Spin and the Stern-Gerlach experiment

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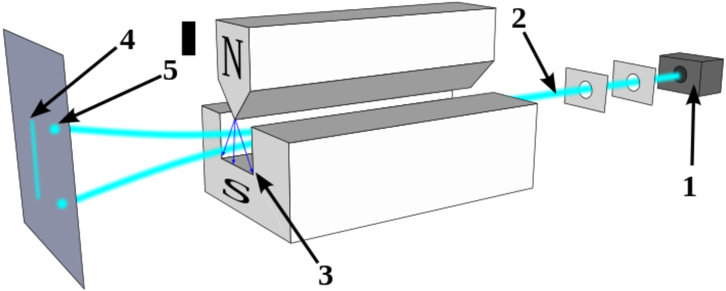
Overview

1. The Stern-Gerlach experiment
2. Reflections on naively constructed hidden spin variables
3. Description of spin on the wave function level
4. Spin in the many worlds interpretation (MWI)
5. Spin in collapse theories
6. Spin in Bohmian mechanics (BM)
7. Contextuality

Stern-Gerlach experiment[1]

O. Stern, W. Gerlach 1922: beams of neutral silver atoms in an inhomogenous magnetic field are sent towards a fluorescent screen.



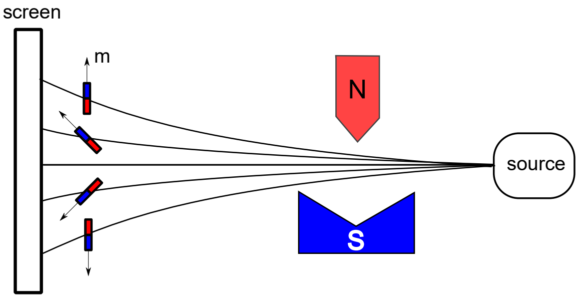
(Beams are not observed before the screen.)

[1]Picture credit: https://en.wikipedia.org/wiki/File:Stern-Gerlach_experiment_svg.svg

Stern-Gerlach experiment

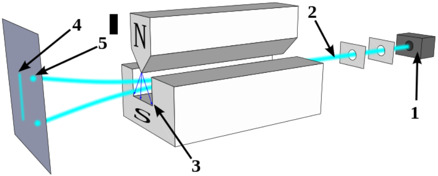
Classical expectation (year 1922!): Particles carrying a magnetic dipole will precess in magnetic fields. In inhomogeneous magnetic fields, they will in addition be deflected (stronger force on one end of the dipole than oppositely on the other).

Crucial: Orientation of dipoles in the beam is random ⇒ **continuous distribution of arrival locations**.



Stern-Gerlach experiment

Experiment: Just two outcomes are possible. (True for every alignment of magnetic field!)



Consequence: Classical picture of spinning magnetic dipole is inadequate.

Conclusions

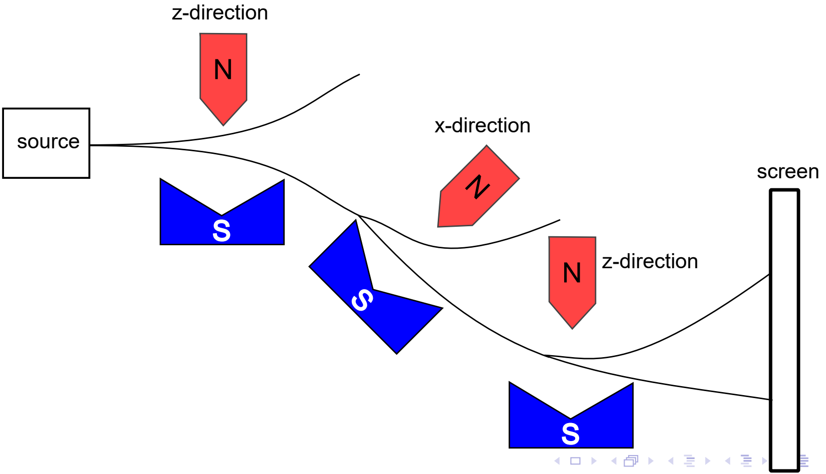
- There is an additional property which (like an intrinsic magnetic moment) deflects the beam: **spin.**
- That property can take only two values, corresponding to the two possible outcomes of the SG experiment. One says: 'spin is quantized.'
- Note: it is a **definition** to say that the particle has spin up (or $+\frac{\hbar}{2}$) if it hits the screen in the SG-expt. in the upper half, and spin down (or $-\frac{\hbar}{2}$) if it hits the screen in the lower half.

