Universität Tübingen, Fachbereich Mathematik
Dr. Stefan Keppeler & Saradha Senthil-Velu

Wintersemester 23/24

Groups and Representations

Homework Assignment 12 (due on 24 January 2024)

Problem 41

Let Γ^j be the irreps of SU(2) (or of SO(3)) as constructed in the lecture.

a) Prove the following formula for the characters

$$\chi^j(\alpha) = \frac{\sin\left((2j+1)\alpha/2\right)}{\sin(\alpha/2)}\,.$$

b) Consider the product representation $\Gamma^j \otimes \Gamma^k$ with $j \geq k$. Show that every irrep Γ^ℓ with $\ell = j-k, \dots, j+k$ appears exactly once in the decomposition of $\Gamma^j \otimes \Gamma^k$, and that all other irreps are absent.

Problem 42

Let Γ be an irrep of SU(2) on $\mathbb{C}^n$. Show that there exists a $T \in \mathrm{GL}(n, \mathbb{C})$ with $T^2 \in \{\pm\mathbb{1}\}$ s.t.

$$\overline{\Gamma(g)} = T\Gamma(g)T^{-1} \quad \forall\, g \in \mathrm{SU}(2).$$

In which cases do we have $T^2 = \mathbb{1}$?

Hint: First find T for the defining representation. Observe that $T \in \mathrm{SU}(2)$ and then investigate $\Gamma(T)$.

Problem 43

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let ad be the adjoint representation of $\mathfrak{g}$, i.e. $\mathrm{ad}_X(Y) = [X, Y]$. The map

$$\begin{aligned} K : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (X, Y) &\mapsto K(X, Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \end{aligned}$$

is called Killing-Form. Show:

a) K is bilinear and symmetric.

b) $K\big(\mathrm{Ad}_g(X), \mathrm{Ad}_g(Y)\big) = K(X, Y) \quad \forall\, X, Y \in \mathfrak{g}$ and $\forall\, g \in G$.

Remark: For semi-simple Lie groups (which we haven't defined, but the classical groups SU(n) and SO(n) are examples) K is positive definite, i.e. it defines a scalar product.

Let G now be such that K is positive definite. We choose an orthonormal basis $\{X_j\}$ with respect to K, i.e. $K(X_j, X_k) = \delta_{jk}$, and define $C_2 \in E(\mathfrak{g})$ by

$$C_2 = \sum_j X_j X_j\,.$$

Show:

c) C_2 is independent of the choice of basis.

d) C_2 is a Casimir operator (the so-called *quadratic Casimir operator*), i.e.

$$\mathrm{Ad}_g(C_2) = C_2 \quad \forall\, g \in G\,.$$