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Groups and Representations

Homework Assignment 5 (due on 22 November 2023)

Problem 18

We consider again the CO_2 molecule of Problem 10.

a) How many non-equivalent irreps does the symmetry group V_4 have, and what are their dimensions?

b) Determine the character table for V_4.

In Problem 10 we found a six-dimensional representation of V_4.

c) Which irreps are contained in this six-dimensional representation?

Problem 19

Three spin-$\frac{1}{2}$ particles[1] define a representation D of S_3 on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^8$ by permutations of the particles, i.e. e.g. $D\left((12)\right)|{\uparrow\downarrow\uparrow}\rangle = |{\downarrow\uparrow\uparrow}\rangle$.

Which irreps of S_3 are contained in D and how often does each of them appear?

Problem 20

Let $g = \begin{pmatrix} u & -\overline{v} \\ v & \overline{u} \end{pmatrix}$, $u, v \in \mathbb{C}$ with $|u|^2 + |v|^2 = 1$.

a) Verify that $g \in \mathrm{SU}(2)$, and explain why every $g \in \mathrm{SU}(2)$ can be written in this way.

The basis vectors $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$ of $\mathbb{C}^2$, as defined in the lecture[1], transform in the two-dimensional representation $\Gamma^2(g) = g \ \forall\, g \in \mathrm{SU}(2)$.

b) Write $\Gamma^2(g)|{\uparrow}\rangle$ and $\Gamma^2(g)|{\downarrow}\rangle$ as linear combinations of $|{\uparrow}\rangle$ and $|{\downarrow}\rangle$.

Consider now $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the product basis $|{\uparrow\uparrow}\rangle = |{\uparrow}\rangle \otimes |{\uparrow}\rangle$ etc. (cf. lecture). Under $\mathrm{SU}(2)$ this basis transforms in $\Gamma^{2\otimes 2} = \Gamma^2 \otimes \Gamma^2$.

c) Expand $\Gamma^{2\otimes 2}|{\uparrow\uparrow}\rangle$ etc. in the product basis.

d) Show: $\mathrm{span}(|0,0\rangle)$ and $\mathrm{span}(|1,1\rangle, |1,0\rangle, |1,-1\rangle)$ (as defined in the lecture) are invariant under $\mathrm{SU}(2)$, and thus carry one- and three-dimensional representations of $\mathrm{SU}(2)$, respectively, i.e. $\Gamma^{2\otimes 2} = \Gamma^1 \oplus \Gamma^3$.

e) Explicitly determine the representation matrices $\Gamma^1(g)$ and $\Gamma^3(g)$.

On $\mathbb{C}^2 \otimes \mathbb{C}^2$ also acts – as in Problem 19 – a representation D of $S_2 \cong \mathbb{Z}_2 = \{e, (12)\}$.

f) In which representations of S_2 do the vectors $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ and $|0,0\rangle$ transform?

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[1] If "spin-$\frac{1}{2}$ particle" doesn't mean much to you, then just ignore the word. We introduced this manner of speaking in Section 2.8, and the only thing you need to know for this homework assignment are the definitions

$$|{\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |{\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |{\downarrow\uparrow}\rangle = |{\downarrow}\rangle \otimes |{\uparrow}\rangle \quad \text{etc.}$$