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Wintersemester 23/24

Groups and Representations

Homework Assignment 6 (due on 29 November 2023)

Problem 21

We consider a rotationally invariant Hamiltonian. Let E be an eigenvalue of H with eigenspace V_E spanned by the spherical harmonics $Y_{1m}(\varphi, \vartheta) = \cos\vartheta\, \mathrm{e}^{\mathrm{i}m\varphi}$ with a fixed radial part R, i.e. $V_E = \mathrm{span}(\{R(r)Y_{1m}(\varphi, \vartheta) : m = -1, 0, 1\})$.[2]

V_E carries a three-dimensional irreducible representation of O(3), defined by $(\Gamma(U)\psi)(x) = \psi(U^{-1}x)$. O(3) contains the subgroup $D_3 = \{e, C, \bar{C}, \sigma_1, \sigma_2, \sigma_3\} \cong S_3$, where C and $\bar{C}$ denote rotations about the z-axis (cf. Section 2.4.1).

Study the effect of perturbations that are only invariant under D_3 or $\mathbb{Z}_3 \cong \{e, C, \bar{C}\}$. Determine the relevant irreducible representations with their multiplicities and sketch the possible splitting of energy levels.

Problem 22

We consider once more the CO_2 molecule of Problems 10 & 18. In Problem 10 we found a six-dimensional representation of V_4. Decompose the six-dimensional carrier space into irreducible invariant subspaces by applying the generalised projection operators.

Problem 23

$V = \mathbb{C}^2$ carries the 2-dimensional irreducible representation of $D_3 \cong S_3$ (cf. Section 2.4.1). On $W = V \otimes V$ we consider the corresponding product representation. Decompose W into irreducible invariant subspaces by applying the generalised projection operators, and determine the Clebsch-Gordan coefficients.

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[2]We use spherical coordinates

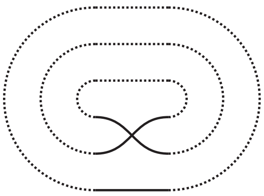
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\vartheta\cos\varphi \\ r\sin\vartheta\sin\varphi \\ r\cos\vartheta \end{pmatrix}.$$

Problem 24
For $\sigma \in S_n$ and $j = 1, \ldots, n$ let $k_j(\sigma)$ be the number of (disjoint) cycles of length j in σ, e.g. $k_1(e) = n$ and $k_j(e) = 0 \; \forall \, j > 1$. Show:

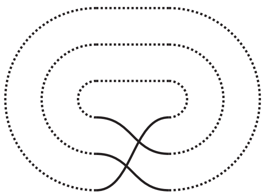
a) The conjugacy class of σ is determined by its cycle structure, i.e.

$$[\sigma] := \{\tau\sigma\tau^{-1} : \tau \in S_n\} = \{\tau \in S_n : k_j(\tau) = k_j(\sigma), j = 1, \ldots, n\}\,.$$

It's almost trivial using the birdtrack notation (see Section 1.4)!
HINT: In order to make the cycle structure visible consider the birdtrack diagram of σ and connect the first line on the left to the first line on the right etc.; e.g. for $(12), (132) \in S_3$ consider



and



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b) The number of elements of a class is given by

$$|[\sigma]| = \frac{n!}{\prod\limits_{j \leq n} k_j! j^{k_j}}\,.$$

c) Fun exercise (optional): Watch the video *An Impossible Bet* by `minutephysics`,

https://youtu.be/eivGlBKlK6M ,

and come up with a good strategy. Don't watch the solution! Think about cycles instead.