But what is going on really in the experiment?

Local hidden spin variables?

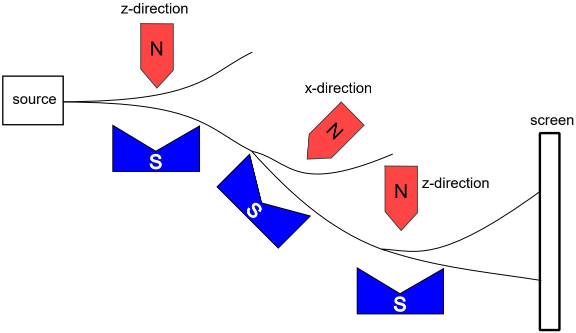
Question: Could it be that each particle carries a local, predefined variable which determines the outcomes of all spin experiments?

To answer the question, consider the following modified SG expt.:



Local hidden spin variables?

Assume (model 1): Each particle has a pre-defined spin z variable $s_z = \pm\hbar/2$ and a pre-defined spin x variable $s_x = \pm\hbar/2$. Furthermore, the SG devices just filter for the respective properties.



Prediction: relative frequences of the two possible values on the screen are 0% for $s_z = +\hbar/2$ and 100% for $s_z = -\hbar/2$.

Local hidden spin variables?

Frequencies of results in experiment: 50% for $s_z = +\hbar/2$ and 50% for $s_z = -\hbar/2$. $\rightarrow$ naive model goes wrong!

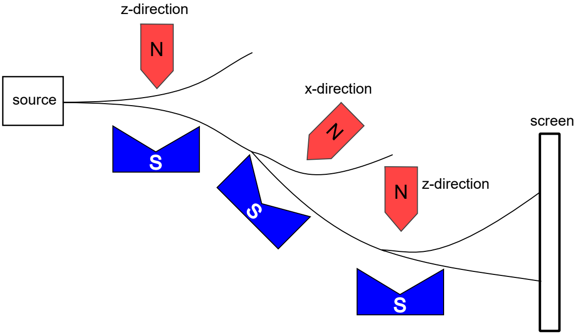
Lesson

The apparatus has an **active role** in determining the outcomes of an experiment.

Local hidden spin variables?

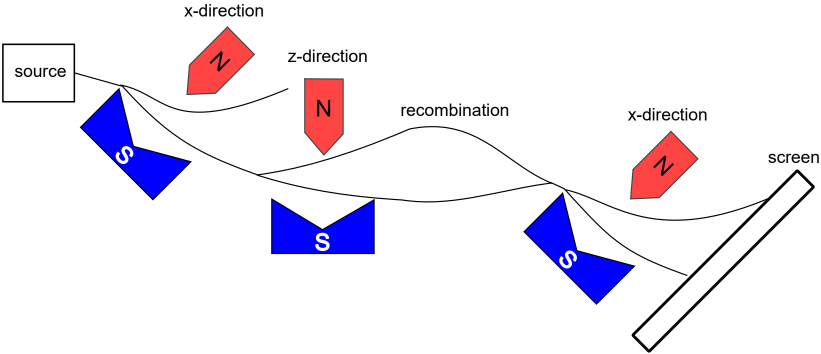
Another attempt (model 2): Maybe the particles carry pre-defined values of s_x, s_z but if one measures s_x, then the s_z is randomized (and the other way around).

