Tropical spin Hurwitz numbers

Master Thesis

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1 Introduction

Classical Hurwitz theory is deeply rooted in mathematical history. It first emerged as an enumerative problem that goes back to Hurwitz: count the number of branched covers of a fixed target curve that exhibit a certain ramification behaviour. This number is called a *Hurwitz number*. Since then, Hurwitz theory has provided a variety of connections between different areas of mathematics such as algebraic geometry, representation theory and mathematical physics.

A modern twist. Spin Hurwitz numbers were introduced by Eskin-Okounkov-Pandharipande in 2008 for certain computations in the moduli space of differentials on a Riemann surface [Gun16]. Similarly to Hurwitz numbers they are defined as a weighted count of branched coverings of a smooth algebraic curve D with fixed degree and branching profile. In addition, they include information about the lift of a theta characteristic of fixed parity on the base curve D. A seemingly small change that makes the problem all the more exciting, a true gem in combinatorics.

Eventually an active and interdisciplinary field of research emerged, *spin Hurwitz theory* with latest results published in 2021 [GKL21]. While the computation of spin Hurwitz numbers was approached from very different angles, e.g. via integrable systems or representation theory of the Sergeev group [EOP08], many questions remain open, in particular a tropical approach is still missing.

In tropical geometry we extract combinatorial data from algebraic objects via a degeneration process and present it in a simplified form: tropical curves for example are metric graphs obtained as result of a degeneration of algebraic curves. Any such process is called *tropicalization*. Ideally, we wish to define for each object in algebraic geometry a (combinatorial) tropical counterpart in such a way, that we can address enumerative problems on the tropical side first [Gat06], where the picture is less cluttered. We then hope to obtain new insights by transferring results back to the algebraic side:

Procedure 1.1. *How to address algebraic problems*

- State a problem in algebraic geometry.
- Tropicalize the objects involved by extracting relevant combinatorics and define an analogous problem on the tropical side.
- Solve it in the "tropical world" using methods from discrete mathematics.
- Transfer the solution back to the algebraic side.

The last step is often the most complicated one, it addresses the counterpart of tropicalization, the question of *realizability*, i.e. can we find algebraic objects that tropicalize to the synthetically defined tropical ones we considered?

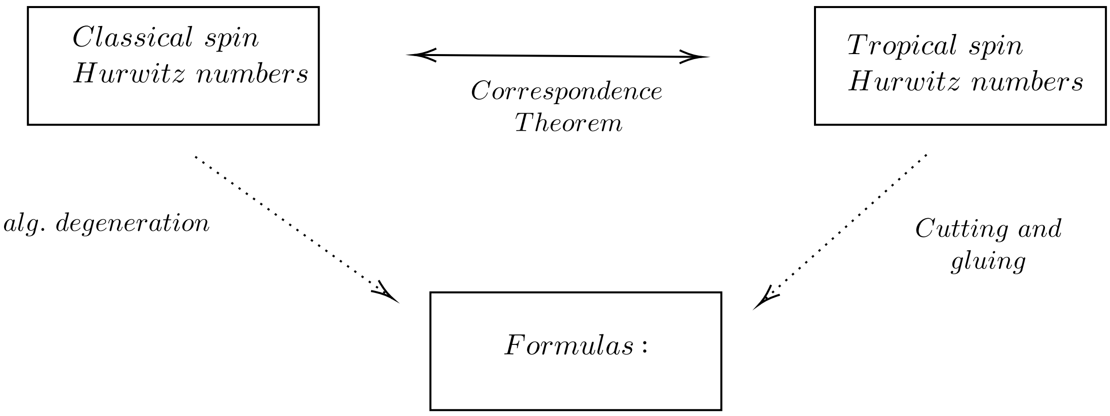
The power of this approach lies beyond doubt. Promising results as well as new insights in existing problems appear each year and give this young theory a boost in prominence. *Tropical Hurwitz theory* for example, mainly developed by Cavalieri, Markwig and Johnson, allowed for new insights in the piecewise polynomial structure of double Hurwitz numbers ([CJM10]) and is essential for the concepts developed in this thesis.

The present work is dedicated to the development of *tropical spin Hurwitz theory*. Under the slogan

Tropical geometry is the combinatorial shadow of algebraic geometry [MS15].

we define *tropical spin Hurwitz numbers* as result of an algebraic degeneration procedure. But soon these tropical objects develop a life of their own. That is they have a natural place in the tropical world as tropical covers with tropical theta characteristics on source and target curve. The difficulty of tropicalizing theta characteristics has already been noted by Caporaso, Melo and Pacini in [CMP20]: odd and even theta characteristics may tropicalize to the same object. To compensate for this loss of information they introduce a sign function on a graph obtained after contraction of a cycle. Our definition of *parity of tropical spin Hurwitz covers* admits an interpretation in terms of their theory of tropical spin curves.

Tropical spin Hurwitz covers are interesting in their own right. As a new combinatorial tool to produce spin Hurwitz numbers or to study their properties, but also as a way to demonstrate the feasibility of tropical geometry to organize degenerations of theta characteristics in spite of the difficulties mentioned above. More generally, tropical geometry provides a way to study the moduli space of spin covers and its intersection theory by means of combinatorics. In light of the success story of tropical geometry we also believe that these will help to deepen the understanding of classical spin Hurwitz numbers and facilitate the study of their combinatorial properties: the philosophy of this approach is justified by the two correspondence theorems established in section 5.1.



$H^{(h,+)}_{(3)^k} = 3^{2h-2}\left((-1)^k 2^{h+k-1} + 1\right)$ $\qquad$ $\mathbb{T}H^{(h,+)}_{(3)^k} = 3^{2h-2}\left((-1)^k 2^{h+k-1} + 1\right)$

Figure 1: Diagram illustrating the relationship between tropical and algebraic spin Hurwitz numbers. Their equality can be established either directly via a correspondence theorem (top arrow) or indirectly via computation in the tropical and in the algebraic world (bottom part of the triangle): Formulas for classical spin Hurwitz numbers of degree 3 on the left ([Lee13]) and for the corresponding tropical ones on the right (subsection 5.1.2).

In section 4.4 we state a recursion formula obtained by Lee in 2012 via spin degeneration, first for degree 3 in [Lee13] and then together with Parker for arbitrary spin Hurwitz numbers in [LP13]. From a tropical perspective it is natural to iterate their degeneration procedure to get tropical covers as maps between the dual graphs of source and target curve. We need to endow these with an additional structure that encodes the parity of the "dual" covering maps and define *tropical spin Hurwitz numbers*, just like in the "algebraic world", as a signed sum of *tropical spin Hurwitz covers.* These are tropical covers together with an *admissible parity function*- a notion we develop as by-product of the algebraic degeneration procedure- on source and target curve that are in some sense compatible. We distinguish two cases: the case where the target curve is *even* and the case where the target curve is *odd.* The first allows us to use already established

theory, the theory of tropical Hurwitz covers, since working with maximally degenerate base curves is still possible. For the second we consider *almost maximally degenerate* base curves, i.e. a curve that is maximally degenerate except for a 1-valent vertex of genus 1. Next, we define the *parity* and *multiplicity* of a tropical spin Hurwitz cover that will determine its weight and introduce tropical spin Hurwitz numbers as weighted count of tropical spin Hurwitz covers. Our main results are two correspondence theorems stating the equality of the tropical spin Hurwitz number with the classical one, theorem 5.37 for even spin Hurwitz numbers and theorem 5.41 for odd ones.
Concrete computational results can be found in subsection 5.1.1-5.1.4, which leads to elementary proofs of Lee and Parkers results for degree 3 and 4 in [Lee13] and [LP13] (see figure 1). It also provides an indirect proof of the correspondence theorems that relies on purely combinatorial methods. For this purpose we need a more general version of tropical covers of elliptic curves than is considered in [BM14]. We develop this in section 3.2.
We obtain new numbers for covers of the tropical line of arbitrary degree with at most 4 branch points, proposition 5.36, that carry over to the classical world via our correspondence. These numbers were not computed by Lee and Parker in [LP13] and, as far as the author knows, are also new to the classical world. Restricting to the special cases considered in subsection 5.1.1-5.1.4 has two advantages. An admissible parity function on the cover curve is unique and the multiplicity coincides with the usual definition for tropical Hurwitz covers.

Organization of this thesis. Being a tropical geometer is little bit like living in two worlds, the algebraic and the tropical one. It is only with a deep understanding of the first that we get to unleash our full creativity in the second. Hence, this thesis is a journey through both worlds.
While section 2-4 mostly reviews existing results (apart from section 3 where we generalize some of the results in [BM14]), section 5 contains the newly established results by this thesis. In section 2 we start with a recall of everything algebraic (related to Hurwitz theory of course), i.e. we collect definitions, tools and set up relevant notation. A strong base for future work is tropical Hurwitz theory, a collection of the main results is given in section 3. Section 4 is a good place to get acquainted with spin Hurwitz theory and section 5, finally, introduces tropical spin Hurwitz numbers. After investigating some special cases in the tropical world (subsection 5.1.1-5.1.4), we conclude with the proof of our correspondence theorems in subsection 5.2.

2 Preliminaries

Building a solid foundation. In order to build a tropical counterpart of spin Hurwitz numbers from a solid foundation, we recall and repeat the algebraic knowledge we will need moving forward. Let us set up two intentions for this section:

1. Develop a *working* knowledge of algebraic objects.
2. Appeal to the geometric *intuition* of the reader.

Hurwitz theory lies at the intersection of various fields of mathematics including topology, differential and algebraic geometry methods and definitions occur in more than just one flavour. In fact, each area has its one way of putting things. It is useful at times to include these different perspectives.

2.1 Curves and Riemann surfaces

Curves lie at the heart of algebraic geometry. Intuitively, a curve is just the zero set of a polynomial in two variables, i.e. $C = \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$ for $f \in \mathbb{C}[x,y]$. Over the course of time, however, it became clear that things get easier if one works over a compact space. Thus, we prefer to deal with *projective curves* in $\mathbb{P}^2$, that is zero sets of homogeneous polynomials in three variables. This is not too far of from the previous notion (on affine subset of $\mathbb{P}^2$ these agree) and offers all the advantages of a compact space. But once people wanted to consider *deformations of curves*, complications began to arise. It became obvious that clinging to the old notion of a curve as vanishing locus of polynomial equations was the root of all difficulties. And thus, curves were ripped out of their ambient space and were replaced by abstract objects:

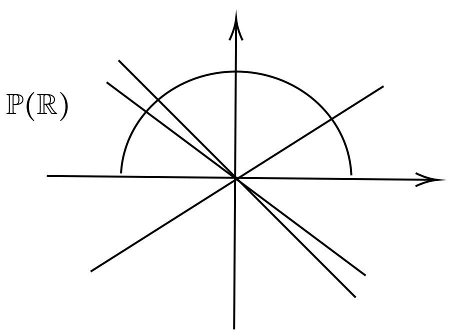
Definition 2.1 (section 1.5. [CMP20]). A *curve* C is a reduced, projective variety of dimension one, not necessarily connected, over $\mathbb{C}$. A pointed curve is a pair (C, σ), where σ is an ordered and finite set of smooth and distinct points of C.

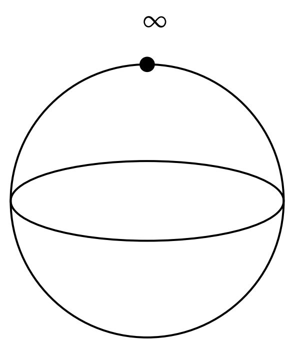
So far definition 2.1 only describes the algebraic "way out", a *Riemann surface*, on the other hand, is the corresponding abstract object on the analytic side. Their relation is a beautiful one, though we will only get a glimpse of it.
Curves and Riemann surfaces. For this paragraph we will implicitly assume each curve C to be smooth. As mentioned before we are interested in the equivalent of a projective curve in differential geometry, a Riemann Surface.

Definition 2.2 (Definition 3.0.4.[RC]). A *Riemann Surface* is a complex analytic manifold of dimension 1.

There is a beautiful correspondence between compact Riemann Surfaces and smooth curves (chapter 1.3 [GGD12]). It can be shown that after embedding an algebraic curve into projective space there is a unique way to turn its image into a Riemann surface, i.e. there exists a unique complex structure thereon. In fact, this construction defines a functor $\mathcal{F}$ between the categories of compact Riemann surfaces (with holomorphic maps as morphisms) and smooth algebraic curves (with birational morphisms as morphisms). Moreover, $\mathcal{F}$ induces a bijection from the set of proper algebraic curves (taken up to isomorphism) to the set of compact Riemann surfaces (taken up to isomorphism)(chapter 1.3 [GGD12]). This justifies that we will use both terms interchangeably. It is worth pondering over this for a second: This connection allows us to view objects and definitions from both fields from completely different perspectives.

Example 2.3 (Two perspective of $\mathbb{P}^1$)**.** We can draw two different pictures of the projective line.





The left was drawn by an algebraic geometer and reflects set theoretic interpretation of $\mathbb{P}^1$ as lines through the origin in $\mathbb{C}^2$ (here only a real picture), the right one by differential geometer, who thinks of $\mathbb{P}^1$ as the Riemann sphere obtained by gluing together two copies of $\mathbb{C}$.

The genus. From a topological point of view the genus $g(C)$ arises as a (topological) invariant that counts "the number of handles" on a compact surface (Theorem 2.4.3 [RC]), i.e. a compact smooth orientable manifold of real dimension 2. Note, any curve C is in particular a real surface and, hence, has a topological genus. From an algebraic perspective we can view the (geometric) genus as counting the number of linearly independent holomorphic differentials on C, that is $g(C) = dim_{\mathbb{C}}(H^0(C, \omega_C))$ as we will see later on. It is a non-trivial fact that geometric and topological genus agree (if C is smooth). If C is reducible we can compute the genus of its irreducible components instead: Let $C := \bigcup_{i=1}^{k} C_i$ its decomposition into irreducible components then $g(C) = 1 - k + \sum_{i=1}^{k} g(C_i)$.

Example 2.4. The curve

$$C := \{(x : y : z) \in \mathbb{P}^2 \mid 0 = y^2 z - x(x - z)(x + z)\}.$$

is an example of an elliptic curve. Since C is smooth and of genus 1, it is a Riemann surface that topologically is a torus.

The canonical bundle. Recall, the cotangent bundle of a smooth manifold M is (as a set) the disjoint union of the cotangent spaces $T_p^* M$:

$$T^* M := \bigcup_{p \in M} \{p\} \times T_p^* M.$$

There is a canonical way to equip $T^* M$ with a holomorphic structure. Hence, the pair $(\pi, T^* M)$, where the map π is just the projection to M, is a vector bundle of rank $dim_{\mathbb{C}}(M)$. If $M = C$ is a smooth curve, $(\pi, T^* M)$ is a line bundle, i.e. a vector bundle of rank 1, and agrees with the *canonical bundle* denoted by ω_C (p.114 [Gat03]). Its space of global sections $H^0(C, \omega)$ is the space of holomorphic 1-forms/differentials on C.
The algebro-geometric analogue of this construction is the *cotangent sheaf.* Though we work with projective curves, we content ourselves with a definition in the case of an affine variety X: the *module of differentials* Ω_X. Why? A "sheafification" of Ω_X that generalizes to projective varieties only shows that a global object Ω_X exists (Remark 7.4.8.[Gat03]). Since it restricts to the affine definition on affine open subsets, it is far more useful for practical computations.

Definition 2.5 (Definition 7.4.1. and Example 7.4.2. [Gat03]). Let X be an affine variety with coordinate ring $R = \mathbb{C}[x_1, ..., x_n]/\langle f_1, ..., f_m \rangle$. We define the R-module Ω_X, the module of differentials, to be the free R-module generated by formal symbols $\{dr; r \in R\}$, modulo the relations

1. $d(r_1 + r_2) = dr_1 + dr_2$ for $r_1, r_2 \in R$,
2. $d(r_1 r_2) = r_2 dr_1 + r_1 dr_2$ for $r_1, r_2 \in R$
3. $dc = 0$ for $c \in \mathbb{C}$.

In this case we have

$$\Omega_X = \langle dx_1, ..., dx_n \rangle_R / \langle \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} dx_i, j = 1, ..., m \rangle_R \text{ and}$$

$$T_P^* X \cong \langle dx_1, ..., dx_n \rangle_R / \langle \sum_{i=1}^{n} \frac{\partial f_j(P)}{\partial x_i} dx_i \rangle \text{ for } P \in X.$$

The last two statements follow from the fact that R is generated by the equivalence classes $\bar{x}_i$ for $i = 1, ..., n$ (in R) and by applying the rules of differentiation (1)-(3). Strictly speaking

Ω_X is not a sheaf. For the purpose of this thesis, however, we will pretend that it is and call it the *cotangent sheaf.* This is justified by Proposition 7.4.11. ([Gat03]) that shows that the generalization only glues these affine constructions together. Moreover, if $X = C$ for a curve C cotangent sheaf and canonical sheaf agree. Hence, we write ω_C in this case.

Example 2.6. (21.2.7. in [Vak17]) Consider the affine curve

$$\tilde{C} := \{(x,y) \in \mathbb{C}^2 : 0 = y^2 - (x^3 - x)\}$$

given by restricting the curve from example 2.4 to the affine set $\{(x:y:z) \in \mathbb{P}^2 | z \neq 0\}$ by choice of coordinates $\frac{x}{z}$ and $\frac{y}{z}$. The cotangent sheaf of $\tilde{C}$ is given by

$$\omega_{\tilde{C}} = \langle dx, dy \rangle_R / \langle 2ydy - (3x^2 - 1)dx \rangle_R, \text{ where } R = \mathbb{C}[x,y]/\langle y^2 - (x^3 - x) \rangle.$$

We see $2ydy = (3x^2 - 1)dx$ in $\omega_{\tilde{C}}$ and conclude that $\omega_{\tilde{C}}$ is locally, i.e on the set $x \neq 0$ ($y \neq 0$), generated by dy (dx). Since both sets form an open cover of $\tilde{C}$ (i.e. $\tilde{C} \subset \{x \neq 0\} \cup \{y \neq 0\}$), the vector spaces $T_P^* \tilde{C}$ for $P \in \tilde{C}$ are 1-dimensional. We conclude that Ω_X is of rank 1, i.e. a line bundle.

