Universität Tübingen, Fachbereich Mathematik
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Groups and Representations

Homework Assignment 10 (due on 10 January 2024)

Problem 37

The elements of $\mathfrak{su}(2)$ can be written as $X = \vec{\sigma}\cdot\vec{x}$ with $\vec{x} \in \mathbb{R}^3$ (cf. Problems 31 & 36). The action of SU(2) on $\mathfrak{su}(2)$ by conjugation (see Problem 36) then defines a homomorphism

$$\begin{aligned} \varphi &: \mathrm{SU}(2) \to \mathrm{GL}(3,\mathbb{R}) \\ \vec{\sigma}\cdot\varphi(U)\vec{x} &:= U(\vec{\sigma}\cdot\vec{x})U^\dagger \,. \end{aligned}$$

Show that

a) $\varphi(U)_{ij} = \frac{1}{2}\,\mathrm{tr}(\sigma_i U \sigma_j U^\dagger)$,

b) $\varphi(U)^T = \varphi(U)^{-1}$, and

c) $\det(\varphi(U)) = 1$. Hint: Recall the connectedness properties of SU(2).

Hence $\varphi(\mathrm{SU}(2)) \subset \mathrm{SO}(3)$.

d) Determine the kernel of φ.

e) Calculate $\varphi(U_\alpha)$ for $U_\alpha = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\alpha\sigma_3}$, $\alpha \in [0, 2\pi)$ and explain that $\varphi(\mathrm{SU}(2)) = \mathrm{SO}(3)$. What can we now conclude using the homomorphism theorem (Problem 8)?

Problem 38

We define $\mathfrak{sl}(2,\mathbb{C}) := \{A \in \mathbb{C}^{2\times 2} : \mathrm{tr}\,A = 0\}$. Then the matrix exponential is a map

$$\exp : \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{SL}(2,\mathbb{C}) = \{B \in \mathbb{C}^{2\times 2} : \det B = 1\}\,.$$

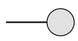
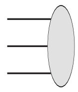
a) Show that the matrix

$$S_a = \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix}$$

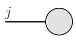
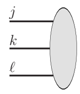
is in the image of exp iff $a = 0$.

b) Is $\mathrm{SL}(2,\mathbb{C})$ compact?

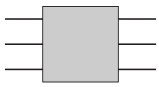
Problem 39
In birdtrack notation we write vectors $v \in V$ or tensors $t \in V^{\otimes n}$, as blobs with one or several lines attached, e.g.

$v =$  $\in V$ or $t =$  $\in V^{\otimes 3}$.

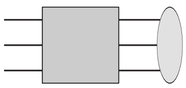
Expressed in a basis, components are

$v_j =$  or $t_{jk\ell} =$  ,

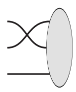
i.e. we write indices on the lines. Linear maps $A : V^{\otimes n} \to V^{\otimes n}$ are represented by blobs with n legs on each side, e.g.

$A =$  $: V^{\otimes 3} \to V^{\otimes 3}$,

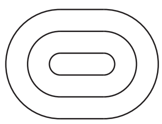
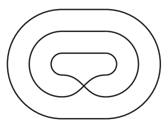
and with $t \in V^{\otimes 3}$ we have

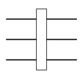
$At =$  $\in V^{\otimes 3}$.

Birdtracks for permutations $p \in S_n$, or $\in \mathcal{A}(S_n)$, see Problems 24 & 28, can now be applied (as linear maps) to elements of $V^{\otimes n}$, e.g.

$(12)t =$  .

We obtain traces of linear maps by connecting the first line on the left to the first line on the right etc. (cf. also Problem 24), with each loop contributing a factor of $\dim V =: N$ (why?), e.g. for $e, (12) \in \mathcal{A}(S_3)$ we get

$\operatorname{tr} e =$  $= N^3$ and $\operatorname{tr}(12) =$  $= N^2$

a) Calculate the trace of $(132) \in S_3$ and the trace of  $\in \mathcal{A}(S_3)$.

b) Normalise the Young operators $e_{\square\square\square}$, $e_{\substack{\square\square\\ \square}}$, $e^{(23)}_{\substack{\square\square\\ \square}}$, $e_{\substack{\square\\ \square\\ \square}} \in \mathcal{A}(S_3)$ of Section 5.3 s.t. they are idempotent. Determine the traces of these primitive idempotents. Later we will see that these are the dimensions of $\mathrm{GL}(N)$ irreps contained in $V^{\otimes 3}$.

HINT: Some identities from Problem 28 are useful.

Merry Christmas and Happy New Year!