That would explain the previous result of a 50-50 distribution.



Local hidden spin variables?

But now consider the following experiment:



Prediction of model 2: 50 % $s_x = +\hbar/2$, 50 % $s_x = -\hbar/2$.

Local hidden spin variables?

Experimental frequencies: 0% $s_x = +\hbar/2$, 100 % $s_x = -\hbar/2$.
→ Also model 2 goes wrong.

Note: These frequencies would have been the prediction of model 1.

Foreboding: local hidden variables seem to be problematic.

Indeed (see lecture on no hidden variables theorems):

Impossibility of LHV for spin

There cannot be any local hidden variables in the sense that each particle carries a set of such variables which is just revealed during a measurement, and which agree with the quantum formalism for spin (which agrees with experiments).

Wave function description of spin

We now recall the usual description of spin in the quantum formalism.

(Non-relativistic) 2-component spinors: wave fn. for a single quantum particle:

$$\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathcal{S} \simeq \mathbb{C}^2, \qquad (t, \mathbf{x}) \mapsto (\psi_1, \psi_2)(t, \mathbf{x}).$$

Wave fn. is a spinor instead of a scalar.
That means, under a rotation $R \in SO(3)$, ψ transforms as

$$\psi(t, \mathbf{x}) \quad \xrightarrow{R} \quad \psi'(t, \mathbf{x}) = S[R]\psi(t, R^{-1}\mathbf{x})$$

where $S[R]$ are matrices forming a (spinorial) representation of $SO(3)$.

(More precisely: projective Hilbert space representation, or representation of double cover of $SO(3)$.)

Wave function description of spin

Spin vector: With every spinor $\psi \in \mathbb{C}^2$, we can associate a vector $\boldsymbol{\omega} \in \mathbb{R}^3$ according to:

$$\boldsymbol{\omega}(\psi) = \psi^\dagger \boldsymbol{\sigma} \psi.$$

Curious fact: If we rotate ψ in spin space by an angle θ, then $\omega(\psi)$ rotates by 2θ.

Angles between ϕ, χ in spin space are here defined by:

$$\theta = \cos^{-1}\left(\frac{|\langle\phi|\chi\rangle|}{\|\phi\|\|\chi\|}\right)$$

Evolution equation

Implement magnetic field $\mathbf{B}(x)$ in Schödinger eq. for spinor-valued ψ:

Pauli equation

$$i\hbar\partial_t\psi = \frac{1}{2m}\left(-i\hbar\nabla - \mathbf{A}(\mathbf{x})\right)^2\psi - \mu\boldsymbol{\sigma}\cdot\mathbf{B}(\mathbf{x})\psi$$

μ: magnetic moment, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, $\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x})$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notation: eigenvectors of Pauli matrices

$$|\uparrow_z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow_z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |\downarrow_x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Reduction to spin degrees of freedom

Qualitative result of time evolution via Pauli eq. for SG experiment in z-direction: Initial wave fn. $\psi(\mathbf{x}) = \begin{pmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \end{pmatrix}$

Assume: Experiment is such that $\psi_1(\mathbf{x})$, $\psi_2(\mathbf{x})$ get deflected in different directions (negligible dispersion and deformation).

Consider special initial wave fn. (spin and position disentangled)

$$\psi(\mathbf{x}) = \chi \otimes \varphi(\mathbf{x}), \quad \chi \in \mathbb{C}^2 : \text{ fixed spinor.}$$

Wave fn. after passing the detector (screen at $x = l$)

$$\begin{pmatrix} \chi_1 \, \varphi(\mathbf{x} - (l, 0, d)) \\ \chi_2 \, \varphi(\mathbf{x} - (l, 0, -d)) \end{pmatrix}.$$

Probabilities: according to Born rule:

$$\begin{aligned} \text{Prob}(s_z = +\hbar/2) &= \|\chi_1 \varphi(\mathbf{x} - (l, 0, d))\|^2 = |\chi_1|^2 = |\langle \chi, \uparrow_z \rangle|^2, \\ \text{Prob}(s_z = -\hbar/2) &= \|\chi_2 \varphi(\mathbf{x} - (l, 0, -d))\|^2 = |\chi_2|^2 = |\langle \chi, \downarrow_z \rangle|^2. \end{aligned}$$

Reduction to spin degrees of freedom

Relation to general measurement formalism: Recall general Born rule: If an observable A is measured for a system with wave fn. ψ, then the outcomes α are random with prob. distr.