Nodal curves. Why should we care about curves with singularities? A lot of rather difficult problems involve smooth curves of high genus, e.g. the computation of spin Hurwitz numbers. Many times the following strategy has proven useful: Take a curve C of choice, deform until it "breaks up" into simpler components of lower genus, that is until it becomes singular, and analyze these components instead. Hopefully, new insight are easier to achieve and "carry over" nicely to the original case. We will only work with the simplest singular curves, *at worst nodal* ones.

Definition 2.7 (Definition 2.2.[Ong14]). A *node* is a singularity on the curve which is locally complex-analytically isomorphic to a neighborhood of the origin in the zero locus $\{xy = 0\} \subset \mathbb{C}^2$ A *nodal curve* is a curve such that every one of its points is either smooth or a node.

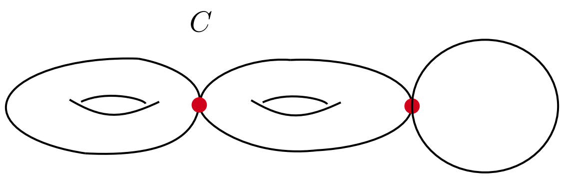


Figure 2: A topological picture of a nodal curve.

Both the genus and the canonical bundle can be extended to nodal curves, where we talk about the arithmetic genus and dualizing sheaf instead.
The arithmetic genus. Formally the arithmetic genus is defined as $p_a(C) := 1 - \chi(\mathcal{O}_C)$ where $\mathcal{O}_C$ is structure sheaf on C, i.e. the sheaf of holomorphic functions (or regular function if you are an algebraic geometer) on C, and χ its (algebraic not topological see [Gat03]) Euler characteristic (section 2.1. [Ong14]). Intuitively, we can think of the arithmetic genus of a nodal curve as the genus of the curve obtained after smoothing the nodes. This captures the feature that it stays constant in families of curves with possibly singular fibres (see definition 2.9). If C has k connected components denoted by C_i, we have $g(C) = 1 - k + \sum_{i=1}^{k} g(C_i)$.
The dualizing sheaf. The following example motivates why we need a different object than the cotangent sheaf when working with nodal curves.

Example 2.8. Consider $C := \{(x,y) \in \mathbb{C}^2 : 0 = yx\}$. We have

$$\omega_C = \langle dx, dy\rangle_R / \langle ydx + xdy\rangle_R, \text{ where } R = \mathbb{C}[x,y]/\langle yx\rangle.$$

For the same reasons as in example 2.6 the vector spaces $T^*_P C$ are one dimensional if $x \neq 0 \neq y$. At the node $P = (0,0)$, however, the fibre of ω_C over the origin is $T^*_P C = \langle dx, dy\rangle_R / \langle 0dx + 0dy\rangle_R$, a vector space of dimension 2.

We see, if C is a nodal curve the dimension of $T^*_P C$ at the nodes might jump. This means Ω_C is not line bundle. Defining the alternative object, the dualizing sheaf, in full generality requires a deeper dive into algebraic geometry. We content ourselves with a description in the case of a nodal curve C with set of nodes $N := \{p^1, ..., p^k\}$ (see pg 32 in [Cav16]):
Let $\nu : C^\nu \to C$ be the normalization of C and $\{r^i, s^i\} = \nu^{-1}(p^i)$. The dualizing sheaf ω_C associates to an open subset U of C meromorphic differentials η on $\nu^{-1}(U)$ having at worst simple poles at r^i, s^i for nodes $p^i \in U$ such that the residues match, i.e. we have

$$Res_{r^i}(\eta) + Res_{s^i}(\eta) = 0$$

for every pair r^i, s^i.
If C is a connected curve of arithmetic genus g, then $h^0(C, \omega_C) := dim_{\mathbb{C}}(H^0(C, \omega_C)) = g$. We make the notion of deforming a curve precise.

Definition 2.9 (adaptation of pg. 19 in [CMP20]). Let Δ be a regular, connected curve with a marked point, $t_0 \in \Delta$. A *1-parameter family* of (pointed) nodal curves over Δ is a flat morphism $\pi : \mathcal{C} \to \Delta$ whose fibres are the (pointed) nodal curves $C_t := \pi^{-1}(t)$. We denote by C_0 the fibre over t_0 and we will always assume that C_t is isomorphic to $C_{t'}$ for $t \neq t'$ and $t, t' \in \Delta \setminus \{t_0\}$. We shall call C_t the "generic" and C_0 the special fibre of the family.

Example 2.10 (see pg. 188 [RC]). Consider the family

$$\pi : \mathcal{C} := \{(x : y : z) \in \mathbb{P}^2 : xy - tz^2 = 0\} \to \Delta$$

over a complex disk $\Delta \subset \mathbb{C}$ with parameter t shown in figure 2.10. The generic fibre is a smooth curve isomorphic to $\mathbb{P}^1$. The special fibre is a nodal curve given by the union of coordinate axes.

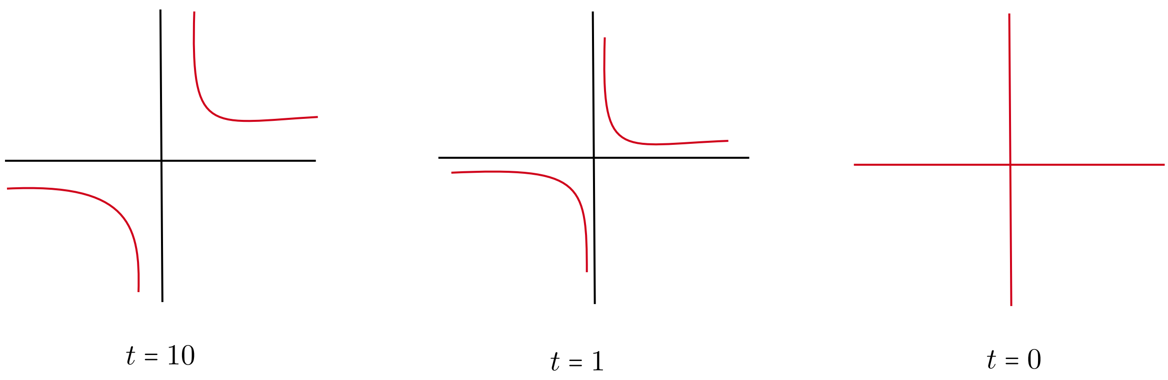


Figure 3: Real picture of the fibre of π over $t = 0, 1, 10$ in the $z = 1$ plane.

Maps between Riemann surfaces. In the category of compact Riemann surfaces the structure-preserving maps are precisely the holomorphic maps. Note that compactness of both target and base space together with holomorphicity are highly restrictive conditions, we expect them to carry a lot of structure within.

Definition 2.11. Let C and D be compact Riemann surfaces. We call a map $f : C \to D$ *holomorphic* if and only if for every point $p \in C$ and every choice of charts ϕ_p around p and $\phi_{f(p)}$ around $f(p)$ the map $F := \phi_{f(p)} \circ f \circ {\phi_p}^{-1}$ is a holomorphic map between open sets of $\mathbb{C}$ (figure 4). We call F the *local expression* for f.

In fact, by carefully choosing our charts we can always achieve that (if f is non-constant) F is power function $z \mapsto z^{k_p}$ where the positive integer k_p is uniquely determined. Here, carefully choosing means that we need to impose a restriction on the choice of charts ϕ_p and $\phi_{f(p)}$: We require that they are centred at $p, f(p)$, i.e. $\phi_p(p) = 0 = \phi_{f(p)}(f(p))$ (see Theorem 4.2.1. [RC]). In this case we call k_p the *ramification index* of f at p.
Generically, i.e. except at finitely (since C is compact) many points, $k_p = 1$. Points, where $k_p > 1$ are called *ramification points* of f and their images $f(p) \in D$ *branch points.* Except at branch points, each $q \in D$ has exactly d inverse images under f. This constant is called the *degree* of f. Figure 5 shows a local image of f around a generic point (on the left) and a branch point (on the right). The collection of ramification indices of preimages of branch points are called *ramification profile* and form a partition of the degree of f.

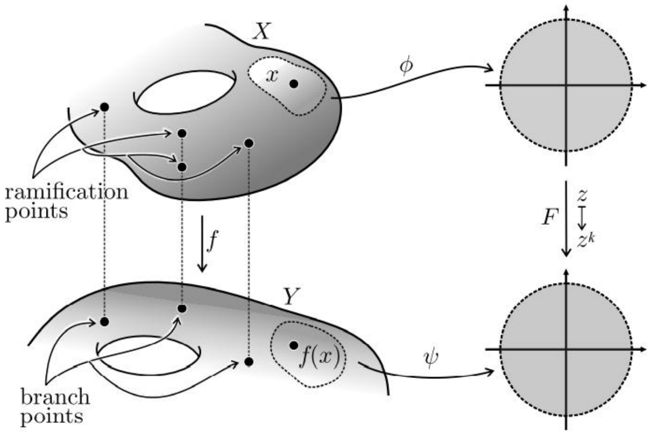


Figure 4: Local structure of a map between two Riemann surfaces [RC].

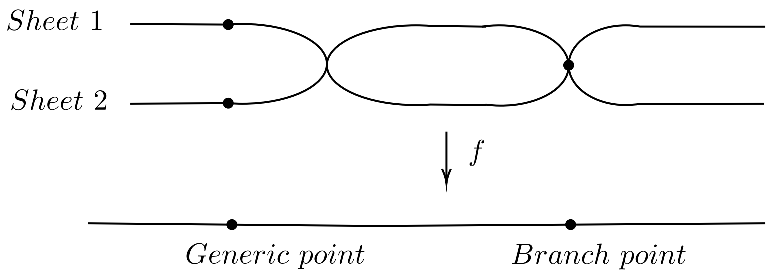


Figure 5: Real (local) picture of a degree 2 branched covering of $\mathbb{P}^1$ with ramification profiles (2).

Line bundles and divisors. A divisor on a curve C is a formal linear combination of points on C with integer coefficients. Let $f : C \mapsto \mathbb{C}$ a meromorphic function, i.e. f corresponds to a non trivial element of the field of rational functions $K(C)$ of C (Definition 3.3.3. [Gat03]). To f we associate a divisor

$$(f) \coloneqq \sum_{P \in C} \nu(P)P,$$

where $\nu(P)$ is the order of the lowest non trivial coefficient in the Laurent series of f. A point P is a zero of f if $\nu(P) > 0$ and a pole if $\nu(P) < 0$. Such divisors are called *principle* and have degree 0. Similarly, we can associate a divisor to a rational map g on C that is only locally defined and thus to a global section s of an arbitrary line bundle L.

Definition 2.12. (13.1.2. [Vak17]) Given a global section s of a line bundle L of C, we can pick an open cover U_i of C and local trivialization $\phi : U_i \to U_i \times \mathbb{C}$ of L for every U_i. Then s can be naturally identified with a functions $g : U_i \to \mathbb{C}$ and the corresponding divisor is $(s) \coloneqq \sum_i (g_i)$.

We can think of (s) as the divisor of zeros and poles of (s). There is a correspondence between the group of line bundles Pic_C with group structure induced by the usual tensor product and the group of divisors on C denoted by Div_C:

$$u : Div_C \to Pic_C, D \mapsto \mathcal{O}_C(D), \text{ where } H^0(C, \mathcal{O}_C(D)) \coloneqq \{f \neq 0 \text{ meromorphic} : (f) + D \geqslant 0\}.$$

The *Abel-Jacobi map u* assigns to any divisor the line bundle $\mathcal{O}_C(D)$ whose global sections consist of meromorphic functions on C whose zeroes and poles are controlled by D. Conversely, it can be shown that every line bundle L admits a global section s and, thus, can be written in the form $L = \mathcal{O}_C(D)$ with $D = (s)$. In this light we can find an equivalent construction on the side of divisors for the space of global sections $H^0(C, L)$ of L.

Definition 2.13 (and Theorem section 1.2. in [Bar14]). We denote by $|D|$ the set of all effective divisors which are linearly equivalent to D, the so called *complete linear system* of D. We have an isomorphism

$$\mathbb{P}H^0(C, \mathcal{O}_C(D)) \cong |D|, f \mapsto D + (f).$$

Definition 2.14. Let ω_C be the cotangent bundle of C. The divisor K_C associated to a global section s of ω_C by the above correspondence is called a *canonical divisor*. All canonical divisors on a curve are linearly equivalent and, as it follows from the Riemann-Roch Theorem (theorem 2.17), have degree $2g - 2$ (example 2.18) (Theorem section 1.3. in [Bar14]).

Example 2.15. Let $C = \mathbb{P}^1$. In order to compute $(\omega_{\mathbb{P}^1})$ we have to find a global section s of $\omega_{\mathbb{P}^1}$ and determine (s). We use the identification of global sections of the cotangent bundle with meromorphic differentials on C. Let $U_1 \coloneqq \{(X : Y)\ X \neq 0\}$ and $U_2 \coloneqq \{(X : Y)\ Y \neq 0\}$ together with local coordinates $x \coloneqq \frac{Y}{X}$ and $y \coloneqq \frac{X}{Y}$. On U_1 (its identification with affine space $\mathbb{A}^1$ to be more precise) we know that Ω_C is locally generated by dx (definition 2.5). This is a meromorphic differential without poles or zeroes (on $\mathbb{A}^1$). We have to figure out if we can extend dx to infinity (i.e to $X = 0$). On $U_1 \cap U_2$ we can obtain its expression in y−coordinates by using the transition function $x \mapsto \frac{1}{x}$. We get a meromorphic differential $dx = \frac{-1}{y^2} dy$ on $\mathbb{P}^1$ with a pole of order 2 at infinity. Hence, $(\omega_{\mathbb{P}^1}) = -2(0 : 1)$ and $deg(\omega_{\mathbb{P}^1}) = -2$.

The canonical embedding. The canonical divisor can be used to embed the a curve into projective space.

Definition 2.16 (Definition 4 [Ser17]). Let C be a curve of genus g and ω_C its cotangent bundle. Let $V \coloneqq H^0(C, \omega_C)$ be the space of global sections of ω_C or, equivalently, the vector space of holomorphic differential forms on C. For each $p \in C$ let $V_p \coloneqq \{s \in V | s(p) = 0\}$ be the subvector space of V consisting of all holomorphic differentials vanishing at p. Let $\mathbb{P}(V)^*$ denote the the dual projective space of V. The map

$$\phi : C \to \mathbb{P}(V)^*, p \mapsto V_p$$

is called the *canonical map*. If C is not hyperelliptic then ϕ is an embedding and $\phi(C)$ is a curve of degree $2g-2$ called the canonical model of C, or simply a *canonical curve*.

Since $|K_C|$ is base-point-free, i.e. there is no point in C common to all divisors in $|K_C|$, the condition $s(p) = 0$ is non trivial. Hence, it defines a hyperplane or equivalently a point in the dual space. The canonical embedding can be extended to effective divisors via

$$D = \sum_i nP_i \mapsto span\{\phi(P_1), ..., \phi(P_n)\}.$$

This embedding gives us a nice way to visualize the canonical class of a curve:
Observe that a global section $s \in V$ (i.e. a point) corresponds to a hyperplane H_s in the dual space. By identifying C with its image $\phi(C)$ we see that the divisor (s) arises from the intersection $H_s \cdot C$. All canonical divisors are linearly equivalent, that is there exists a meromorphic function f such that $(s_1) + (f) = (s_2)$ for two canonical divisors (s_1) and (s_2). Geometrically, we can interpret f as a transformation which moves the hyperplane H_{s_1} to H_{s_2} ([Bar14]) (see figure 6).

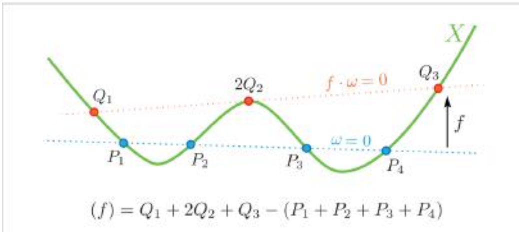


Figure 6: Figure 1.2 in [Bar14]

Tools to compute $h^0(C, L)$. We are interested in the dimension of the space of global sections of a line bundle. A powerful tool to address this is *the Riemann-Roch theorem* for line bundles (alternatively with our identification, for divisors) on smooth curves.

Theorem 2.17. Let D be a divisor on a smooth genus g curve C and K_C the canonical divisor. The Riemann-Roch formula states

$$h^0(C, D) - h^0(K_C - D) = deg(D) - g + 1.$$

Example 2.18. An easy consequence is $deg(K_C) = 2g - 2$. Choosing $D := K_C$ and $D = 0$ in the above formula yields the result.

Aside from our interest in computing $h^0(C, D)$, the Riemann-Roch Theorem has an important application to Hurwitz theory. It relates all discrete invariants we have associated to a map of compact Riemann Surfaces.

Theorem 2.19. [(Riemann Hurwitz formula) [Theorem 2.2.2 [Ong14]]] Let $f : C \to D$ be a non constant holomorphic map between two smooth curves. Then

$$K_C \cong f^*(K_D) + \mathcal{R}_f,$$

where $\nu(P)$ is the vanishing order of f at p and $\mathcal{R}_f := \sum_{p \in C}(\nu(P) - 1)p$ the ramification divisor of f. The numerical formula appears as a corollary:

$$2 - 2g(C) = d(2 - 2g(D)) + \sum_{p \in C} \nu_p - 1.$$

For the moment we renounce to giving an example since countless applications of this formula wait for us in the next chapters.

2.2 Hurwitz theory

The (real) starting point of our journey. The roots of *Hurwitz theory* date as far back as 1900, where Hurwitz published an enumerative problem, the computation of so-called *Hurwitz numbers.* It soon emerged into a rich and beautiful theory that is concerned with the enumerative study of holomorphic maps between Riemann surfaces.
As usual (in mathematics) we do not care about their concrete realization, but want to identify objects that are "essentially the same" .

Definition 2.20 (Definition 6.1.1 [RC]). Two holomorphic maps of Riemann surfaces $f : C \to D$ and $g : \tilde{C} \to D$ are called *isomorphic* if there is an isomorphism of Riemann surfaces $\phi : C \to \tilde{C}$ such that $f = g \circ \phi$. An *automorphism* of f is an isomorphism $\phi : C \to C$ such that $f = f \circ \phi$. The group of automorphisms of f is denoted $Aut(f)$.

