Universität Tübingen, Fachbereich Mathematik Wintersemester 23/24
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# Groups and Representations

Homework Assignment 13 (due on 31 January 2024)

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**Problem 44**

Let $|j\rangle$, $j = 1, \ldots, N$, be the canonical basis for $V \cong \mathbb{C}^N$, and let $\mathrm{GL}(N)$ act on $V$ in the defining representation, i.e. $g|j\rangle = |k\rangle g_{kj}$ for $g \in \mathrm{GL}(N)$ (recall that we sum over repeated indices). For $V \otimes V$ choose the product basis $|jk\rangle = |j\rangle \otimes |k\rangle$, $j, k = 1, \ldots, N$, and let $\mathrm{GL}(N)$ act as follows,

$$g|jk\rangle = |j'k'\rangle g_{j'j}\, g_{k'k}\,.$$

Specialise to $N = 2$ and consider $e_{\substack{\square\\\square}} V \otimes V$. We obtain a basis for $e_{\substack{\square\\\square}} V \otimes V$ by applying $e_{\substack{\square\\\square}}$ to basis tensors $|jk\rangle$.

**Show** that $e_{\substack{\square\\\square}} V \otimes V$ is invariant under $\mathrm{GL}(N)$ by explicitly calculating the representation of $G$ carried by $e_{\substack{\square\\\square}} V \otimes V$. To this end apply $g \in \mathrm{GL}(2)$ to a basis of $e_{\substack{\square\\\square}} V \otimes V$ and read off the representation. Is this representation irreducible?

Repeat with $\mathrm{GL}(2)$ replaced by $\mathrm{SU}(2)$. Which representation do we obtain now?

**Problem 45**

We show that the $\mathrm{GL}(N)$ irrep corresponding to the Young diagram $\Theta_\mathrm{a} = \begin{array}{|c|}\hline \phantom{x} \\ \hline \phantom{x} \\ \hline \vdots \\ \hline \phantom{x} \\ \hline \end{array}$ with $N$ rows is given by the determinant:

- First recall that for vectors $|i_1, \ldots, i_N\rangle$ contributing to $e_\mathrm{a} g|\alpha\rangle$ all $i_k$ are different.
- Write these vectors as $p|1, \ldots, N\rangle$ with a permutation $p$.
- Then calculate $e_\mathrm{a} g|1, \ldots, N\rangle$ for $g \in \mathrm{GL}(N)$.

Which irrep corresponds to $\Theta_\mathrm{a}$ if we replace $\mathrm{GL}(N)$ by the subgroup $\mathrm{SU}(N) \subset \mathrm{GL}(N)$?