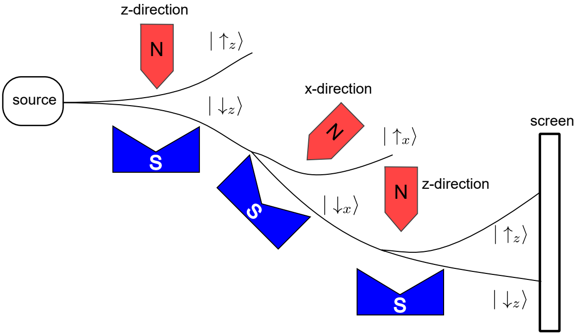
$$\rho(\alpha) = \sum_{\lambda} |\langle \phi_{\alpha,\lambda} | \psi \rangle|^2$$

where $\phi_{\alpha,\lambda}$ is an orthonormal basis (ONB) of eigenvectors of A.

Comparison with SG expt.: Probabilities agree with general Born rule for observable $A = \hbar/2\,\sigma_z$ on the Hilbert space $\mathscr{H} = \mathbb{C}^2$.

Spinors $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ form an ONB of eigenvectors of σ_z (eigenvalues $\pm\hbar/2$).

Example: the z-x-z SG experiment again



(Coefficients of the wave fns. are not shown.)

Probabilities for last s_z-expt.: Use $|\chi\rangle = |\downarrow_x\rangle$ in previous formula:

$$\text{Prob}(s_z = +\hbar/2) = |\langle\downarrow_x | \uparrow_z\rangle|^2 = \left|\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\rangle\right|^2 = \frac{1}{2}.$$

Similarly: $\text{Prob}(s_z = +\hbar/2) = |\langle\downarrow_x | \downarrow_z\rangle|^2 = \frac{1}{2}.$

Lesson

Spin gets implemented as a property of the wave function, not of the particles.

Question: OK, we can calculate the probabilities correctly. But what is really happening in the SG experiment? (We know from the discussion of the measurement problem that wave functions are not the full story.)

→ Discuss that for the **precise versions of quantum theory** which we have got to know!

Spin in collapse theories

We consider **GRWm** here.

Use the spinor wave fn. as before.

Modify Pauli eq. by an additional stochastic term which generates collapses (frequency of collapses proportional to degrees of freedom).

Primitive ontology: mass density function

$$m(t,\mathbf{x}) = \sum_{i=1}^{N} m_i \int d^3\mathbf{x}_1 \cdots \widehat{d^3\mathbf{x}_i} \cdots d^3\mathbf{x}_N \, (\psi^\dagger\psi)(t,\mathbf{x}_1,...,\mathbf{x}_i=\mathbf{x},...,\mathbf{x}_N)$$

Important: Both apparatus and object need to be modeled according to GRWm to avoid the measurement problem.

The SG experiment in GRWm

Consider a system consisting of one silver atom and an apparatus consisting of 10^{23} atoms which registers the outcome (up/down).

Time evol. of wave fn.:

$$\begin{aligned} & \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle) \otimes |\text{detector ready}\rangle \\ \longrightarrow \quad & \frac{1}{\sqrt{2}}|\uparrow_z\rangle \otimes |\text{detector up}\rangle + \frac{1}{\sqrt{2}}|\downarrow_z\rangle \otimes |\text{detector down}\rangle. \end{aligned}$$

Once this superposition is generated, there is a great probability that a stochastic collapse will reduce it to one of the wave packets.