The counting problem. Even if we fix the target curve, say $D = \mathbb{P}^1$, the number of isomorphism classes of maps to D is infinite. This is why, in addition to fixing a curve D of genus h, we introduce some additional geometric constraints. These make our count finite. Thus, fix

1. a collection of points $q_1, ..., q_n \in D$, the prescribed branch points,
2. a positive integer d, the prescribed degree,
3. a collection $\lambda_1, ..., \lambda_n$ of partitions of d, the prescribed ramification profiles

and count the weighted number of (isomorphism classes of) maps $f : C \to D$, denoted by $[f]$, from a connected Riemann surface C, such that f is of degree d and displays the ramification behaviour at points q_i prescribed by the partitions λ_i. We say, f is a *Hurwitz map or cover* for the discrete data $(g, h, d, \lambda_1, ..., \lambda_n)$.

Definition 2.21. The number defined above is called a *Hurwitz number* for the discrete data $(g, h, d, \lambda_1, ..., \lambda_n)$ and denoted by $H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_n)$. We have

$$H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_n) := \sum_{[f]} \frac{1}{|Aut(f)|}$$

where the sum is over all isomorphism classes of Hurwitz covers. If we allow C to be disconnected, we denote the corresponding Hurwitz number by $H^{\bullet}_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_n)$.

Remark 2.22. The notation $H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_n)$ implicitly assumes that a Hurwitz number only depends on the discrete data $(g, h, d, \lambda_1, ..., \lambda_n)$. But what about the choices we made, e.g. the choice of a Riemann surface D or the choice of a configuration of points on D? *Riemann's existence theorem* (Theorem 6.2.2 [RC]) resolves these issues. He observes that a topological cover of finite degree $f^0 : C^0 \to D \setminus \{q_1, ..., q_n\}$ extends uniquely (up to isomorphisms) to a holomorphic map of compact Riemann surfaces $f : C \to D$.

The existence problem. The most basic question, a priori, is: For which data does a such a map exist? Surprisingly, it turns out to be a famous problem, known as the *Hurwitz existence problem.* We already saw necessary condition in the previous section: the data $(g, h, d, \lambda_1, ..., \lambda_n)$ has to satisfy the Riemann Hurwitz formula 2.19, i.e.

$$2 - 2g = d(2 - 2h) + \sum_{i=1}^{n}(d - l(\lambda_i)),$$

where we used that

$$\sum_{P\in C}(\nu(P)-1)=\sum_{\tilde{P}\in D}(d-l(\lambda_{\tilde{P}})) \text{ and } \lambda_{\tilde{P}} \text{ denotes the ramifcation profile of } f \text{ at } \tilde{P}.$$

However, it is far from being a sufficient condition as example 2.26 shows. The work of Mednykh provides us with a complete solution of Hurwitz's existence problem in terms of irreducible representation of the symmetric group ([Med84]). However, his formula is only useful in very specific cases since largely intractable sums make computation infeasible. Hence, finding practicable sufficient and necessary conditions is still an open problem ([RC]).

Example 2.23. Let $D=\mathbb{P}^1$, $d\in\mathbb{N}$, $q^1=(0:1)$,$q^2=(1:0)$ and $\lambda_1=(d)=\lambda_2$. We compute the Hurwitz number $H_{0\xrightarrow{d}0}((d),(d))$. One can show (see Example 6.1.7. [RC]) that there exists only one map $f:\mathbb{P}^1\to\mathbb{P}^1$ satisfying $(0,0,d,(d),(d))$.
We can either describe f by using the chart $\phi_1:U_1=\{(X:Y):Y\neq 0\}\to\mathbb{C},(X:Y)\mapsto\frac{X}{Y}=:x$ for both target and source and give a local expression on $\phi(U_1)$, i.e. $x\mapsto x^d$. Extending to infinity by setting $\infty\mapsto\infty$ (in the chart ϕ_1 we call $q^2=(1:0)$ the point at infinity and zero $q^1=(0:1)$) yields a holomorphic map on $\mathbb{P}^1$ that is fully ramified over 0 and ∞ and unramified else.
Alternatively, we can define f directly by using projective coordinates, i.e. set $f:\mathbb{P}^1\to\mathbb{P}^1,(X:Y)\mapsto(X^d:Y^d)$. The automorphism group of f is given by

$$Aut(f)=\{\phi(x)=ax,a\in\mathbb{C} \text{ and } a^d=1\}.$$

Indeed, recall that the automorphisms of $\mathbb{P}^1$ are the Möbius transformations that is maps of the form $\phi(x)=\frac{(ax+b)}{(cx+d)}$ where $a,b,c,d\in\mathbb{C}$ satisfy $ac-db\neq 0$. The condition

$$(\phi(x))^d=f(\phi(x))=f(x)=x^d$$

yields the claim. Thus we have

$$H_{0\xrightarrow{d}0}((d),(d))=\frac{1}{|Aut(f)|}=\frac{1}{d}.$$

Notice that the computation of these numbers is in general quite hard if we look at the question from the lens of differential geometry only. In simple cases it is possible to write down the corresponding Hurwitz maps explicitly, but as genus and degree increase this approach is hardly fruitful.
A change of perspective. A change of perspective from differential geometry to group theory enabled Hurwitz to determine the first class of Hurwitz numbers, so-called *simple Hurwitz numbers* which count covers of the projective line with arbitrary ramification profiles λ over 0 and only simple branch points else (i.e. points with ramification profile $(2,1,...,1)$). If we allow for an arbitrary ramification over two points, we count *double Hurwitz numbers.* For us simple ramification does not work. We consider the next more complicated behaviour *almost simple* ramification, i.e. $\lambda=(3,1,...,1)$. Why? Let us just say for the moment, because we need our partitions to be odd. The analogue of a double Hurwitz number in the case of almost simple ramification is still called a double Hurwitz number:

Definition 2.24. Let $d,g\in\mathbb{N}$ and μ,ν two special partitions of d. We denote the double Hurwitz number by $H_g^d(\mu,\nu)$.

The translation into a group-theoretic problem has many advantages. It settles the question of finiteness for example. Our approach, however, is a different one. Since we do not wish to deprive the reader from this important part of Hurwitz theory, we state the equivalent formulation in the symmetric group below. For details we refer the interested reader to chapter in [RC] for a beautiful introduction. In short, Hurwitz covers define homomorphisms from the fundamental group $\pi_1(C \setminus B, c_o)$ of the target surface C, punctured at the branch locus B, to a symmetric group. These homomorphism are uniquely determined by their value on a generating set of $\pi_1(C \setminus B, c_o)$ and thus are in bijective correspondence with tuples of permutations. The set of generators of $\pi_1(C \setminus B, c_o)$ includes loops around the punctures of C whose image encodes the ramification profile of f at the corresponding branch point. More precisely, the image of such a loop is a permutation with cycle type given by the ramification profile of f.

Definition 2.25. Fix d, h, n, g positive integers and partitions $\lambda_1, ..., \lambda_n$ of d such that the Riemann-Hurwitz Formula is satisfied. We have

$$H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_n) = \frac{|M|}{d!}$$

where

$$\begin{aligned} M := \{(\sigma_1, ..., \sigma_n, a_1, b_1, ..., a_h, b_h) \in {S_d}^{n+2h} : \sigma_i \text{ has cycle type } \lambda_i, \\ \prod_{i=1}^{n} \sigma_i \prod_{i=1}^{h} [a_i, b_i] = id \text{ where } [a_i, b_i] := a_i b_i {a_i}^{-1} {b_i}^{-1}, \\ \langle \sigma_1, ..., \sigma_n, a_1, b_1, ..., a_h, b_h \rangle \text{ acts transitively on } \{1, ..., d\}\}. \end{aligned}$$

We call such a factorization of the identity $(\sigma_1, ..., \sigma_n, a_1, b_1, ..., a_h, b_h)$ a *monodromy representation* of type $(g, h, d, \lambda_1, ..., \lambda_n)$.

The take-away message is: counting branched coverings of type $(g, h, d, \lambda_1, ..., \lambda_n)$ is as good as counting tuple of permutation in S_d.

Example 2.26. (Exercise 100. [RC]) We claim $H_{0 \xrightarrow{4} 0}((3,1), (2,2)^2) = 0$. The easiest way to see this is to count monodromy representations. A monodromy representation is determined by a tuple $(\sigma_1, ..., \sigma_3)$ in S_4 such that $\sigma_1 \circ \sigma_2 \circ \sigma_3 = id$ where σ_1 is a 3-cycle and σ_i for $i = 1, 2$ is a composition of two disjoint transpositions denoted τ_j^i for $j = 1, 2$. Note that $\tau_1^1 \circ \tau_2^1 \circ \tau_1^2 \circ \tau_2^2$ can never be a 3-cycle since S_4 does not have enough disjoint transpositions. Choose two disjoint τ_1^1 and τ_2^1. For the next transposition we have 3 possibilities. First, τ_1^2 is disjoint from τ_2^1, hence $\tau_1^2 = \tau_1^1$. But this forces $\tau_2^2 = \tau_2^1$. Second, $\tau_1^2 = \tau_2^1$ and thus $\tau_2^2 = \tau_1^1$. The last chance to get a three cycle is $\tau_2^1 \circ \tau_1^2$ is a 3-cycle. But because of the disjointness condition τ_2^2 cannot be τ_1^1 since τ_1^2 contains only one element of τ_2^1 and thus one of τ_1^1 as well.

Degeneration formulas We motivated our interest for nodal curves by identifying these as limits of a degeneration procedure. This procedure is what will simplify the count of Hurwitz numbers and give rise to the so-called *Degeneration formulas*.

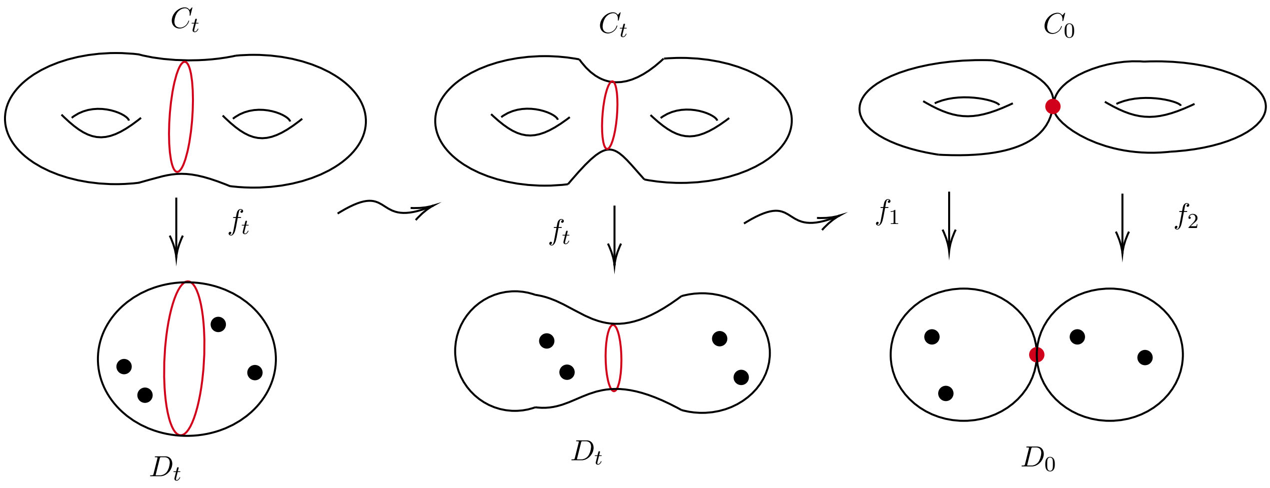


Figure 7: Degeneration of a Hurwitz cover.

*The intuitive idea.*Take the Hurwitz cover $f : C \to D$ that satisfies the discrete data $(g, h, d, \lambda_1, ..., \lambda_n)$. Draw a loop around two of the branch points on the base curve, shrink it simultaneously with its preimage on the source curve until both, C and D, become singular. We end up with a cover f_0 of nodal curve $D = D_1 \cup D_2$ by a nodal curve $C = C_1 \cup C_2$. Such a *degenerate cover* consists of a glued pair of Hurwitz covers $(f_1 : C_1 \to D_1, f_2 : C_2 \to D_2)$ for the separate discrete data $(g(C_1), g(D_1), d, \lambda_1, ...\lambda_{n-2}, m)$ and $(g(C_2), g(D_2), d, m, \lambda_{n-1}, \lambda_n)$ where $g(C) = g(C_1) + g(C_2)$ and $g(D) = g(D_1) + g(D_2)$. Of course, these maps fit together over the nodes, i.e. they have matching ramification profile m there.
This suggests: Instead of counting Hurwitz covers of D, rather count covers of D_1 and D_2 separately. However, we need to take into account different possibilities of gluing these Hurwitz cover together or, in put the other way around, keep track of the fact that different Hurwitz covers can degenerate to the same nodal cover. No matter how you choose to look at it, this gives rise to the factors in the formulas below.

Theorem 2.27 (Theorem 7.5.1[RC]). Let $m = (m_1, ..., m_l)$ be a partition of a positive integer d. The order of the centralizer of any permutation of cycle type m in S_d is given by $m!|m|$, where $m! = |Aut(m)|$ and $|m| = \prod_{i=1}^{l} m_i$. Then the following formulas hold for all Hurwitz data:

1. Reducing the genus of a higher genus base curve

$$H^{\bullet}_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s) = \sum_m m!|m| H^{\bullet}_{g-l(m) \xrightarrow{d} h-1}(\lambda_1, ..., \lambda_s, m, m)$$

2. Base curve of genus zero: reducing branch points

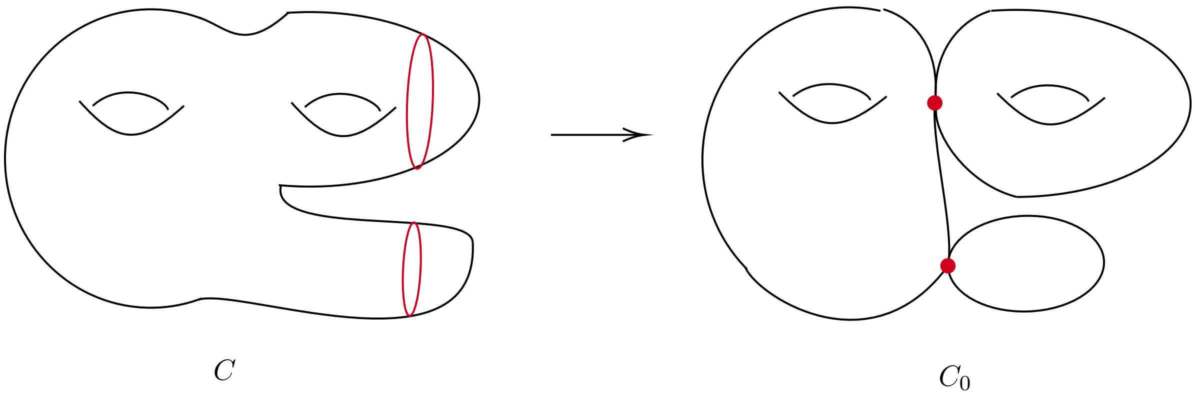
$$H^{\bullet}_{g \xrightarrow{d} 0}(\lambda_1, ..., \lambda_s, \mu_1, ..., \mu_t) = \sum_m m!|m| H^{\bullet}_{g_1 \xrightarrow{d} 0}(\lambda_1, ..., \lambda_s, m) H^{\bullet}_{g_2 \xrightarrow{d} 0}(m, \mu_1, ..., \mu_t).$$

Here, g_1 and g_2 are determined by the Riemann-Hurwitz formula and satisfy the condition $g_1 + g_2 + l(m) - 1 = g$, where $l(m)$ denotes the length of the partition m.

Thus, (disconnected) Hurwitz numbers display a recursive structure, i.e. they can be expressed as a product and sum of Hurwitz numbers either of lower genus or with fewer ramification profiles. Iterating these degeneration steps yields the following important result.

Lemma 2.28 (Theorem 7.5.3. [RC]). All disconnected Hurwitz numbers can be expressed in terms of Hurwitz numbers $H^{\bullet}{}_{g\xrightarrow{d}0}(\lambda_1, \lambda_2, \lambda_3)$, i.e. where the genus of the base curve is 0 and there are only 3 ramification conditions imposed.

Remark 2.29. There are degeneration formulas for connected Hurwitz numbers as well. Expressing connected Hurwitz numbers in terms of connected Hurwitz numbers is more challenging since degenerating a connected curve C can produce more than two connected components.



Write somewhere marking of all rams and branchs.

Dual graph A first step towards tropical geometry is to define an object that encodes combinatorial information about a curve.

Definition 2.30 (Dual graph 2.1. [CMR16]). Let C be a nodal curve. We construct the dual graph Γ_C in the following way::

- For each irreducible component C_v draw a vertex v with weight $g(v)$ defined as the genus of C_v.
- For each node draw an edge e, where e is incident to a vertex v if the node is contained in the irreducible component dual to v.

If C is a marked curve, we add an end for each marked point p and attach it to the vertex v dual to the component containing p.

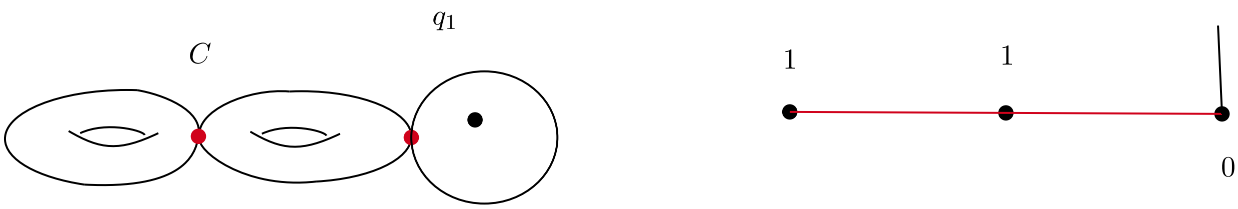


Figure 8: Dual graph of a nodal curve with one marked point.

As usual when associating a new object to an old one, it is crucial to know exactly what information is preserved and what is lost in the process. The dual graph remember:

1. the number of irreducible components together with their genus.
2. how these components intersect.
3. how marked points are distributed on the curve.

The dual graph does not remember the isomorphism type of the curve.

3 Tropical Hurwitz theory

By filtering out the noise of algebraic complexity and extracting the relevant combinatorial data, Hurwitz was able to reduce the problem of counting covers with fixed ramification data to a purely combinatorial one of the symmetric group. This “extraction of combinatorial data” and simplification of algebraic geometry is what tropical geometry is all about.

