Foundations of Quantum Mechanics: Assignment 7

Exercise 26: Essay question (20 points)
Explain why the interference pattern in the double-slit experiment disappears if a detector measures through which slit the electron went, even if we ignore the outcome of the detection. Derive this fact from the projection postulate, using the observable P_B defined by $P_B\psi(\boldsymbol{x}) = 1_B(\boldsymbol{x})\psi(\boldsymbol{x})$ as a model of a detector, where B is a suitable neighborhood of one slit. Explain why the pattern on the screen must be the same as if one slit were closed in each run, each slit in 50% of the runs. (Use formulas where appropriate.)

Exercise 27: Projections (30 points)
We defined a projection to be an operator P such that there is an ONB $\{\phi_n : n \in \mathbb{N}\}$ diagonalizing P, $P\phi_n = \lambda_n\phi_n$, with eigenvalues λ_n that are 0 or 1.

(a) Show that the projections are exactly the self-adjoint operators P with $P^2 = P$.

(b) Suppose that $P : \mathscr{H} \to \mathscr{H}$ is a projection with range $\mathscr{K}$; one says that P is the projection to $\mathscr{K}$. Show that $I - P$ is the projection to the orthogonal complement of $\mathscr{K}$, i.e., to $\mathscr{K}^\perp = \{\phi \in \mathscr{H} : \langle\phi|\psi\rangle = 0 \,\forall\psi \in \mathscr{K}\}$.

(c) Suppose that P is the projection to $\mathscr{K}$. Show that the element in $\mathscr{K}$ closest to a given vector $\psi \in \mathscr{H}$ is $P\psi$.

Exercise 28: Iterated Stern-Gerlach experiment (20 points)
Consider the following experiment on a single electron. Suppose it has a wave function of the product form $\psi_s(\boldsymbol{x}) = \phi_s\,\chi(\boldsymbol{x})$, and we focus only on the spinor. The initial spinor is $\phi = (1, 0)$.

(a) A Stern–Gerlach experiment in the y-direction (or σ_2-measurement) is carried out, then a Stern–Gerlach experiment in the z-direction (or σ_3-measurement). Both measurements taken together have four possible outcomes: up-up, up-down, down-up, down-down. Find the probabilities of the four outcomes.

(b) As in (a), but now the z-experiment comes first and the y-experiment afterwards.

Please turn over.

Exercise 29: Lie algebras of $SO(3)$ and $SU(2)$ (30 points. Level: difficult)
A *Lie group* G, named after Sophus Lie (1842–1899), is a group that is also a manifold (a curved surface) such that the group multiplication and inversion are smooth mappings. Examples of Lie groups include $GL(n), SO(n), U(n), SU(n)$. The elements infinitesimally close to 1 in G form the *Lie algebra* g of G; more precisely, g is the tangent space of 1, which is here the set

$$\left\{ \frac{dA}{dt}(t=0) \middle| A : (-1,1) \to G \text{ smooth}, A(0) = 1 \right\}.$$

(a) Determine the Lie algebras $so(3)$ and $su(2)$ as subspaces of the space of all real 3×3 (complex 2×2) matrices.

(b) The *exponential mapping* $\exp : g \to G$ can be heuristically understood as follows: For $X \in g$, a corresponding group element infinitesimally close to 1 can be written as $1 + X/n$ with n a large natural number (so $1/n$ serves as an infinitesimal dt). Hence, roughly speaking, $(1 + X/n) \in G$, hence $(1 + X/n)^n \in G$; take the limit $n \to \infty$ to obtain $\exp(X) =: e^X$. Verify that the matrix exponential (defined by the exponential series) actually maps $so(3)$ to $SO(3)$ and $su(2)$ to $SU(2)$. (Hint: diagonalize $X \in g$.)

(c) We now consider the question what the group multiplication of e^X and e^Y looks like for $X, Y \in g$. We know that the solution Z of $e^Z = e^X e^Y$ is $Z = X + Y$ if X and Y commute, but not in general. A version of the *Baker–Campbell–Hausdorff formula* says that

$$\text{the solution of } e^Z = e^{-tX} e^{-tY} e^{t(X+Y)} \text{ is } Z = \tfrac{1}{2}t^2[X,Y] + \mathcal{O}(t^3)$$

as $t \to 0$, with $[X,Y] = XY - YX$ the *commutator* or *Lie bracket*. The Lie bracket is an operation on g that encodes how the group multiplication deviates from addition in g. Thus, one defines a *Lie algebra* in general as a vector space together with a bracket $[\cdot,\cdot] : g \times g \to g$ that is anti-symmetric, bilinear, and satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\,.$$

Verify that $so(3)$ and $su(2)$ (with commutators as Lie brackets) are isomorphic Lie algebras. (Hint: The Pauli matrices have something to do with rotations about the x-, y-, and z-axis.)

Hand in: by Tuesday December 5, 2023, 8:15am

Reading assignment due Thursday December 7, 2023: A. Einstein, *Reply to Criticisms*, pages 665–688 in P. Schilpp (editor): *Albert Einstein, Philosopher–Scientist* (1949). Read pages 665–672 and the first quarter of 673.