Collapse:

$$\begin{aligned} & \frac{1}{\sqrt{2}}|\uparrow_z\rangle \otimes |\text{detector up}\rangle + \frac{1}{\sqrt{2}}|\downarrow_z\rangle \otimes |\text{detector down}\rangle. \\ \longrightarrow \quad & |\uparrow_z\rangle \otimes |\text{detector up}\rangle. \end{aligned}$$

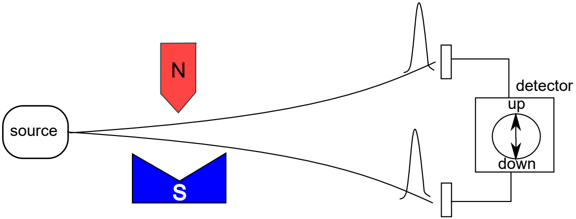
The mass density function projects this configuration space picture into physical space.

The SG experiment in GRWm

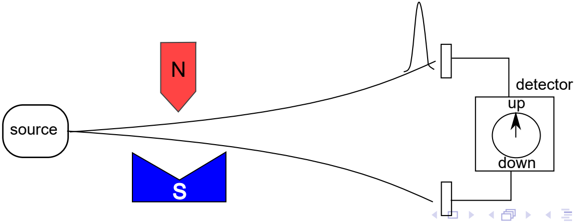
Note: The large number of degrees of freedom of the detector is essential, otherwise collapses would be very infrequent (not enough to likely ensure only one outcome).

The SG experiment in GRWm

Before collapse: (detector and particle made up from mass density)



After collapse: (random outcome: up)



Spin in many worlds

Use Pauli eq. without modifications.

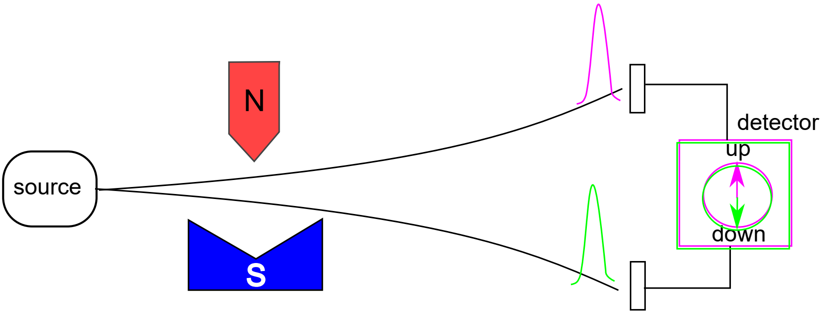
Mass density function: (as in GRWm).

$$m(t, \mathbf{x}) = \sum_{i=1}^{N} m_i \int d^3\mathbf{x}_1 \cdots \widehat{d^3\mathbf{x}_i} \cdots d^3\mathbf{x}_N \, (\psi^\dagger \psi)(t, \mathbf{x}_1, ..., \mathbf{x}_i = \mathbf{x}, ..., \mathbf{x}_N)$$

The SG experiment in the MWI

Wave fn. evolution: as before.

Picture: (particle and detector made up from mass density)



Crucial: As the detector has many degrees of freedom, the two wave packets in configuration space cannot be brought to interference anymore, i.e., they behave independently. This allows us to regard the respective contributions to $m(t, \mathbf{x})$ as separate worlds (pink and green).

Spin in Bohmian mechanics

Difficulty: How can a particle theory cope with spin after all?