Where tropical geometry comes into play...

Following the step-by-step procedure 1.1 to tackle Hurwitz’s problem we end up with a *tropical Hurwitz theory*. Step 1 is clear since we have our algebraic problem. For step 2 we ask: What are the objects we care about and should tropicalize? Riemann surfaces and maps between them. This leads us to the notion of *tropical Hurwitz covers*, i.e. maps between graphs satisfying some additional conditions. In the easiest case, branched coverings of $\mathbb{P}^1$, we consider maps, whose target is just a line. Step 3 is where the magic happens. The definition of *tropical Hurwitz numbers* is just a weighted count of graphs. Appreciate this, the original task, e.g. determine $H_{1\xrightarrow{3}0}((3),(3),(3),(3))$ only using the geometric definition, tropicalized, becomes: draw all graphs with two vertices, two ends of weight 3 plus some other conditions. This is doable. Figure 9 shows all graphs counting towards $H_{1\xrightarrow{3}0}((3),(3),(3),(3))$. Remains the last step. How are these numbers related to Hurwitz numbers? The answer is they are equal! This is the result of a non-trivial theorem, a *correspondence theorem*, which is a strive-to result in tropical geometry.

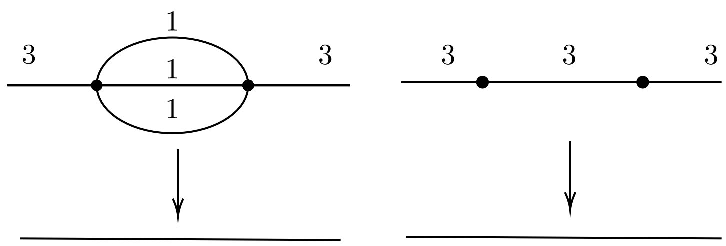


Figure 9: Tropical covers that contribute to $H_{1\xrightarrow{3}0}((3),(3),(3),(3))$.

3.1 Tropical covers

Let us take a step back for a minute and start by introducing analogous objects on the tropical side. Consider a connected graph Γ and denote by $E(\Gamma)$ the set of edges and by $V(\Gamma)$ the set of vertices. The valence $val(v)$ of a vertex $v \in V(\Gamma)$ is the number of edges adjacent to v. Half-edges, i.e. edges that are adjacent to one vertex only, are called ends. Edges that are not ends are called bounded.

Definition 3.1. An abstract tropical curve (Γ, l, g) is a graph Γ with

1. a metric $l : E(\Gamma) \to \mathbb{R} \cup \infty$ on the edges, such that $l(e) = \infty$ if e is an end, and $l(e) \in \mathbb{R}$ otherwise.

2. a weighting $g : V(\Gamma) \to \mathbb{Z}$ on the vertices.

The value of g at a vertex v is called the genus of v.

The *genus* of Γ is $g(\Gamma) := b_1(\Gamma) + \sum_{v \in V(\Gamma)} g(v)$, where $b_1(\Gamma) := |E(\Gamma)| - |V(\Gamma)| + c(\Gamma)$ and $c(\Gamma)$ is the number of connected components of Γ.

We say Γ is *stable*, if for all $v \in V(\Gamma)$ we have

$$2g(v) - 2 + val(v) > 0.$$

This means that each genus 0 vertex has to be at least trivalent and 2-valent vertices have to be of higher genus.

Remark 3.2. We want to view Γ as the dual graph of an algebraic curve C and $g(v)$ as the genus of an irreducible component of C dual to v. The definition of stability for graphs is just a tropical version of the one for curves, i.e. if C is stable (stability in algebraic geometry is given by finiteness of the automorphism group), so is its dual graph. The metric on Γ adds data beyond the information given by the dual graph. It encodes the "speed" at which a node in a family of algebraic curves that degenerate to C is formed.

Next we are interested in structure preserving maps.

Definition 3.3 ([CJM10]). Two abstract tropical curves Γ_1 and Γ_2 are called *isomorphic* if there is a homeomorphism $\Gamma_1 \to \Gamma_2$ such that every edge of Γ_1 is mapped bijectively onto an edge of Γ_2 by an affine map of slope ± 1, i.e. by a map of the form $t \to a \pm t$ (where $a = 0$ or $a = l(e)$, and we again identify an edge of length $l(e)$ with the interval $(0, l(e))$).
The *combinatorial type* of an abstract tropical curve is the equivalence class obtained by identifying any two isomorphic tropical curves such that the genus function is preserved.

If we work with tropical curves whose ends are labelled, we require, in addition, that the labelling is preserved.

Example 3.4. Figure 3.4 shows an abstract tropical curve of genus 2 with the metric written next to each edge on the right and the corresponding combinatorial type on the left.

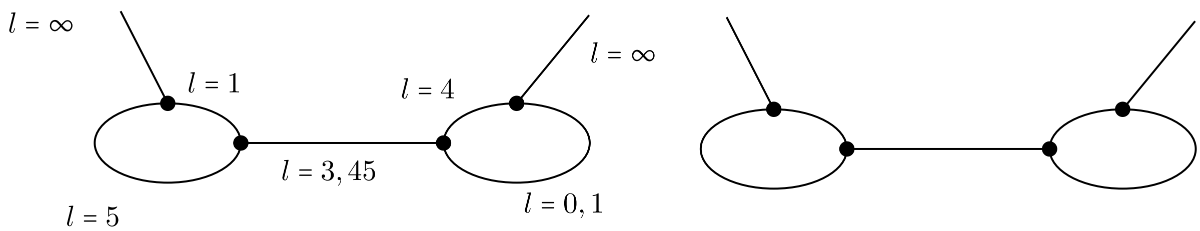


Figure 10: Abstract tropical curve and its combinatorial type. Vertices without labelling are assumed to have genus 0.

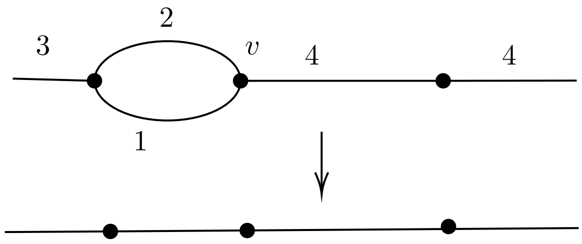
Definition 3.5 (Definition 3.1 [Mar20]). Let Γ and B be abstract tropical curves. We call Γ a *tropical cover* of B if there exists a surjective map $h : \Gamma \to B$ such that h is

1. locally integer affine linear, i.e. h restricts on each edge e to an affine linear function $h_{|e} : [0, l(e)] \to \mathbb{R} : t \mapsto a + \omega(e)t$ whose slope is a positive integer called the weight (or expansion factor) $\omega(e)$.

2. balanced/harmonic: Let $v \in \Gamma$ and e' adjacent to $h(v)$. We require the integer $d_v := \sum_{e \mapsto e'} \omega(e)$ to be independent of the choice of e'. In this case call d_v the *local degree* of h at v.

In addition, we require images and preimages of vertices to be vertices.

Example 3.6. If we consider covers of the tropical line, the balancing condition can be rephrased as: for every vertex v we have that the sum of in-going weights is equal to the sum of out-going weights. Below is an example of a non-harmonic map. The balancing condition is violated at v.



A tropical cover h is a *tropical Hurwitz cover* if at every point $v \in \Gamma$ the local *Riemann Hurwitz condition* (3.2.1. in [CMR16]) is satisfied, i.e. we have

$$0 \leqslant d_v(2 - 2g(v')) - \sum(\omega(e) - 1) - (2 - 2g(v)) =: r_v, \tag{1}$$

where $h(v) = v'$ and the genus function g is extended on Γ by mapping points that are not vertices to 0.

Remark 3.7. This condition is necessary for the existence of a degenerate algebraic Hurwitz cover of degree d_v locally (i.e. at v) "dual" to h (dual in the sense of definition 3.23). It is an example of a *realizability condition.*
In the following we consider two cases:

1. $r_v = 0$ for all $v \in V(\Gamma)$ (subsection 3.3).

2. $r_v = 2$ for all $v \in V(\Gamma)$ (subsection 3.2).

In case 1 we move all the ramification to the ends of the tropical curve. This approach makes translating the algebraic count of Hurwitz numbers to tropical geometry (via degeneration) particularly clear since it allows us to see *all* the ramification behaviour happening on the level of algebraic curves on the level of graphs as well. We have a bijection between the algebraic ramification/branch points and the ends of the respective tropical curves. This is the setting in which Bertrand, Brugallé and Mikhalkin define *general tropical Hurwitz numbers.* We will adopt this convention when working with base curves of arbitrary genus (in section 5, especially subsection 5.1.3).
In case 2 we allow for almost simple ramification (with ramification profile equal to $(3, 1, ..., 1)$) at the interior of the curve (see definition 3.8). This approach is useful when defining tropical Hurwitz numbers in analogy to classical Hurwitz numbers as intersection products ([BM14]) (and saves us from drawing all additional ends). This is the definition we use when working with tropical (spin) Hurwitz numbers for elliptic curves and for $\mathbb{TP}^1$ (subsection 5.1.1). Proposition 3.25 shows that both definitions are compatible, i.e. adding additional ends in case 1 does not affect the count.

Just like in the classical world we can define the degree, branch and ramification points of a tropical Hurwitz cover.

Definition 3.8. Let $h : \Gamma \to B$ be a tropical Hurwitz cover. We call $d := \sum_{p_1 \in h^{-1}(p_2)} d_p$, where p_2 is an arbitrary point of B, the *degree* of h. In case 1 of remark 3.7 we call a vertex of B adjacent to an end e a (unmarked) *branch point.* Its preimages are called (unmarked) *ramification points* with *ramification profile* given by the collection of weights of ends mapping to e. These form a partition of d.[1] If for $v \in V(\Gamma)$ the integer r_v defined in (1) is positive, we say that v is an (unmarked) *ramification point* ([BM14], Definition 2.2). In this case its image $h(v)$ is called an (unmarked) *branch point* of B. If $r_v = 2$ (case 2 of remark 3.7), the vertex v is an almost simple ramification.

[1]This is a consequence of the balancing condition.

By forgetting about the metric on base and cover curve we obtain the combinatorial type of a Hurwitz cover. This is the data we want to work with. Since choosing a metric on the base curve together with a weighting on the edges of the cover curve determine a metric on the cover in a unique way, we do not need to encumber drawings with this additional information. Indeed, remember that a tropical Hurwitz covers restricts to an affine integer linear function on each edge $e \in E(\Gamma)$ whose slope is just given by the weight $\omega(e)$. This yields the relation $l(e') = l(e)\omega(e)$, where $e \mapsto e'$.
Figure 11 shows an example of a tropical Hurwitz cover of degree 3 with 2 branch points, in the top row on the left, with 4 branch points with ramification profile (3), in the top row on the right and a tropical cover in the bottom row. Note that the local Riemann Hurwitz condition is violated at the right vertex (denoted by v) since

$$d_v(2 - 2g(v')) - \sum(\omega(e) - 1) - (2 - 2g(v)) = 2(2 - 0) - 3(2 - 1) - (2 - 0) = -1 < 0.$$

The balancing condition, however is satisfied.

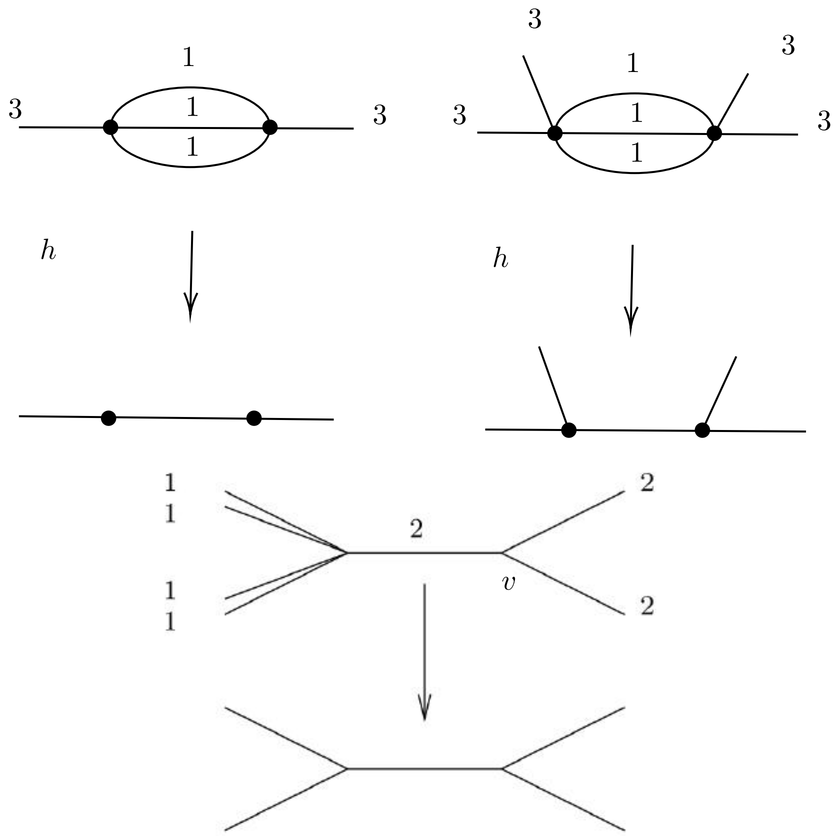


Figure 11: Fig.6 in [Mar20] on the bottom.

Remark 3.9. Just like in the classical world *tropical Hurwitz numbers* count tropical Hurwitz covers. There are two approaches in the literature that make this statement precise. These correspond to the two cases considered in remark 3.7. The first goes back to Bertrand, Brugallé and Mikhalkin (see case 1) and the second to Markwig, Cavalieri and Johnson (case 2 for simple ramification). Along a different vein, they introduce tropical Hurwitz numbers via tropical intersection theory: They establish a moduli space of tropical covers together a tropical branch map that records the images of the simple ramifications ([BM14]) and whose degree is defined to be the tropical Hurwitz number. A correspondence theorem was proved by matching tropical Hurwitz covers to *monodromy graphs*. These are graphical representations of the cut and join equations in the symmetric group and can be used to count monodromy representations. Transferred to the tropical world they give rise to tropical Hurwitz covers that satisfy the Riemann Hurwitz condition with "$\leqslant$" rather than "=". The transition to case 1 of remark 3.7 simply requires adding ends where needed and declaring *all* preimages of vertices to be vertices. This was done in the top row of figure 11.

For the purpose of this thesis we mostly consider 3-valent base (2-valent respectively when suppressing additional ends) curves with genus 0 vertices only. Inspired by the degeneration perspective, we call such a tropical curve B *maximally degenerate*. Indeed, in this case the lift of B to a nodal curve is a maximal stable degeneration, i.e. the result of an algebraic degeneration ad extremum.
The treatment of the general case, i.e. the analysis of tropical covers of an arbitrary but fixed maximally degenerate base curve B (together with a set of n ends and n ramification profiles), is difficult for several reasons (see section 3.3). We will restrict to selected special cases.

3.2 Tropical Hurwitz numbers for $\mathbb{TP}^1$ and $\mathbb{T}E$

In this section we consider the two special cases where the target curve is either $B := \mathbb{TP}^1$ or $B := \mathbb{T}E$ (figure 12). The first is an already existing result in [Hah14] and the second a generalization of tropical covers of elliptic curves with only simple ramification considered in [BGM18] to almost simple ramification. Both results are established in the context of case 2 of remark 3.7.

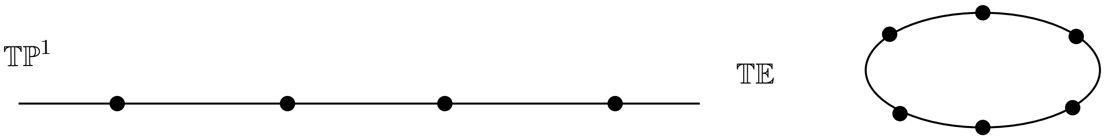


Figure 12: The tropical line on the left and a tropical elliptic curve on the right.

Tropical $3-$*cycle covers of* $\mathbb{TP}^1$*.*

Definition 3.10 (Connected 3-cycle coverings based on Definition 4.9. [Hah14]). Fix integers s, g, d and partitions μ and ν of d satisfying $2s = 2g - 2 + l(\mu) + l(\nu)$. Set $B = \mathbb{TP}^1$ with s vertices of genus 0. A *tropical* 3-*cycle* cover of $\mathbb{TP}^1$ is a tropical cover $h : \Gamma \to B$, such that

1. Γ is of genus g,
2. the ramification profile over the left end is μ and over the right end ν,
3. the preimage of one of the s vertices is either a 4-valent vertex of genus 0 (type (i),(ii),(iii) in figure 13) or a 2-valent vertex of genus 1 (type (iv) in figure 13).

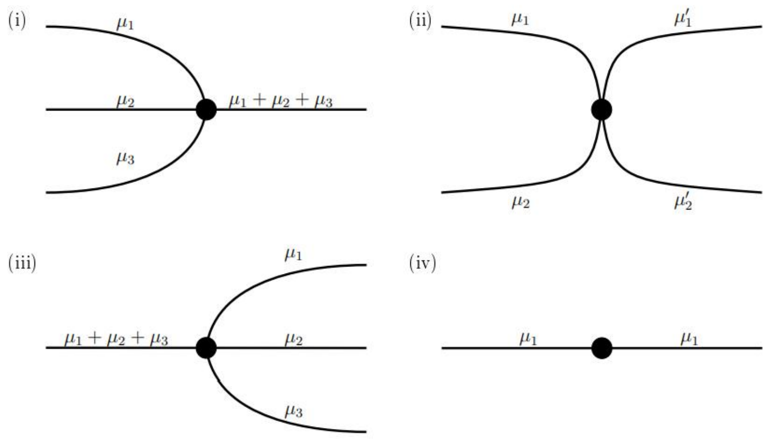


Figure 13: Possible inner vertices for a tropical 3-cycle cover (Figure 4.4 in [Hah14]).

To alleviate notation we introduce the following:

Definition 3.11 (Definition 6.3. [Hah14])*.* Let Γ be a tropical curve such that there is a tropical 3-cycle cover $h : \Gamma \to \mathbb{TP}^1$.

