Prof. Dr. Roderich Tumulka
Eberhard Karls University Tübingen

Winter semester 2023/24
October 24, 2023

Foundations of Quantum Mechanics: Assignment 2

Exercise 5: Essay question: What is surprising about the double-slit experiment?

Exercise 6: Unitary operators.
(a) For any vector $\boldsymbol{a} \in \mathbb{R}^3$, the *translation operator* $T_{\boldsymbol{a}}$ is defined on $L^2(\mathbb{R}^{3N})$ by

$$\big(T_{\boldsymbol{a}}\psi\big)(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) = \psi(\boldsymbol{x}_1-\boldsymbol{a},\ldots,\boldsymbol{x}_N-\boldsymbol{a})\,. \tag{1}$$

It shifts the wave function by $\boldsymbol{a}$ in every $\boldsymbol{x}_i$. Show that $T_{\boldsymbol{a}}$ is unitary.

(b) For any rotation matrix $R \in SO(3)$, the *rotation operator* U_R is defined on $L^2(\mathbb{R}^{3N})$ by

$$\big(U_R\psi\big)(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N) = \psi(R^{-1}\boldsymbol{x}_1,\ldots,R^{-1}\boldsymbol{x}_N)\,. \tag{2}$$

It rotates the wave function according to R in every $\boldsymbol{x}_i$. Show that U_R is unitary.

Exercise 7: Orthonormal system.
For $n \in \mathbb{Z}$, let the function $\varphi_n : [-\pi,\pi] \to \mathbb{C}$ be defined by

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}\,. \tag{3}$$

Show that they form an *orthonormal system* in $L^2([-\pi,\pi])$, i.e., that

$$\langle\varphi_n|\varphi_m\rangle = \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{4}$$

with

$$\langle\psi|\phi\rangle = \int_{-\pi}^{\pi} \psi(x)^*\,\phi(x)\,dx\,. \tag{5}$$

Exercise 8: Dense subspace.
The space $\ell^2 = \{(x_1,x_2,\ldots) : x_n \in \mathbb{C}, \sum |x_n|^2 < \infty\}$ of all square-summable sequences is a Hilbert space with inner product

$$\langle x|y\rangle = \sum_{n=1}^{\infty} x_n^* y_n\,. \tag{6}$$

A subset S of a Hilbert space $\mathscr{H}$ is called *dense* if for every $\psi \in \mathscr{H}$ and $\varepsilon > 0$ there is $\phi \in S$ with $\|\psi - \phi\| < \varepsilon$. (For example, the set $\mathbb{Q}$ of rational numbers is dense in $\mathbb{R}$.) Let S be the subspace of ℓ^2 consisting of all sequences $(x_1,x_2,\ldots)$ with only finitely many nonzero entries.

(a) Show that S is dense in ℓ^2.

(b) Show that in a finite-dimensional Hilbert space (that is, without loss of generality, in $\mathbb{C}^n$), the only dense subspace is the full space $\mathbb{C}^n$.

Hand in: Tuesday October 31, 2023, at 8:15 AM

Reading assignment due Thursday November 2, 2023: First three pages of J. Bell: De Broglie–Bohm, delayed-choice double-slit experiment, and density matrix. *International Journal of Quantum Chemistry* **14**: 155–159 (1980)