Basic equations of BM with spin:

1. Pauli equation
2. Modified guidance law for the particles:

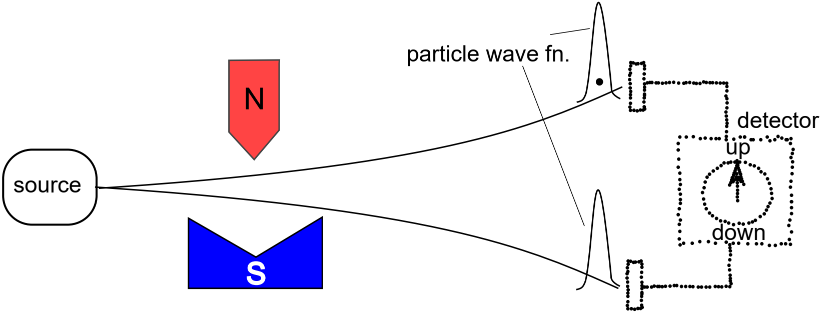
$$\frac{dQ}{dt} = \hbar m^{-1} \Im \frac{\psi^\dagger(\nabla - i\mathbf{A})\psi}{\psi^\dagger\psi}(t, Q(t))$$

Note: no spin variables introduced in addition to the particles!

Paradox: As a theory with particles, and with nothing spinning, how can BM reproduce the results of the SG experiment?

The SG experiment in BM

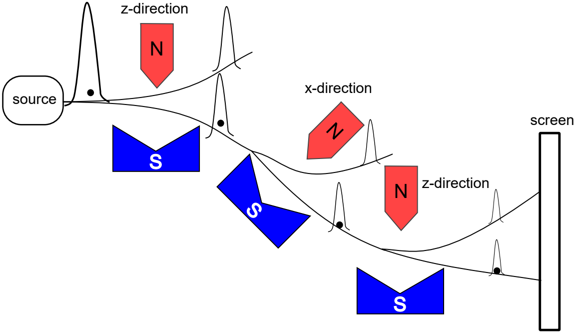
Depending on the initial configuration, the Bohmian config. of particle and detector evolves (in a deterministic way) either to a config. where the particle is in the upper part and the detector displays the result 'up'



... or to a config. where the particle in the lower part and the detector displays the result 'down'.

z-x-z SG experiment in BM

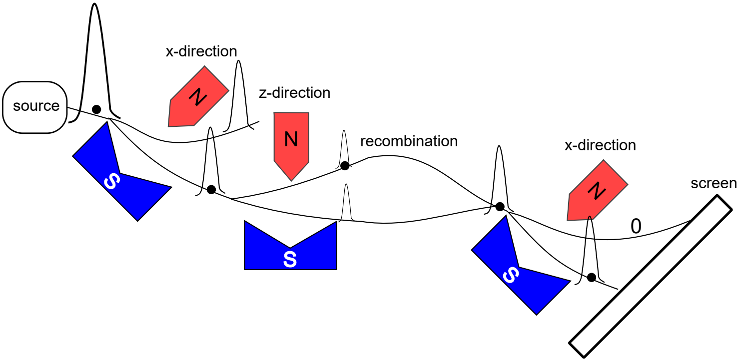
Before, we saw that naive particle theories had problems with more complicated SG experiments.
How does BM explain e.g. the z-x-z SG experiment?



Answer: Particles travel with the waves, which one depends on initial conditions. Probabilities come out right because of equivariance property (for initial quantum equilibrium distribution).

SG recombination experiment

And what about the recombination experiment?



Again, the particles travel with the waves (which packet depends on initial position). But as the waves interfere destructively in the last upper x-branch, the particle (assuming it got that far) has to travel with the wave in the lower branch.

Contextuality

In BM, it becomes apparent that the quantum formalism has the following feature:

Contextuality

There can be many different experiments which nevertheless yield the same statistics of outcomes.

Here: illustration at the example of SG-type experiments.

Contextuality in SG-experiments

We consider two SG-type experiments for the same initial wave fn.