1. A vertex of type (ii) in figure 13 is called a *butterfly vertex.*
2. A *balanced single fork* consists of two ends of weight n sharing the same vertex (see figure 14 (viii)-(ix)).
3. A *balanced double fork* consists of three ends of weight n sharing the same vertex (see figure 14 (vi)-(vii)).
4. A *single wiener* consists of two bounded edges of weight n, sharing the same two end vertices (see figure 14 (ii)-(iv)).
5. A *double wiener* consists of three bounded edges of weight n, sharing the same two end vertices (see figure 14 (i)).

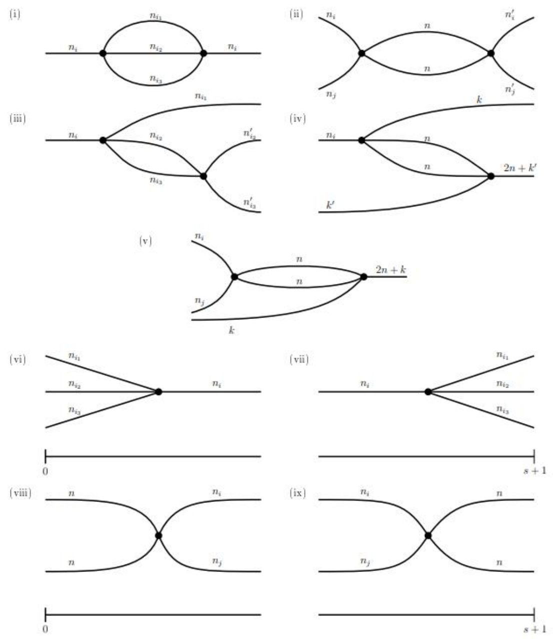


Figure 14: Figure 6.5. [Hah14]

Remark 3.12. Tropical 3-cycle covers are shadows of algebraic Hurwitz covers of $\mathbb{P}^1$ with $k := s+2$ branch points. Over two of them we allow for arbitrary ramification behaviour, otherwise we settle for almost simple ramification profile that corresponds to the ramification at the interior of the tropical source curve.

Like algebraic Hurwitz numbers, *tropical Hurwitz numbers* count tropical Hurwitz covers with a certain multiplicity.

Definition 3.13 (adapted from Definition 4.11. in [Hah14]). Let $h : \Gamma \to \mathbb{TP}^1$ be a tropical 3-cycle cover of $\mathbb{TP}^1$. We define its multiplicity, $mult(h)$, to be

$$mult(h) := \frac{1}{|Aut(\Gamma)|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \prod_{v \in V^1(\Gamma)} \frac{(\omega(e_v) - 1)((\omega(e_v) - 2))}{3!} 2^{|I|+|J|} 0^{|K|},$$

where

- $V^1(\Gamma)$ is the set of genus 1 vertices and for $v \in V^1(\Gamma)$ e_v denotes the incoming edge.
- $|Aut(\Gamma)| = 3!^{W_1+W_2} 2^{F_1+F_2}$, where W_1, respectively F_1 is the number of single Wiener, respectively single balanced forks and W_2, respectively F_2 is the number of double Wiener, respectively double balanced forks.
- J is the set of 4-valent vertices as in figure 13 (ii) with $\mu_1 \neq \mu_1'$ and $\mu_1 \neq \mu_2'$.
- K is the set of 4-valent butterfly vertices with $\mu_1 = \mu_1' = \mu_1 = \mu_2'$.
- I is the set of 4-valent vertices as in figure 13 (i) and (iii).

Remark 3.14. We changed the multiplicity definition in [Hah14] by a factor of $\frac{1}{3!}$ for each genus 1 vertex to take into account that the order of elements in the 3-cycle is fixed by permutation it acts upon (see Lemma 5.8. (iv) [Hah14]).

The case of degree 3 covers with ramification profile (3) over $\pm\infty$ is of special interest to us.

Example 3.15. Fix a positive integer $k > 2$ and consider a tropical 3-cycle cover Γ of $\mathbb{TP}^1$ with branching profile (3) over $\pm\infty$ and $s = k-2$ almost simple branch points. Note, if Γ has an inner vertex of type (ii) (figure 13), than $mult(\Gamma) = 0$. We see, if $mult(\Gamma) > 0$ than Γ consists of the two building blocks shown in figure 15 that is a double Wiener and genus 1 vertex. These can be glued to each other in any order to obtain a graph with $k-2$ inner vertices and thus all possible covers. The multiplicity simplifies to

$$\begin{aligned} mult(h) &= \frac{1}{3!^{W_2}} \frac{1}{3}^{|\{v:g(v)=1\}|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) 2^{2W_2} \\ &= \frac{2}{3}^{W_2} \frac{1}{3}^{|\{v:g(v)=1\}|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e). \end{aligned}$$

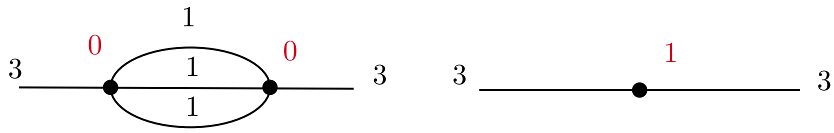


Figure 15: The two buildings blocks with genus function in red.

Tropical 3−cycle covers of $\mathbb{T}E$. Intuitively, tropical 3−cycle covers of an elliptic curve are just 3−cycle coverings of $\mathbb{TP}^1$ with ramification profile (3) over $\pm\infty$ glued together along the two outer ends.

Definition 3.16. Fix an integer g and set $B = \mathbb{T}E$, a circle with $s = g-1$ vertices of genus 0. A tropical 3-cycle cover of $\mathbb{T}E$ is a tropical Hurwitz cover $h : \Gamma \to \mathbb{T}E$, such that

1. Γ is of genus g,
2. the preimage of each of the s vertices is just like in definition 3.10 except that balanced forks cannot occur.

The multiplicity of h is analogous to the one in definition 3.13.

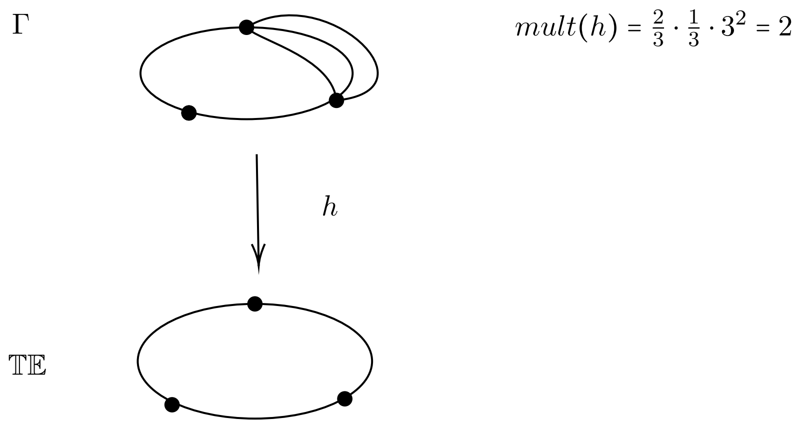


Figure 16: Tropical elliptic cover of degree 3 for $s = 3$ and multiplicity.

Definition 3.17. Fix integers s, g, d and partitions μ and ν of d satisfying $2s = 2g-2+l(\mu)+l(\nu)$. Let B be as in definition 3.10. The *tropical double Hurwitz number* $\mathbb{T}H_{g\xrightarrow{d}0}(\mu,\nu)$ is defined as

$$\mathbb{T}H_{g\xrightarrow{d}0}(\mu,\nu) := \sum_h mult(h),$$

where the sum goes over all tropical 3-cycle covers h as in definition 3.10.
Fix integers s, g, d satisfying $s = g-1$. Let B be an elliptic curve with s inner vertices. The *tropical Hurwitz number* with almost simple branching $\mathbb{T}H_{g\xrightarrow{d}1}$ is defined as

$$\mathbb{T}H_{g\xrightarrow{d}1} := \sum_h mult(h),$$

where the sum goes over all tropical 3-cycle covers h as in definition 3.16.

We have the following correspondence.

Theorem 3.18 (Theorem 6.7. [Hah14]). Fix integers s, g, d and partitions μ and ν of d satisfying $2s = 2g-2+l(\mu)+l(\nu)$. Then:

$$H_{g\xrightarrow{d}0}(\mu,\nu) = \mathbb{T}H_{g\xrightarrow{d}0}(\mu,\nu).$$

Theorem 3.19. Fix integers s, g, d satisfying $s = g-1$. Then:

$$H_{g\xrightarrow{d}1}((31...1)^s) = \mathbb{T}H_{g\xrightarrow{d}1}.$$

Hahn proves theorem 3.18 by matching graphical representations of generalized *cut-and-join relations* in the symmetric group to combinatorial types of tropical covers. From the degeneration perspective the idea of the proof is following:
Let us first interpret tropical covers as shadows of degenerate Hurwitz cover . To do this we go back to the degeneration formulas 2.27. Any Hurwitz number can be expressed in terms of Hurwitz numbers of type $H_{g\xrightarrow{d}0}(\lambda_1,\lambda_2,\lambda_3)$. Algebraically this corresponds to counting degenerate Hurwitz covers of a maximal stable degeneration D of $\mathbb{P}^1$, i.e. the base curve D consists only of spheres, where each sphere is glued to other spheres at exactly three points. We are just one step away from the tropical world. Taking dual graphs of base and cover curve together with the obvious map of graphs yields a tropical Hurwitz cover. The weights encode the ramification profiles over the nodes. Think of it as clever book-keeping of the factors coming out of the degeneration formulas. This process is not injective, we may have many maps lying over a tropical cover. However, we know exactly how many! This information is given by the multiplicity. We will see precisely how the tropical multiplicity is linked to the degeneration formulas in the next sections.

3.3 Tropical Hurwitz numbers for arbitrary base curves

In [BBM11], Bertrand Brugallé and Mikhalkin define general Hurwitz numbers tropically and prove a correspondence theorem using topological degeneration.
Fix integers s, h, g, d and partitions $\lambda_1, ..., \lambda_s$ of d satisfying the Riemann-Hurwitz formula.

Definition 3.20. Fix a maximally degenerate base curve B of genus h together with a set of s ends. A tropical Hurwitz cover that counts towards $\mathbb{T}H_{g\xrightarrow{d}h}(\lambda_1, ..., \lambda_s)$ is a Hurwitz cover $h : \Gamma \to B$ as in definition 3.8 such that

1. Γ is of genus g,

2. ends of Γ mapping to the i-th end of B are labelled with the parts of λ_i.

Let v be a vertex of Γ. Since the base curve B is trivalent, $h(v)$ is adjacent to 3 edges, e_1, e_2, e_3. By the balancing condition edges mapping to e_i define a partition n_i^v of d_v, where d_v is the local degree at v and $i = 1, 2, 3$. We call the number

$$H_v := H_{g(v) \xrightarrow{d_v} 0}(n_1^v, n_2^v, n_3^v)$$

the local Hurwitz number of Γ at v.

Definition 3.21. The *tropical Hurwitz number* $\mathbb{T}H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s)$ is defined as

$$\mathbb{T}H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s) := \sum_h mult(h),$$

where $h : \Gamma \to B$ is a tropical Hurwitz cover and

$$mult(h) := \frac{1}{|Aut(h)|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \prod_{v \in V(\Gamma)} n_1^v! \, n_2^v! \, n_3^v! \, H_v$$

is the multiplicity of h given in Definition 2.6. ([BBM11]).

Note that this notion of multiplicity is natural in light of our the degeneration perspective. We recognize that it is just the collection of all factor produced by the degeneration formulas (see section 5).

We have a correspondence.

Theorem 3.22. (Theorem 2.11. [BBM11]) For any integers s, h, g, d and partitions $\lambda_1, ..., \lambda_s$ of d satisfying the Riemann-Hurwitz formula we have

$$\mathbb{T}H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s) = H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s).$$

Proof idea. A usual, thanks to the degeneration formulas we can count Hurwitz covers of a maximal stable degeneration of the base curve instead. To such a map we may associate a map of graphs.

Definition 3.23 (Dual graph of covers 3.2. [CMR16]). Let $f : C_0 \to D_0$ be a degenerate Hurwitz cover. We construct the dual cover $h : \Gamma_{C_0} \to \Gamma_{D_0}$ in the following way::

- Take the dual graph of the source and target curves in the sense of definition 2.30 and call them Γ_{C_0} and Γ_{D_0}.
- The map between them is a well-defined map of graphs: For a Hurwitz cover, a component of the source maps onto precisely one component of the target, yielding a map of vertices. Since nodes map to nodes, edges map to edges.
- Weighting. We mark edges of Γ_{C_0} with integers recording the ramification at the corresponding node or marked point of the source curve.

The corresponding tropical Hurwitz cover is essentially a metrization of the dual map and its multiplicity equals the number of covers which degenerate to it.

Ends or no ends. We want to point out that the definition of tropical Hurwitz covers given in subsection 3.2 and subsection 3.3 differ as remarked in 3.7. A priori it is not obvious that both counts agree.

Example 3.24. The multiplicity of tropical 3−cycle coverings of degree 3 given in example 3.15 agrees with the one in definition 3.21. By inspection of the degeneration formulas we can list all possible local Hurwitz numbers: $H_{1\xrightarrow{3}0}((3)^3) = \frac{1}{3}$, $H_{0\xrightarrow{3}0}((3)^2,(1^3)) = \frac{1}{3}$, $H_{0\xrightarrow{3}0}((3),(1^3)^2) = 0$, $H_{0\xrightarrow{3}0}((3),(2,1)^2) = 1$, $H_{0\xrightarrow{3}0}((3)^2,(2,1)) = 0$. Note that at least one of the ramification profiles (i.e. one of the three partitions) of a local Hurwitz number has to be the partition (3) since all vertices of $B := \mathbb{TP}^1$ are branch points with prescribed ramification profile (3). Indeed, under the degeneration process branch points are distributed onto the irreducible components of the nodal curve such that each one contains at least one branch point.
This gives us construction guidelines for tropical Hurwitz covers of degree 3: All possible inner vertices are given by the local Hurwitz numbers from above. The matching tropical pictures are shown in figure 17. Note, whenever we have a vertex of type (5) (figure 17), we also have a vertex of type (4) (figure 17) that contributes with a factor of 0 to the multiplicity of the respective cover. For this reason we do not consider this case: In order to obtain a vertex of type (5), we have to put ourselves into the condition of having one strand of weight 2 and one of weight 1. We start from a strand of weight 3 over $-\infty$. We can either split it into three strands of weight 1. To get a strand of weight 2 from here requires us to join two of them, but such a vertex has local Hurwitz number 0. Hence, we can only create a strand of weight 2 and a strand of weight 1 by splitting the strand of weight 3 directly according to a vertex of type (4).
Using $|Aut(h)| = 3!^{W_2}$, $(1^3)! = 3!$ and $(3)! = 1$ and substituting in definition 3.21 we obtain

$$\begin{aligned} mult(h) &= \frac{1}{3!^{W_2}} \prod_{e\in E(\Gamma)\text{bounded}} \omega(e) \, \frac{1}{3}^{|\{v:g(v)=1\}|} \underbrace{(3!)^{|\{v:g(v)=0\}|}\frac{1}{3}^{|\{v:g(v)=0\}|}}_{=2^{2W_2}} \\ &= \frac{2}{3}^{W_2} \frac{1}{3}^{|\{v:g(v)=1\}|} \prod_{e\in E(\Gamma)\text{bounded}} \omega(e). \end{aligned}$$

The last equality follows by observing $|\{v : g(v) = 0\}| = 2W_2$.

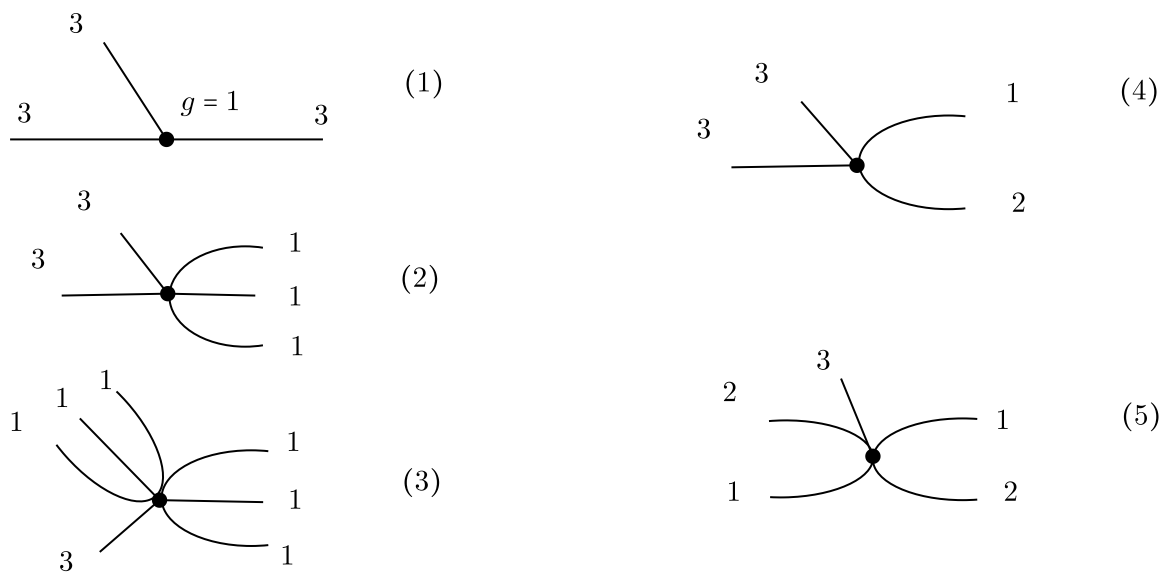


Figure 17: Vertices with corresponding local Hurwitz numbers.

Proposition 3.25. For $B = \mathbb{TP}^1$ or $B = \mathbb{TE}$ the Hurwitz numbers defined in subsection 3.2 and subsection 3.3 agree.

Proof. Let $h : \Gamma \to B$ be a 3-cycle covering as in subsection 3.2. To obtain a tropical Hurwitz cover in the sense of definition, we need to make the following adjustments:

- add an end at each inner vertex of B such that B is trivalent.
- for each $v' \in V(B)$ add one end of weight 3 and $d(v) - 3$ ends of weight 1 at the vertex $v \in h^{-1}(v')$.
- for each point $p \in h^{-1}(v') \setminus V(\Gamma)$ (i.e. p is a point on an edge e) create a genus 0 vertex on e and add $\omega(e)$ ends of weight 1.

