statements about operators on $\mathbb{C}^d$ true or false? Justify your answers.

a) $R^\dagger R$ is always a positive operator.

b) If E is a positive operator, then so is $R^\dagger E R$.

c) The positive operators form a subspace of the space of self-adjoint operators.

d) The sum of two projections is positive only if they commute.

e) e^{At} is a positive operator for every self-adjoint A and $t \in \mathbb{R}$.

Exercise 45: Angles in Hilbert space (25 points)
In Chapter 2, I said that the angle θ between two vectors ϕ, χ in Hilbert space is $\theta = \arccos \frac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}$. I take it back. In this exercise, we look at the reasons why the natural definition for θ actually is

$$\theta = \arccos \frac{\mathrm{Re}\langle\phi|\chi\rangle}{\|\phi\|\,\|\chi\|}\,, \tag{1}$$

while the one for the angle α between the 1d subspaces $\mathbb{C}\phi, \mathbb{C}\chi$ is

$$\alpha = \arccos \frac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}\,. \tag{2}$$

(a) For the most natural identification mapping $J : \mathbb{C}^d \to \mathbb{R}^{2d}$ given by $J(\phi_1, \ldots, \phi_d) = (\mathrm{Re}\,\phi_1, \mathrm{Im}\,\phi_1, \ldots, \mathrm{Re}\,\phi_d, \mathrm{Im}\,\phi_d)$, show that

$$\langle J\phi|J\chi\rangle_{\mathbb{R}^{2d}} = \mathrm{Re}\langle\phi|\chi\rangle_{\mathbb{C}^d} \quad \text{and} \quad \|J\phi\|_{\mathbb{R}^{2d}} = \|\phi\|_{\mathbb{C}^d}\,, \tag{3}$$

where $\langle x|y\rangle_{\mathbb{R}^n} = \sum_{i=1}^n x_i y_i$ and $\langle\phi|\chi\rangle_{\mathbb{C}^d} = \sum_{j=1}^d \phi_j^* \chi_j$, and the norms are accordingly defined.

(b) Explain by means of an example in $\mathbb{R}^3$ why the intuitive notion of the angle β between two subspaces U, V of $\mathbb{R}^n$ is given by

$$\beta = \inf_{u \in U\setminus\{0\}} \inf_{v \in V\setminus\{0\}} \theta(u, v)\,, \tag{4}$$

where $\theta(u, v)$ means the angle between u and v. We adopt the same definition in $\mathbb{C}^d$.

(c) Show that for $\phi, \chi \in \mathbb{C}^d \setminus \{0\}$,

$$\inf_{u \in \mathbb{C}\phi\setminus\{0\}} \inf_{v \in \mathbb{C}\chi\setminus\{0\}} \arccos \frac{\mathrm{Re}\langle u|v\rangle}{\|u\|\,\|v\|} = \arccos \frac{|\langle\phi|\chi\rangle|}{\|\phi\|\,\|\chi\|}\,. \tag{5}$$

Exercise 46: Sum of projections (25 points)
Let $\mathscr{H}$ be a Hilbert space of finite dimension, let P_1 and P_2 be projections in $\mathscr{H}$, $P_i = P_i^\dagger$ and $P_i^2 = P_i$, and let $\mathscr{H}_i$ be the range of P_i. Show that if $Q := P_1 + P_2$ is also a projection ($Q = Q^\dagger$ and $Q^2 = Q$), then **(a)** $\mathscr{H}_1 \perp \mathscr{H}_2$, and **(b)** the range $\mathscr{K}$ of Q is the span of $\mathscr{H}_1 \cup \mathscr{H}_2$.

Hand in: by Tuesday January 23, 2024, 8:15am

Reading assignment due Thursday January 25, 2024:
N. Bohr: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Physical Review* **48**: 696–702 (1935)