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_1(\mathbf{x}) \\ \psi_2(\mathbf{x}) \end{pmatrix}.$$

Experiment 1: usual spin-z SG experiment

Outcome statistics: $\text{Prob(up)} = \|\psi_1\|^2, \quad \text{Prob(down)} = \|\psi_2\|^2.$

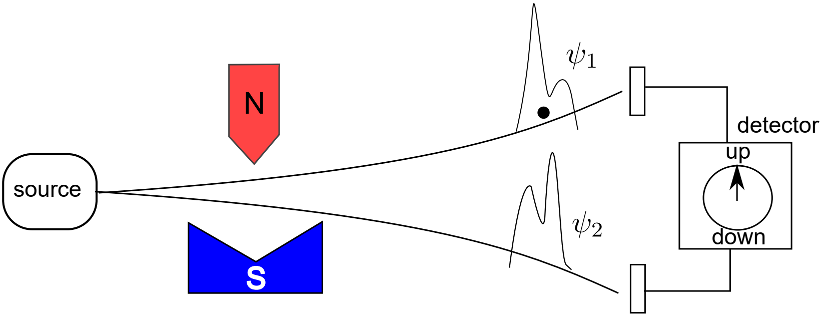
Experiment 2: spin-z SG experiment with gradient of magnetic field reversed and relabeling $\text{up}^* = \text{down down}^* = \text{up}.$

Now: $\psi_1(\mathbf{x})$ will be deflected in negative z-direction and $\psi_2(\mathbf{x})$ in positive z-direction (exactly opposite when compared to expt. 1).

Outcome statistics:
$\text{Prob}(\text{up}^*) = \text{norm of lower wave packet} = \|\psi_1\|^2$
$\text{Prob}(\text{down}^*) = \text{norm of upper wave packet} = \|\psi_2\|^2.$
$\rightarrow$ Exactly the same!

Contextuality in SG-experiments

Bohmian paths for expt. 1:



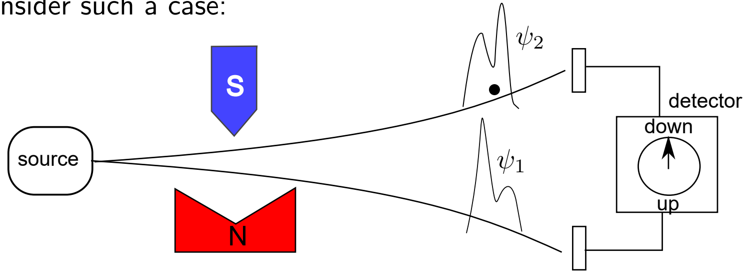
Bohmian paths for expt. 2:

The relevant process is the splitting of the wave packets in z-direction which is effectively one-dimensional.

→ Trajectories cannot cross.

→ There will be trajectories which end up in the upper half in both experiments. (If the distribution of results is 50-50, then the 50% of initial condition with greater z-value will be such.)

Consider such a case:



→ The same initial conditions lead to two different results in the two experiments!

Contextuality in SG-experiments

Conclusions:

- Bohmian particles do not have an intrinsic spin value.
- Spin is a property of the wave packet by which the particle is guided.
- Different experimental setups can make the same initial conditions lead to different outcomes for spin measurements, even though the outcome statistics are the same.

Contextuality

There can be many different experiments which nevertheless yield the same statistics of outcomes.

Lesson

Operator obervables represent **equivalence classes** of experiments with the same outcome statistics.

Additional spin variables in variants of BM

We have seen that in BM, spin is a property of the wave fn. which guides the particles (and leads us to say that a particle has spin up/down depending on the way it comes out in a SG experiment).

Question: What happens if we insist on introducing an actual spin vector in addition to the position?

Suggestion by Bohm, Schiller, Tiomno:

$$\mathbf{S}(t) = \frac{\psi^\dagger \boldsymbol{\sigma} \psi}{\psi^\dagger \psi}(t, Q(t))$$

Then:

- Spin vector always points in the direction (up/down) associated with the end position (upper half/lower half) in the SG expt.
- $\mathbf{S}(t)$ does not influence the position $Q(t)$.
- Contrary to $Q(t)$, $\mathbf{S}(t)$ is redundant: In both versions of BM, the experimental device reacts in the same way – whether or not $\mathbf{S}(t)$ is introduced.

Questions?