Call $h' : \Gamma' \to B'$ the new cover. It is obvious that this process yields a bijection between isomorphism classes of tropical covers that contribute to the Hurwitz numbers defined in subsection 3.2 and the ones defined in subsection 3.3.
We can rewrite the contribution of h to the first count (definition 3.13) in terms of local Hurwitz numbers as

$$mult(h) = \frac{1}{|Aut(\Gamma)|} \prod_{e \in E(\Gamma) \text{bounded}} \omega(e) \prod_{v \in V(\Gamma)} n_v^1! \, n_v^2! \, H_v,$$

where n_v^1 and n_v^2 are the partitions of $d(v)$ given by the weights of the incoming and outgoing edges. Indeed, we only have to notice that the vertex contributions from definition 3.13 match the product of the respective local Hurwitz number H_v (computed in Lemma 7.8. ([Hah14])) with an automorphism factor from the local partitions n_v^1 and n_v^2.
The new cover h' is counted with the multiplicity defined in 3.21:

$$mult(h') = \frac{1}{|Aut(\Gamma')|} \prod_{e \in E(\Gamma') \text{bounded}} \omega(e) \prod_{v \in V(\Gamma')} n_v^1! \, n_v^2! \, n_v^3! \, H_v.$$

We have: $|Aut(\Gamma')| = |Aut(\Gamma')_i| \cdot |Aut(\Gamma')_e|$, where $Aut(\Gamma')_e$ denotes the automorphism that only permute ends and map interior edges identically and $Aut(\Gamma')_i$ denotes the automorphism that map ends identically. It holds that $|Aut(\Gamma')_i| = |Aut(\Gamma)_i|$, whereas $|Aut(\Gamma')_e|$ changes according to the possibilities to permute the additional ends of weight 1, i.e. by a factor of $(d(v) - 3)!$ for each vertex $v \in V(\Gamma) \cap V(\Gamma')$ and by a factor of $\omega(e)!$ for each newly created vertex on an edge e.
Newly created vertices are weighted by

$$\omega(e)! H_{0 \xrightarrow{\omega(e)} 0}((\omega(e)), (\omega(e)), (1, ..., 1)) = \omega(e)! \cdot \frac{1}{\omega(e)} = (\omega(e) - 1)!$$

and provide an additional bounded edge of weight $\omega(e)$. For a vertex $v \in V(\Gamma) \cap V(\Gamma')$ we get a new factor of $(d(v) - 3)!$ for the partition $(3, 1, ..., 1)$. In total we see that new contributions cancel out. It follows that the two expressions agree. □

Note that proposition 3.25 together with correspondence theorem 3.22 prove theorem 3.19 in section 3.2.

3.4 Tropical disconnected Hurwitz numbers

Tropical disconnected Hurwitz numbers are defined in analogy to algebraic disconnected Hurwitz numbers, i.e. we allow tropical covers of B to be disconnected.

Definition 3.26. Fix a maximally degenerate base curve B of genus h together with a set of s ends. A tropical Hurwitz cover that counts towards $\mathbb{T}H_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s)$ is a Hurwitz cover $h : \Gamma \to B$ as in definition 3.8 such that

1. Γ is a (possibly disconnected) curve of genus g,
2. ends of Γ mapping to the i-th end of B are labelled with the parts of λ_i.

The *disconnected tropical Hurwitz number* $\mathbb{T}H^\bullet_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s)$ is defined as

$$\mathbb{T}H^\bullet_{g \xrightarrow{d} h}(\lambda_1, ..., \lambda_s) := \sum_h mult(h),$$

where $h : \Gamma \to B$ is a tropical Hurwitz cover and

$$mult(h) := \frac{1}{|Aut(h)|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \prod_{v \in V(\Gamma)} n_1^v! n_2^v! n_3^v! H_v.$$

Remark 3.27 (see [BGM18]). The multiplicity of a disconnected cover $h : \Gamma \to \mathbb{TP}^1$ or a disconnected cover $\tilde{h} : \Gamma \to \mathbb{TE}$ is given by

$$\begin{aligned} mult(h) &= \frac{1}{|Aut(h)|} \prod_K \frac{1}{\omega_K} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \prod_{v \in V(\Gamma)} n_1^v! n_2^v! n_3^v! H_v \\ mult(\tilde{h}) &= \frac{1}{|Aut(\tilde{h})|} \prod_K \frac{1}{n_K} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \prod_{v \in V(\Gamma)} n_1^v! n_2^v! n_3^v! H_v \end{aligned}$$

where the first product in the first row goes over all connected components K of Γ that just consist of one single edge with weight ω_K. The first product in the second row runs over all connected components K of Γ that just consist of a single circle that winds around the circle n_K times.

Just like in the the connected case we have a correspondence between the tropical and the classical count.

4 Spin Hurwitz theory

Spin Hurwitz numbers are, quite similar to Hurwitz numbers, a weighted count of ramified covers. The subtle difference lies in the weighting where for each ramified cover we include the additional information of a sign. Where does this sign come from? It is given by the parity of a "pullback theta characteristic". We see that we need further introduction into the world of *spin curves and theta characteristics.*

4.1 Theta characteristics and theta hyperplanes

A modern view. From the modern point of view theta characteristics are roots of the canonical bundle.

Definition 4.1. Let C be a smooth curve of genus g with canonical class K_C of C. Any divisor L, such that $2L = K_C$, is called a theta characteristic of C. A theta characteristic is even respectively odd according to the parity of the dimension $h^0(C, L)$ of the vector space $H^0(C, L)$.
Equivalently (Definition 9 [Ser17]): Denote by ω_C the canonical bundle. We call a line bundle L on C such that $L^{\otimes 2} \cong \omega_C$ a theta characteristic of C.

Example 4.2. Let $C = \mathbb{P}^1$. The canonical class on $\mathbb{P}^1$ is given by $-2P$ for any point $P \in \mathbb{P}^1$. Thus, up tp linear equivalence there is only one theta characteristic $-P$ of even parity since $deg(-P) < 0$ implies that $h^0(\mathbb{P}^1, -P) = 0$.

A classical view. Having a definition in terms of abstract curves is useful in many ways ([Ser17]). To develop an intuition, however, a visual approach is better suited. Thus, let us look at the "embedded" version of theta characteristics, *theta hyperplanes.* In general, there are many ways to embed a curve into projective space. It turns out, however, that the canonical embedding (and its multiples) is the only one useful for deformation theoretic arguments since it is the only one that can be deformed together with the curve (see p.4 [Ser17]).

Definition 4.3 ([Ser17], Definition 7). Let C be a non-hyperelliptic genus g curve with $\phi : C \to \mathbb{P}(V) \cong \mathbb{P}^{g-1}$ the canonical embedding (definition 2.16). Then for any hyperplane $H \subset \mathbb{P}(V)$ the intersection $K_C := \phi^*(H) = H \cdot C$ gives a divisor K_C of degree $2g-2$ on C called a canonical divisor. If $H \cdot C = 2N$ for a divisor N in C then H is called a *theta hyperplane.*

Example 4.4. Let C be a non-hyperelliptic curve of genus 3. Then $\phi : C \to \mathbb{P}^2$ is an embedding whose image is a curve of degree $2 \cdot 3 - 2 = 4$, a plane quartic (figure 18). Its theta hyperplanes are the classical 28 bitangents. (p.5 [Ser17]).

Thus, we can think of theta characteristics as intersection divisor that arise from the intersection of $\phi(C)$ with a hyperplane H if all intersection multiplicities are even.

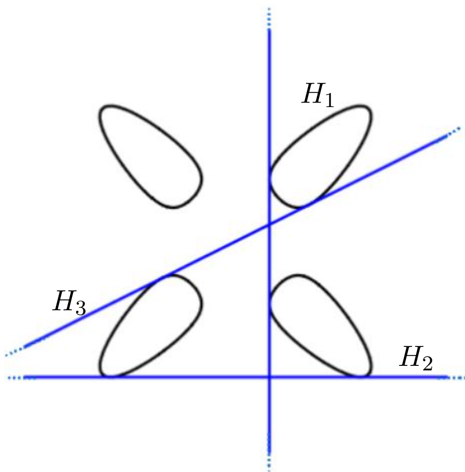


Figure 18: The real part of the Edge quartic (Figure 1. [PSV11]) C together with 3 theta hyperplanes H_1, H_2 and H_3 and theta characteristics given by $L_i = P_1^i + P_2^i$, where P_1^i and P_2^i are the intersection points of multiplicity 2 of H_i with C for $i = 1, 2, 3$.

Denote by $|L|$ the complete linear system of the line bundle L. Using the correspondence of line bundles and divisors we can think of $|L|$ as the set of all effective divisors D' linearly equivalent to D, where $\mathcal{O}_C(D) \cong L$. We have a bijective correspondence

$$\{(L, D) : L \text{ is a theta characteristic and } D \in |L|\} \leftrightarrow \{H \subset \mathbb{P}(V) : H \text{ is a theta hyperplane }\}.$$

Take $D \in |L|$. Since $L^{\otimes 2} \cong \omega_C$ and addition of divisors corresponds to taking tensor products of the corresponding line bundles, we get $2D \in |L^{\otimes 2}| = |\omega_C|$. Hence $2D$ is a canonical divisor and $H := \phi(2D)$ is a hyperplane intersecting $C \cong \phi(C)$ in $2D$, i.e. H satisfies $\phi^*(H) = 2D$. Conversely, given a theta hyperplane H the line bundle $O_C(N)$ is a theta characteristic, where N is the divisor in definition 4.3.

Definition 4.5 (Definition 13([Ser17])). A tuple (C, L, α), where C is a curve of genus g, L is an odd (or even) theta characteristic and $\alpha : L^{\otimes 2} \to \omega_C$ is an isomorphism, is called an odd (or even) *spin curve.*

Following Lee and Parker in [LP13] we will sometimes refer to the tuple (C, L) as a spin curve and omit mentioning the isomorphism α.

Theorem 4.6 (Theorem 10. in [Ser17]). On a genus g curve C there are $\binom{2^g}{2}$ odd and $\binom{2^g+1}{2}$ even theta characteristics.

Example 4.7. Going back to the case of genus 3 theorem 4.6 states that we should have $2^{2\cdot 3} = 64$ bitangents. However, there are only 28. Observe that $\binom{2^3}{2} = 28$. It seems as though only odd theta characteristics are visible as theta hyperplanes. Indeed, it can be shown that for a "general curve" with a theta characteristic L (a notion that can be made precise see [Ser17]) the dimension $h^0(C, L)$ is in minimal, i.e. either 0 or 1. This is what happened to the remaining 36 even theta characteristics of our curve of genus 3. They do not admit any non-trivial global sections.

4.2 Spin Hurwitz numbers

We introduce the counting problem: Fix

- a spin curve (D, N) of genus h with parity $p := h^0(N) \mod 2$.
- a collection of points $q^1, ..., q^k \in D$, the prescribed branch points,
- a positive integer d, the prescribed degree,
- a collection $m^1, ..., m^k$ of odd partitions of d, the prescribed ramification profiles.

Let $f : C \to D$ be a Hurwitz cover for the discrete data $(h, d, m^1, ...m^k)$ (in our notation we omit the genus of the cover curve since we allow our domain curve to be disconnected). The Euler characteristic $\chi(C)$ of C is fixed by the Riemann Hurwitz formula:

$$\chi(C) = d(2 - 2h) + \sum_{i=1}^{k}(l(m^i) - d).$$

Bare in mind that we allow ramification over the branch points, $q^1, ..., q^k$, only.
Next, define the line bundle

$$L_f := f^\star(N) \otimes \mathcal{O}(\underbrace{\sum_{i,j} \frac{1}{2}({m^i}_j - 1){x_j}^i}_{\frac{1}{2}\mathcal{R}_f}),$$

where $m^i = ({m^i}_1, ..., {m^i}_{l(m^i)})$ and $f({x_j}^i) = q^i$ for $i = 1, ..., k$ and $j = 1, ..., l(m^i)$. The divisor $\mathcal{R}_f$ is the *branch divisor* of f. Denote by $\tilde{L_f}$ and $\tilde{N}$ the associated divisors. By theorem 2.19 we have

$$2\tilde{L_f} = 2f^\star(\tilde{N}) + \underbrace{\sum_{i,j}({m^i}_j - 1){x_j}^i}_{=\mathcal{R}_f} \cong K_C.$$

Thus, L_f defines a theta characteristic on C.

Definition 4.8. Let $f : C \to D$ be a Hurwitz cover satisfying the discrete data $(h, d, m^1, ..., m^k)$ together with its associated theta characteristic L_f, as above. We call the pair (f, L_f) a *spin Hurwitz cover* of parity $p(f)$ given by $p(f) := h^0(C, L_f) \mod 2$.

Definition 4.9. Fix the the discrete data (4.2) as above. The *spin Hurwitz number* of genus h and parity p is defined as weighted sum of isomorphism classes of spin covers $f : C \to D$ with sign determined by the parity $p(f)$:

$$H^{(h,p)}_{m^1,\dots,m^k} = \sum_f \frac{(-1)^{p(f)}}{|Aut(f)|}.$$

Example 4.10. Set $(h, d, m^1, ...m^k) = (0, 3, (3)^3)$ and let $f : C \to \mathbb{P}^1$ be a map that contributes to $H^{(0,+)}_{(3)^3}$. Considering the ramification profile $(3) = (d)$ forces C to be connected. In this case $\chi(C) = 2 - 2g(C)$ and the Riemann Hurwitz formula yields $g(C) = 1$. The canonical class on a genus 1 curve is $K_C = 0$ (p.2. [EOP08]). For $i = 1, 2, 3$ let x_i be a ramifications point of f. Recall from example 4.2 that $\mathbb{P}^1$ has only one theta characteristic (unique up to equivalence) given by $-P$ for any point $P \in \mathbb{P}^1$. The choices $-x_i$ for $i = 1, 2, 3$ yield three equivalent divisors:

$$\tilde{L_f} \sim -2x_1 + x_2 + x_3 \sim x_1 - 2x_2 + x_3 \sim x_1 + x_2 - 2x_3,$$

where $\tilde{L_f}$ is the divisor associated to L_f. Note, $3\tilde{L_f} \sim 0$ implies $\tilde{L_f} = 0$ because $2\tilde{L_f} = 0$ (L_f is a theta characteristic). Thus L_f is the trivial bundle $C \times \mathbb{C}$ whose space of global section is a 1-dimensional vector space. This yields $L_f = \mathcal{O}(\tilde{L_f}) \cong \mathcal{O}$, i.e. $p(f) = 1$. Hence,

$$H^{(0,+)}_{(3)^3} = -\sum_f \frac{1}{|Aut(f)|} = -H_{1 \xrightarrow{3} 0}((3)^3) = -\frac{1}{3}.$$

Deformations of theta characteristics We have seen how useful it is to deform curves until they break up into simpler components. In order to make a sensible use of such a degeneration method for spin curves we need to clarify what we mean by "degeneration".
We introduce the notion of a *family of spin curves.* Similar to the intuitive notion of a family of curves, we want to view a family of spin curves as a collection of curves over a base together with theta characteristics on each fibre. This can be formalized in the following way:

Definition 4.11 ([Ser17], Definition 14). If $\pi : \mathcal{C} \to \Delta$ is a smooth family of curves and ω_π is the relative cotangent bundle of π, then a line bundle $\mathcal{L}$ on $\mathcal{C}$, such that $\mathcal{L}^{\otimes 2} \cong \omega_\pi$, is a *family of theta characteristics.* If we fix $\alpha : \mathcal{L}^{\otimes 2} \to \omega_\pi$, then the triplet $(\mathcal{C}, \mathcal{L}, \alpha)$ is called a *family of spin curves.*
One each fibre $C_t := \pi^{-1}(t)$, for $t \in \Delta$, the relative cotangent bundle ω_π, restricts to the cotangent bundle ω_{C_t} of the fibre and $\mathcal{L}$ to a theta characteristic thereon.

Keeping the idea of applying a degeneration method in mind to simplify the count of spin Hurwitz numbers, we need to know what happens to the parity of a theta characteristic under such a deformation. Luckily, Atiyah and Mumford establish the following important result.

Theorem 4.12 (Theorem 15 ([Ser17])). The parity of a spin curve of genus g is a deformation invariant.

However, degenerating spin curves, i.e. deforming spin curves into **singular** ones, is more involved. We need to extend the notion of a theta characteristic to nodal curves. What is the right way to do so?
Reframing the question. Just as one can consider the moduli space $\mathcal{M}_{g,n}$ of $n-$pointed curves of genus g, one can construct a moduli space $\mathcal{S}_{g,n}$ parametrizing spin curves (C, L), where C is a smooth $n-$pointed curves of genus g and L a theta characteristic of C. The space $\mathcal{S}_{g,n}$ comes together with a forgetful morphism $\phi : \mathcal{S}_{g,n} \to \mathcal{M}_{g,n}, (C, L) \mapsto C$. In this context we can reframe the question, how to define a theta characteristics on a nodal curve, as how to define "the right"

compactification of $\mathcal{S}_{g,n}$. Of course one would like such a compactification to be compatible with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, i.e. there should be a finite morphism $\bar{\phi}$ that makes the diagram below commute.

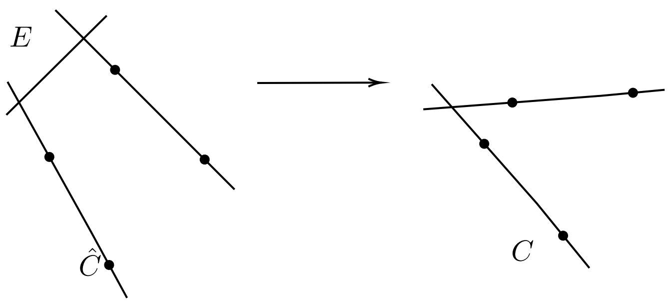
$$\begin{array}{ccccc} & S_{g,n} & \longrightarrow & \overline{S}_{g,n} & \\ \phi & \Big\downarrow & & \Big\downarrow & \overline{\phi} \\ & \mathcal{M}_{g,n} & \longrightarrow & \overline{\mathcal{M}}_{g,n} & \end{array}$$

This was first achieved by Cornalba in [Cor89]. The main difference to $\overline{\mathcal{M}}_{g,n}$ is that he had to enlarge the class of singular curves. Instead of considering at worst stable curves, he had to allow *quasistable* curves to enter the story.

