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Winter semester 2023/24
November 14, 2023

Foundations of Quantum Mechanics: Assignment 5

Exercise 18: Essay question. (20 points)
Describe what the Heisenberg uncertainty relation asserts.

Exercise 19: Pauli matrices (30 points)
The three *Pauli matrices* are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) For each of σ_1 and σ_2, find an orthonormal basis of eigenvectors in $\mathbb{C}^2$.

(b) Show that for every unit vector $\boldsymbol{n} \in \mathbb{R}^3$, the Pauli matrix in direction $\boldsymbol{n}$, $\sigma_{\boldsymbol{n}} := \boldsymbol{n} \cdot \boldsymbol{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$, has eigenvalues ± 1. (Hint: compute det and trace.)

(c) Show that every self-adjoint complex 2×2 matrix A is of the form $A = cI + \boldsymbol{u} \cdot \boldsymbol{\sigma}$ with $c \in \mathbb{R}$ and $\boldsymbol{u} \in \mathbb{R}^3$.

Exercise 20: Spectral theorem (20 points)
In this problem we use without proof the following generalization of the spectral theorem (formulated here in finite dimension): If the self-adjoint $d \times d$ matrices A and B commute, then they can be simultaneously unitarily diagonalized, i.e., there is an orthonormal basis $\{\phi_1, \dots, \phi_d\}$ such that each ϕ_j is an eigenvector of A and an eigenvector of B.

Show that a $d \times d$ matrix C can be unitarily diagonalized iff C commutes with $C^\dagger$. Such a matrix is called "normal." (Hint: write $C = A + iB$.)

Exercise 21: Potential step (30 points)
Consider a potential step $V(x) = V_0 \, 1_{0 \leq x}$ in 1d with $V_0 > 0$. We want to compute the reflection and transmission probabilities as a function of the incoming momentum $\hbar k_0$, assuming that $E := \hbar^2 k_0^2/2m > V_0$. A recipe for that is to consider a plane wave e^{ik_0x} coming from $x = -\infty$ and see how much gets reflected and transmitted by constructing an eigenfunction ψ of H, $H\psi = E\psi$, from the ansatz

$$\psi(x) = \begin{cases} Ae^{ik_0x} + Be^{-ik_0x} & \text{for } x < 0 \\ Ce^{iK_0x} & \text{for } x > 0 \end{cases} \tag{2}$$

with complex coefficients A, B, C, representing the incoming wave e^{ik_0x}, the reflected wave e^{-ik_0x}, and the transmitted wave e^{iK_0x}. Regarding V as a limit of continuous functions leads to the further requirement that ψ be continuous and continuously differentiable at 0.

(a) Determine K_0 from $H\psi = E\psi$.

(b) Determine B and C from A.

(c) The recipe says that the three waves are associated with probability currents $j_{\text{in}} = \hbar k_0 |A|^2/m$, $j_R = \hbar k_0 |B|^2/m$, and $j_T = \hbar K_0 |C|^2/m$; and that the reflection probability is $P_R = j_R/j_{\text{in}}$, while the transmission probability is $P_T = j_T/j_{\text{in}}$. Compute P_R and P_T.

(d) To justify the recipe, consider a "plane wave packet" $\psi(x) = \phi(x)\, e^{ik_0 x}$ with a (non-Gaussian) envelope profile $\phi(x)$ that is nearly constant over a region of length L much larger than the wave length $2\pi/k_0$ and then drops to 0 over a length much smaller than L but still much larger than the wave length. Let us take for granted that under the free Schrödinger evolution the envelope function ϕ will approximately maintain its shape (in particular, its length) for a long time and simply move at speed $\hbar k_0/m$. Suppose the packet hits the step at $t = 0$; forget about what happens at the edges of the packet and focus on the bulk. A reflected plane wave packet and a transmitted one are generated; at time $\tau > 0$, the incoming plane wave packet is used up, and the two outgoing packets end. During $0 < t < \tau$, the region around 0 is well approximated by (2); after τ, the outgoing packets keep moving away from the origin. Determine τ and the lengths L_R and L_T of the reflected and transmitted packets ψ_R and ψ_T.

(e) Determine $P_R = \|\psi_R\|^2$ and $P_T = \|\psi_T\|^2$ and verify that they agree with part (c).

(f) Draw a space-time diagram of the Bohmian trajectories in the bulk (that is, ignoring any edge effects).

Hand in: by Tuesday, November 21, 2023, 8:15am

Reading assignment due Thursday November 23, 2023: Section II of T. Norsen: The Pilot-Wave Perspective on Quantum Scattering and Tunneling. *American Journal of Physics* **81**: 258 (2013)