Definition 4.13 ([CMP20]). Let (C,σ) be a $n-$marked curve, where C has at most nodes as singularities and $\sigma = \{p_1, ..., p_n\}$ is a set of distinct and smooth points of C. We say that a component $E \subset X$ is exceptional, if $E \cong \mathbb{P}^1$, $|E \cap \overline{C \setminus E}| = 2$ and $E \cap \sigma = \varnothing$. Then (C,σ) is

1. *stable*, if C does not contain any exceptional components.
2. *semistable*, if a smooth rational component meets other components in at at least 2 points.
3. *quasistable*, if C is semistable and two exceptional components are disjoint.

Indeed, quasistable curves are not so much worse. Given a quasistable curve $\hat{C}$ we obtain a stable one by contracting all the exceptional components (see figure 4.2)



Conversely, take a stable curve C and a set of nodes R of C. The blow up of C at R is a quasistable curve $\hat{C}$:
Let $\nu_R : C_R^\nu \to C$ be the partial normalization map, where ${\nu_r}^{-1}(r) = \{r_1, r_2\}$ for $r \in R$. Then $\hat{C}$ is defined as

$$\hat{C} := C_R^\nu \cup \bigcup_{r \in R} E_r \text{ with exceptional components } E_r.$$

Why consider quasistable curves in the first place since it is possible to compactify the moduli space of theta characteristics while insisting on working with stable curves alone? It is a trade-off: Insist on having at worst stable curves and you have to "allow for singularities for the sheaves". Insist on local freeness of the sheaves and you will have to accept dealing with slightly more singular curves. Choosing the second path, has one considerable advantage. We can view a degeneration of a theta characteristic as a line bundle on a curve (pg. 7 [Far12])[2]. This is exactly what our intuition would tell us. Let us describe the points of $\overline{\mathcal{S}}_{g,n}$, i.e. *stable spin curves*, in more detail.

[2] Using the correspondence between locally free sheaves and vector bundles

Definition 4.14 (see section 2 of [Cor89]). Let $\hat{C}$ be a quasistable curve. Consider a triple $(\hat{C}, L, \alpha)$, where L is a line bundle of degree $g-1$ and $\alpha : L^{\otimes 2} \to \omega_{\hat{C}}$ a homomorphism satisfying the following conditions:

1. $L_{|E} = \mathcal{O}(1)$ on each exceptional component of $\hat{C}$.
2. α induces an isomorphism $\tilde{\alpha} : L_{|C}^{\otimes 2} \to \omega_C$, where C denotes the curve $\hat{C}$ without exceptional components.

Then (L, α) is called a *spin structure* on $\hat{C}$ and $(\hat{C}, L, \alpha)$ a spin curve.

If $\hat{C}$ is smooth, L is just a theta characteristic. Hence, this definition generalizes the notion of theta characteristics to singular curves. Even better, it extends the classical result in theorem 4.6 to stable curves in the following sense. The map $\bar{\phi} : \overline{\mathcal{S}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ from above sending a stable spin curve $(\hat{C}, L, \alpha)$ to the stable model C of $\hat{C}$ (given by contracting all exceptional components) is of degree 2^{2g}. In other words, we can find 2^{2g} "theta characteristics" (meaning stable spin curves) lying over C. We will see that a stable spin curve over a stable curve C is given by a theta characteristic on a partial normalization of C. Indeed, condition 2 in definition 4.14 means that $L_{|C_R^\nu}$ is a theta characteristic where C_R^ν is the partial normalization of C at R and R a subset of the nodes of C. Now, there is a certain number of acceptable ways (see section 3 in [Cor89]) to blow up C to obtain a quasistable curve $\hat{C}$ that admits a spin structure. This yields the count.

Example 4.15 ([Cor89], Example (3.1))**.** Recall the degeneration formulas for Hurwitz numbers. They enabled the transition to tropical geometry. Given a ramified covering $f : C \to D$ the idea was to deform base and source curve simultaneously until the target curve becomes a singular curve $D_0 := D_1 \cup D_2$ consisting of two smooth components D_1 that meet in one node p. Formalizing this means considering one parameter families of curves whose special fibre is D_0. What is the "right" special fibre in the spin case? Let us look at $\bar{\phi}^{-1}(D_0)$ for possible candidates. Suppose $(\hat{D_0}, L) \in \bar{\phi}^{-1}(D_0)$. Cornalba argues that in order for a spin structure on D_0 to exist one must insert an exceptional component E at the node. To see why this is the case suppose there exists a line bundle L on D_0 such that there is an isomorphism $\alpha : L^{\otimes 2} \to \omega_{D_0}$, where ω_{D_0} is the dualizing sheaf of D_0. Then for $i = 1, 2$ restricting α to the smooth component D_i would have to be an isomorphism as well. The restriction of ω_{D_0} to D_i has degree $2g(D_i) - 2 + 1$ since the associated divisor is $K_{D_i} + p$ (see pg.30 [Cav16]). But $2g(D_i) - 1$ is odd, a contradiction. Hence, $\hat{D} = D_1 \cup E \cup D_2$ such that for $i = 1, 2$ the component D_i meets E at node p_i. A spin structure on $\hat{D}$ is a tuple (L, α) as in definition 4.14. Write $L = (L_{|D_1}, L_{|E}, L_{|D_2})$. By condition 1 we know $L_{|E} = \mathcal{O}(1)$ and condition 2 implies that $L_{|D_1}$ and $L_{|D_2}$ are theta characteristics on D_1 and D_2 respectively.

In retrospect we see what Cornalba means by "acceptable" ways to blow up C. To be able to take a root of the dualizing sheaf on each irreducible component we need the number of nodes that are not blown up to be even (on each component). Thus, determining quasistable curves in $\phi^{-1}(C)$ that carry a spin structure gets quite easy: Choose an even number of nodes on each irreducible component of C and blow up the remaining ones. The number of ways to do this is best computed using the perspective of dual graphs (see next paragraph).

Definition 4.16 (see pg. 19 in [CMP20]). A one-parameter family of curves is a family of (pointed) nodal curves $\pi : \mathcal{C} \to \Delta$ over a regular, connected curve Δ with a marked point, $t_0 \in \Delta$. We denote by C_0 the fibre over t_0 and we will always assume that every fibre C_t for $t \in \Delta \setminus \{t_0\}$ has the same dual graph. We shall call C_t the "generic" fibre.

A *one-parameter family of stable spin curves* is a triplet $(\mathcal{C}, \mathcal{L}, \alpha)$, where $\mathcal{L}$ is a line bundle and $\alpha : \mathcal{L}^{\otimes 2} \to \omega_\pi$ a homomorphism such that for every $t \in \Delta$ the fibre $(C_t, \mathcal{L}_{|C_t}, \alpha_{|C_t})$ is a stable spin curve.

4.3 Dual spin graphs

Just like dual graphs encode the combinatorial data of a stable curve, *dual spin graphs* encode the combinatorial data of a stable spin curve. This discretization is best viewed from the perspective of its stable model. A stable spin curve $\widehat{C_R}$ over a stable curve C is given by ([CMP20], section 3.1.):

1. a partial normalization $\widehat{C^\nu}_R$ of C at a set of nodes R of C.

2. a theta characteristic $\widehat{L}_R$ on $\widehat{C^\nu}_R$.

The existence of $\widehat{L_R}$ forces each irreducible component Z of $\widehat{C^\nu}_R$ to have only an even number of nodes. Otherwise the restriction of the dualizing sheaf $\omega_{\widehat{C^\nu}_R}$ on Z would have odd degree since

$$deg(\omega_Z) = 2 - 2g(Z) + n, \text{ where } n = |R|.$$

Let Γ_C be the dual graph of C and denote by N the set of nodes. We provide an (informal) dictionary between the algebraic operations above and corresponding operations on the dual graph (see figure 4.3 for a visualization):

1. *Blow up* at a node $r \in N \leftrightarrow$ *Adding* a genus 0 vertex on the edge dual to r.

2. *Partial normalization* at $R \subset N \leftrightarrow$ *Erasing* edges dual to R.

We denote the graph obtained by removing edges dual to R by $\Gamma_C - R$.

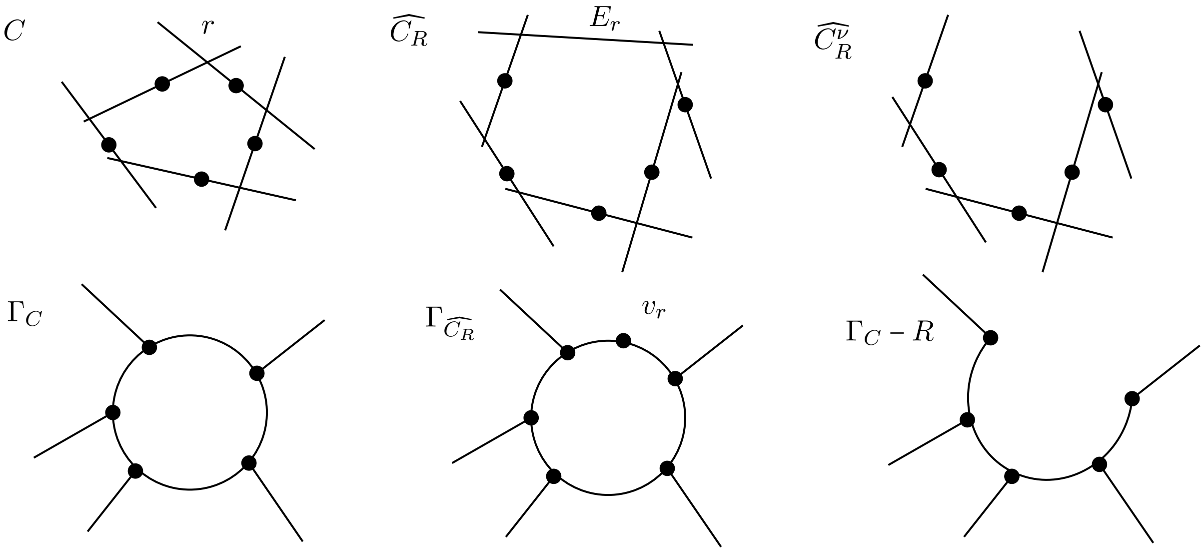


Figure 19: Illustration of the algebraic operations and their graph-theoretic counterpart where $R := \{r\}$

With this in mind we can rephrase the discussion around example 4.15 in the language of graphs: The preimage $\phi^{-1}(C)$ consists of quasistable curves $\widehat{C_R}$ such that the dual graph $\Gamma_C - R$ of $\widehat{C^\nu}_R$ is cyclic, i.e. $\Gamma_C - R$ is an element of the cycle space of $\Gamma_C - R$ denoted by $\mathcal{C}_{\Gamma_C - R}$ ([CMP20], section 1.4). Indeed, requiring the number of nodes that lie on each connected component of $\widehat{C^\nu}_R$ to be even implies that each vertex of $\Gamma_C - R$ has a even number of adjacent edges. This is the case if and only if $\Gamma_C - R$ is cyclic. As promised, counting the number of stable spin curves in $\phi^{-1}(C)$ expressed in graph-theoretic language:

Proposition 4.17 (Proposition 3.5. [Far12]). Let $C \in \overline{\mathcal{M}_g}$ and $b := b_1(\Gamma_C)$ be the Betti number of the dual graph. Then the number of components of the zero-dimensional scheme $\phi^{-1}(C)$ is equal to

$$2^{2g(\Gamma_C)-2b} \sum_P 2^{b_1(P)}$$

A component corresponding to a cycle P of Γ_C appears with multiplicity $2^{b-b_1(P)}$.

Thus, we get a total of $2^{2g(\Gamma_C)}$ spin structures counted with multiplicity. Since connected components of $\widehat{C^{\nu}_R}$ are in natural bijection with the vertices of the graph G/P obtained after contracting the edges of the cycle $P := E(\Gamma_C) \smallsetminus R$ we write (following [CMP20]) the decomposition into connected components of $\widehat{C^{\nu}_R}$ as $\widehat{C^{\nu}_R} = \bigcup_{v \in V(G/P)} Z_v$.

Definition 4.18 (Definition 3.2.1. [CMP20]). Let $(\widehat{C_R}, \widehat{L_R})$ be a stable n-pointed spin curve, where C the stable model of $\widehat{C_R}$ and $\widehat{C_R}$ the quasistable curve associated to a set of nodes R of C. Write $\widehat{C_R} = \widehat{C^{\nu}_R} \cup \bigcup_{p \in R} E_r$ and $\widehat{C^{\nu}_R} = \bigcup_{v \in V(G/P)} Z_v$ the decomposition into connected components of $\widehat{C^{\nu}_R}$. The *dual spin graph* of $(\widehat{C_R}, \widehat{L_R})$ is the spin graph (Γ_C, P, s) defined as follows:

- Γ_C is the dual graph of C.
- $P = E(\Gamma_C) \smallsetminus R_C$, where $R_C \subset E(\Gamma_C)$ corresponds to the set of nodes R.
- $s(v)$ is the parity of $h^0(Z_v, \widehat{L_R}_{|Z_v})$ for all $v \in V(G/P)$.

Dual spin graphs play a similar role for tropical spin Hurwitz covers as dual graphs do for tropical Hurwitz covers (see remark 5.6).

4.4 Spin degeneration formulas

Hurwitz numbers and spin Hurwitz numbers are not so very different. An interesting feature of Hurwitz numbers is their recursive structure. Do spin Hurwitz numbers exhibit a similar behaviour? The theorem below answers this question.

Theorem 4.19 (Theorem 1.1. [LP13]). For $d, m^1, ..., m^k$ as in 4.2

1. If $h = h_1 + h_2$ and $p = (p_1 + p_2) \mod 2$ then for $0 \leqslant k_0 \leqslant k$

$$H^{(h,p)}_{m^1,...,m^k} = \sum_m |m| m! H^{(h_1,p_1)}_{m^1,...,m^{k_0},m} H^{(h_2,p_1)}_{m,m^{k_0+1},...,m^k}$$

2. If $h \geqslant 2$ or if $(h,p) = (1,+)$ then

$$H^{(h,+)}_{m^1,...,m^k} = \sum_m |m| m! H^{(h-1,+)}_{m,m,m^1,...,m^k}$$

where the sums are over all odd partitions m of d,

We have seen in theorem 4.12 that the parity of a theta characteristic is a deformation invariant, i.e. it is constant in a family of spin curves as in definition 4.11. Thus, it is at least worth a try to transfer the degeneration method to the spin case. This is exactly what Lee and Parker do. They consider holomorphic maps between families of *stable* spin curves as in definition 4.16 and express both their parity and number in terms of the parity and number of maps into the irreducible components of the special fibre of the target curve.
Below you find a sketch of the proof:

- Step 0: Express the spin Hurwitz number (definition 4.9) in terms of relative Gromov Witten moduli spaces.
- Step 1: Describe the relative moduli space $\mathcal{M}_{m,0}$ of maps f_0 into a nodal curve D_0.
- Step 2:: Identify maps in $\mathcal{M}_{m,0}$ as limit maps as the target curve D degenerates to D_0.
- Step 3: Make the idea of simultaneously degenerating cover and base curve of a spin Hurwitz cover concrete in terms of families of curves.
- Step 4: Transform theses into families of stable spin curves to keep track of their parity.
- Step 5: Pause to make some "Parity statements".
- Step 6: Prove theorem 4.19.

For us this degeneration is a bridge towards tropical geometry. What we need to understand is the structure of the "end product" of Lees and Parkers degeneration procedure. Hence, we are going to add mpre detail to step 0, 1, 2 and 6. The concrete construction of the algebraic families of stable spin curves, i.e. step 3,4 and 5, is of lesser interest to us. We direct the curious reader to [LP13] for more details.

Step 0: *Express the spin Hurwitz number in terms of relative Gromov Witten moduli spaces.* Working with relative Gromov Witten spaces is similar (not analogous) to working with y_0 labelled maps instead of Hurwitz covers (as is done in [RC]) where we label the preimages of a point. Isomorphisms of y_0 labelled maps have to be isomorphisms of Hurwitz covers and preserve the labelling. In our case Lee and Parker consider *V-regular maps.* These are ramified coverings together with a marking of all ramification points. The advantage is that by imposing "the right" conditions the automorphism group of a V-regular map can be made trivial (see lemma 4.23). This simplifies future computations a lot.

Definition 4.20 ([LP13]). Let D be a smooth curve of genus h and let $V = \{q^1, ..., q^k\}$ be a fixed set of points on D. Given partitions $m^1, ..., m^k$ of d, a degree d holomorphic map $f : C \to D$ from a (possibly disconnected) curve C is called V-regular with contact partitions $m^1, ..., m^k$ if for each $i = 1, ..., k$ $f^{-1}(q^i)$ consists of $l(m^i)$ points q^i_j so that the ramification index of f at q^i_j is m^i_j. We call the number j the label of q^i_j.

Remark 4.21. In the setting of definition 4.20 we do not allow any (i.e. random) labelling of the preimages of q^i, but one that matches the prescribed ramification profile m^i. Consider $m^i = (1, 2, 2)$ and let x, y, z be the ramification points of f with order $1, 2, 2$. An acceptable labelling is $q^i_1 := x, q^i_3 := y, q^i_2 := z$, whereas $q^i_2 := x, q^i_1 := y, q^i_3 := z$ is not allowed.

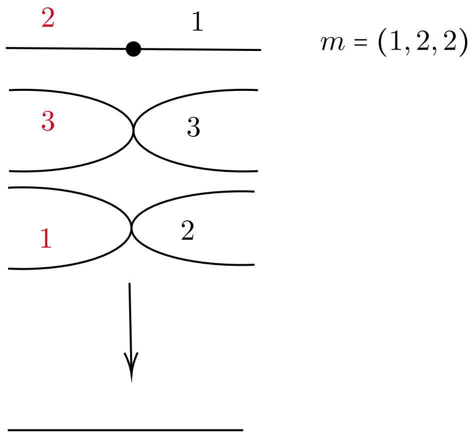


Figure 20: The labelling on the right is acceptable, the one on the left is not.

Definition 4.22 ([Lee13]). Two V-regular maps $(f, C; \{q_j^i\})$ and $(\tilde{f}, \tilde{C}; \{\tilde{q_j^i}\})$ are isomorphic if there is a biholomorphism $\sigma : C \to \tilde{C}$ with $\tilde{f} \circ \sigma = f$ and $\sigma(q_j^i) = \tilde{q_j^i}$ for all i, j.
The automorphism group $Aut(f, V)$ of a V-regular map $(f, C; \{q_j^i\})$ consists of automorphisms $\sigma \in Aut(f)$ with $\sigma(q_j^i) = q_j^i$ for all i, j.
The *relative Gromov Witten moduli space*

$$\mathcal{M}^V_{\chi, m^1, \ldots, m^k}(D, d), \tag{2}$$

consists of isomorphism classes of V-regular maps $(f, C; \{q_j^i\})$ as in definition 4.20 with Euler characteristic $\chi := \chi(C)$.
Relating V-regular maps to spin Hurwitz covers allows us to express spin Hurwitz numbers in terms of V-regular maps.

Lemma 4.23 ([Lee13], Lemma 1.1.).
Fix a genus h and partitions $m^1, \ldots, m^k$ of an integer d.

1. If $m^1, \ldots, m^k$ are all odd partitions, then

$$H^{(h,p)}_{m^1, \ldots, m^k} = \frac{1}{\prod_{i=1}^k m^i!} \sum \frac{(-1)^{p(f)}}{|Aut(f, V)|},$$

   where the sum goes over all $(f, C; \{q_j^i\}) \in \mathcal{M}^V_{\chi, m^1, \ldots, m^k}(D, d)$ (defined in (2).

2. If $m^i = (1^d)$, then $|Aut(f, V)| = 1$.

Proof. Fix the discrete data $h, p, m^1, \ldots, m^k$. Let A be the set of isomorphism classes of V-regular maps $(f, C; \{q_j^i\})$ and B be the set of isomorphism classes of Hurwitz covers f. Consider the map

$$\pi : A \to B, (f, C; \{q_j^i\}) \mapsto f,$$

that simply forgets the contact marked points $\{q_j^i\}$ of f. We have $|\pi^{-1}(f)| = \frac{|Aut(f,V)| \prod_{i=1}^k m^i!}{|Aut(f)|}$ (see [Lee13]) and thus

$$H^{(h,p)}_{m^1, \ldots, m^k} = \sum_f \frac{(-1)^{p(f)}}{|Aut(f)|} = \sum_f \frac{(-1)^{p(f)}}{|Aut(f, V)| \prod_{i=1}^k m^i!} |\pi^{-1}(f)| = \frac{1}{\prod_{i=1}^k m^i!} \sum_{(f, C; \{q_j^i\})} \frac{(-1)^{p(f)}}{|Aut(f, V)|}.$$

For the second part, suppose $m^i = (1^d)$ for some $1 \leqslant i \leqslant d$, then q^i is not a branch point of f. Let B be set of branch points. We know that $f : C \smallsetminus f^{-1}(B) \to D \smallsetminus B$ is a covering map and $\sigma C \smallsetminus f^{-1}(B) \to C \smallsetminus f^{-1}(B)$ satisfies $f \circ \sigma = f$. Hence, σ is a lift of f that fixes $q_1^i, ..., q_d^i$ and by the unique lifting property of coverings spaces (see Exercise 79. in [RC]) has to be the identity on $C \smallsetminus f^{-1}(B)$. Moreover, the set $f^{-1}(B)$ is finite forcing σ to be the identity on C. □

Step 1: *Description of* $\mathcal{M}_{m,0}$. Motivated by example 4.15 we analyse the structure of maps into a quasi-stable curve D_0. Fix discrete data $d, h, \chi, m^1, ..., m^k$ as definition 4.9 and consider $D_0 = D_1 \cup E \cup D_2$ of genus h such that for $i = 1, 2$:

- D_0 contains two nodes p^1 and p^2, such that D_i meets E at node p_i.
- D_i is smooth with $g(D_i) = h_i$ and $h_1 + h_2 = h$.
- E is an exceptional component.

Let $m^{k+1} = m^{k+2} = m^{k+3} = (1^d)$, $m = (m_1, ..., m_l)$ be a partition of d and $1 \leqslant k_0 \leqslant k$. Consider the product space

$$\mathcal{P}_m = \mathcal{M}^{V_1}_{\chi_1, m^{k+1}, m^1, ..., m^{k_0}, m}(D_1, d) \times \mathcal{M}^{V_e}_{\chi_e, m, m^{k+2}, m}(E, d) \times \mathcal{M}^{V_2}_{\chi_2, m, m^{k+3}, m^{k_0+1} ..., m^k}(D_2, d)$$
$$V_1 = \{q^{k+1}, q^1, ..., q^{k_0}, p^1\},\ V_e = \{p^1, q^{k+2}, p^2\},\ V_2 = \{p^2, q^{k_0+1}, ..., q^k, q^{k+3}\}$$
$$\chi_1 + \chi_e + \chi_2 - 4l(m) = \chi$$

Remark 4.24. For later use we call the first, second and third factors of $\mathcal{P}_m$: $\mathcal{M}^1_m, \mathcal{M}^e_m, \mathcal{M}^2_m$.

It follows from Lemma 2.1 and from Lemma 2.2 in [Lee13] that a map $(f_1, f_e, f_2) \in \mathcal{P}_m$ has the following structure:

- f_1, f_2 are ramified at contact marked points q^i and nodal points only.
- the domain of f_e is a disjoint union of $l(m)$ rational curves E_i, i.e. $f_e : \bigcup_{i=1}^{l(m)} E_i \to E$, where $f_{|E_i}$ has degree m_i.

Figure 21 shows a cartoon of such a map.

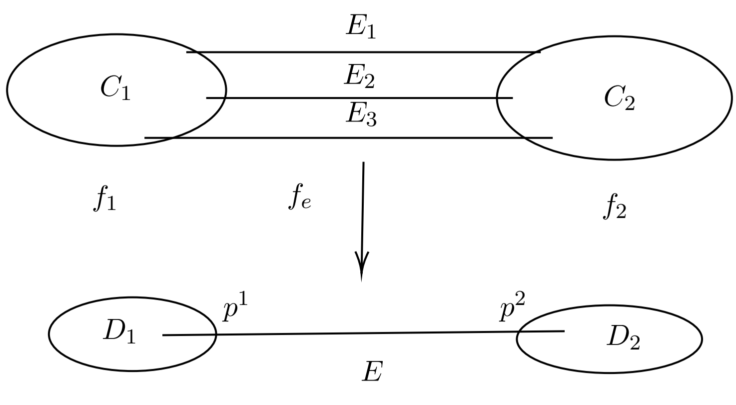


Figure 21: Sketch of a map into D_0.

Remark 4.25. Marking three additional points $q^{k+1}, q^{k+2}, q^{k+3}$ with prescribed ramification behaviour $m^{k+1} = m^{k+2} = m^{k+3} = (1^d)$ ensures that the automorphism group of each component map is trivial (Lemma 4.23). In the following we will always do so. Hence, automorphism group will not appear in future computations.

By Lemma 2.2 [Lee13]) we have that

$$|\mathcal{M}^{V_e}_{\chi_e, m^1,\dots,m^k}(E,d)| = \frac{d!m!}{|m|}.$$

Note, this number is in some sense artificial since forgetting contact marked points yields exactly one map.
Matching ramification orders over the nodes ensure that we can glue the domains of f_1 and f_e and of f_2 and f_e, respectively, together to obtain a map $f : C_0 \to D_0$. Denote by $\mathcal{M}_{m,0}$ the space of such glued maps $f = (f_1, f_e, f_2)$ and discard the information about labelling above the nodes. Thus,

$$\mathcal{P}_m \to \mathcal{M}_{m,0}, (f_1, f_e, f_2) \mapsto f$$

is a cover of degree $m!^2$ (Lemma 2.3 [Lee13]). The factor $m!^2$ appears because according to remark 4.21 we have $m!$ possibilities to label the ramification points above each node.

Step 2:*Identify $\mathcal{M}_{m,0}$ as space of limit maps.* Why did we look at these maps in the first place? They occur as limit maps of maps as the target curve degenerates to D_0. Fix D_0 as in step 1 and construct a 1-parameter family of curves $\mathcal{D} \to \Delta$ (with parameter r) as in section 4 of [Lee13] together with sections Q^i that select $k+3$ special points on each fibre such that $Q^i(0) = q^i$ for $i = 1, \dots, k+3$. The generic fibre D_r for $r \neq 0$ is smooth and of genus h and the special fibre is given by D_0. For $r \neq 0$ consider the moduli space

$$\mathcal{M}_r := \mathcal{M}^{V_r}_{\chi, m^1,\dots,m^{k+3}}(D_r, d) \text{ with } V_r = \{Q^1(r), \dots, Q^{k+3}(r)\}.$$

Of course we can compute the spin Hurwitz number $H^{(h,p)}_{m^1,\dots,m^k}$ by counting maps in $\mathcal{M}_r$ for any $r \neq 0$ (lemma 4.23). What happens if we let r approach 0? To be concrete: We want to relate the spaces $\mathcal{M}_r$ as $r \to 0$ to $\mathcal{M}_{m,0}$. In Lemma 3.1. ([Lee13]) Lee observes that

$$lim_{r\to 0} \mathcal{M}_r \subset \bigcup_m \mathcal{M}_{m,0},$$

where the union is over all partitions m of d with $\mathcal{P}_m \neq \emptyset$ and $lim_{r\to 0} \mathcal{M}_r$ denotes the set of limits of sequences of maps in $\mathcal{M}_r$ as $r \to 0$(for a justification of the existence of such limit maps see [Lee13]). On the other hand, we can reconstruct all maps in $\mathcal{M}_r$ by choosing $f \in \mathcal{M}_{m,0}$ and smoothing the domain curve C_0 ([Lee13]). Recall, for $i = 1, 2$ the preimage of $p^i \in D_i$ consists of $l(m)$ points x^i_j. Each one is a node of C_0 and can be smoothed in m_j ways ([Lee13]). This yields $\underbrace{|m|}_{\text{smoothing over } p^1} \cdot \underbrace{|m|}_{\text{smoothing over } p^2}$ maps in $\mathcal{M}_r$. In total we have

$$\mathcal{M}_r = \bigcup_m \bigcup_{f \in \mathcal{M}_{m,0}} \mathcal{Z}_{m,f,r} \text{ for } r \neq 0,$$

where $\mathcal{Z}_{m,f,r}$ (informally) denotes the set of maps in $\mathcal{M}_r$ having $f \in \mathcal{M}_{m,0}$ as limit and $|\mathcal{Z}_{m,f,r}| = |m|^2$. For a more precise definition of $\mathcal{Z}_{m,f,r}$ see [Lee13].

Step 3: *Simultaneously degenerating cover and base curve.* Fix $f : C_0 \to D_0 \in \mathcal{M}_{m,0}$. We are ready to make the statement "f is the limit of maps into D_r" concrete. This means that we need, in addition to the deformation of D_0 given by $\mathcal{D}$, a deformation of C_0. This construction is done in Theorem 5.1. [LP13]. For each $\zeta \in Q_m := \{\zeta = (\zeta^1_1, \zeta^2_1 \dots, \zeta^1_{l(m)}, \zeta^2_{l(m)}) \text{ with } {\zeta^1_j}^{m_j} = 1 \text{ and } {\zeta^2_j}^{m_j} = 1\}$ there is a 1-parameter family of quasistable curves $\mathcal{C}_\zeta \to \Delta$ (with parameter s) together with a holomorphic map $\mathcal{F}_\zeta : \mathcal{C}_\zeta \to \mathcal{D}$, such that the generic fibre $\mathcal{C}_{\zeta,s}$ for $s \neq 0$ is smooth and the map $f_{\zeta,s} = \mathcal{F}_{\zeta|\mathcal{C}_{\zeta,s}}$ is contained in $\mathcal{M}_r$ (where $s^{|m|} = r$). The special fibre is given by C_0 in the case m is an odd partition of d and $f_{\zeta,0} = f$. If m is even the special fibre is a little bit more complicated (see [LP13] if you are curious). However, we do not need to worry about that since we will see later on when introducing spin structures on both families that the contributions of these maps cancel out.

Note, the families $\mathcal{C}_\zeta$ are indexed by vectors $\zeta \in Q_m$. This accounts for the different possibilities of smoothing the nodes of C_0 and provides us with a way to get all maps in the set $\mathcal{Z}_{m,f,r}$. By Lemma 5.2. ([LP13]) we have for $s \neq 0$ and $s^{|m|} = r$ that

$$\mathcal{Z}_{m,f,r} = \bigcup_{\zeta \in Q_m} \{f_{\zeta,s}\}.$$

Step 4: *Keep track of the parity.* Now is the time for theta characteristics to enter the story. By example 4.15 we know that a 'spin structure on D_0 is given by a collection of theta characteristics on each of its smooth components. Thus, choose parities p_1 and p_2 with $p = p_1 + p_2 \mod 2$. Lee then endows $\mathcal{D} \to \Delta$ with a spin structure $(\mathcal{N}, \Phi)$ such that:

- for $r \neq 0$ $\mathcal{N}$ restricts to a theta characteristic on D_r with a parity p.
- for $i = 1, 2$ the restriction of $\mathcal{N}$ to the smooth component D_i of the special fibre $\mathcal{N}_{|D_i}$ is a theta characteristic of parity p_i.

The tuple $(\mathcal{D}, \mathcal{N}, \Phi)$ is a family of stable spin curves as in definition 4.16. The same can be done for the family $\mathcal{C}_\zeta$. We restrict to the case m is odd.

Theorem 4.26 (Theorem 5.1. [LP13]). For $f = (f_1, f_e, f_2) \in \mathcal{M}_{m,0}$ let $\mathcal{C}_\zeta \to \Delta$ and $\mathcal{F}_\zeta : \mathcal{C}_\zeta \to \mathcal{D}$ be as above. Then, there exists a line bundle $\mathcal{L}_\zeta$ over $\mathcal{C}_\zeta$ satisfying:

1. the restriction $\mathcal{L}_{\zeta|\mathcal{C}_{\zeta,s}}$ is a theta characteristic on $\mathcal{C}_{\zeta,s}$ and the restriction map $f_{\zeta,s} = \mathcal{F}_{\zeta|\mathcal{C}_{\zeta,s}}$ has the associated parity $p(f_{\zeta,s}) = p(\mathcal{L}_{\zeta|\mathcal{C}_{\zeta,s}})$ since $\mathcal{L}_{\zeta|\mathcal{C}_{\zeta,s}} = L_{f_{\zeta,s}}$.
2. the restriction to the special fibre is a collection of theta characteristics $\mathcal{L}_{\zeta|C_1} = L_{f_1}$ and $\mathcal{L}_{\zeta|C_2} = L_{f_2}$.

Step 5: *Parity statements.* A life saving fact about parities is the following.

Theorem 4.27 ([LP13], Theorem 4.2). Let $f = (f_1, f_e, f_2) \in \mathcal{M}_{m,0}$ and $r \neq 0$.

1. If m is odd, then $p(f_r) = p(f_1) + p(f_2) \mod 2$ for all $f_r \in \mathcal{Z}_{m,f,r}$.
2. If m is even, then $\sum_{f_r \in \mathcal{Z}_{m,f,r}} (-1)^{p(f)} = 0$.

This is good news. For even m maps in $\mathcal{M}_{m,0}$ have component maps f_1 and f_2 that do not have a well defined parity (see definition 4.9) since their branch divisor is not divisible by 2.

Step 6:*Proof (theorem 4.19).* We only prove part 1. Part 2 makes use of a different degeneration. We refer to [LP13] for more details.

Luckily, spin Hurwitz numbers depend only on discrete data, i.e. the genus of D, the parity of the theta characteristic N and the ramification profile over the q^i, but neither on the configuration of the points nor on the curve D. This means we can replace $\mathcal{M}^V_{\chi,m^1,\dots,m^k}(D,d)$ (step 0) by any one of the spaces $\mathcal{M}_r$ for $r \neq 0$. By lemma 4.23 we have

$$H^{(h,p)}_{m^1,\dots,m^k} = H^{(h,p)}_{m^1,\dots,m^k,(1^d),(1^d),(1^d)} = \frac{1}{d!^3 \prod_{i=1}^k m^i!} \sum_{f_r \in \mathcal{M}_r} (-1)^{p(f_r)} \tag{3}$$

$$= \frac{1}{d!^3 \prod_{i=1}^k m^i!} \sum_m \sum_{f \in \mathcal{M}_{m,0}} \sum_{f_r \in \mathcal{Z}_{m,f,r}} (-1)^{p(f_r)} \tag{4}$$

where we use $(1^d)! = d!$ for the first equality and $\mathcal{M}_r = \bigcup_m \bigcup_{f \in \mathcal{M}_{m,0}} \mathcal{Z}_{m,f,r}$ for the second. Thanks to theorem 4.27 the last sum vanishes whenever m is an even partition of d. Using $|\mathcal{Z}_{m,f,r}| = |m|^2$ we compute the spin Hurwitz number by counting degenerate coverings instead:

$$H^{(h,p)}_{m^1,\dots,m^k} = \frac{1}{d!^3 \prod_{i=1}^k m^i!} \sum_{m:odd} |m|^2 \sum_{f=(f_2,f_e,f_1) \in \mathcal{M}_{m,0}} (-1)^{p(f_1)+p(f_2)}.$$

Recall

$$\mathcal{P}_m \to \mathcal{M}_{m,0}, (f_1, f_e, f_2) \mapsto f$$

is a cover of degree $m!^2$. We can sum over disjoint tuples in $\mathcal{P}_m$ as well.

$$\begin{aligned}
\sum_{f=(f_2,f_e,f_1)\in\mathcal{M}_{m,0}} (-1)^{p(f_1)+p(f_2)} &= \frac{1}{m!^2} \sum_{f=(f_2,f_e,f_1)\in\mathcal{P}_m} (-1)^{p(f_1)+p(f_2)} \\
&= \frac{1}{m!^2} \sum_{f_e \in \mathcal{M}^e_m} \Big(\sum_{f_1 \in \mathcal{M}^1_m} (-1)^{p(f_1)} \Big) \Big(\sum_{f_2 \in \mathcal{M}^2_m} (-1)^{p(f_2)} \Big) \\
&\underbrace{=}_{|\mathcal{M}^e_m| = \frac{d!m!}{|m|}} \frac{d!}{m!|m|} \Big(\sum_{f_1 \in \mathcal{M}^1_m} (-1)^{p(f_1)} \Big) \Big(\sum_{f_2 \in \mathcal{M}^2_m} (-1)^{p(f_2)} \Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}^1{}_m :=& \mathcal{M}^{V_1}_{\chi_1,(1^d),m^1,\dots,m^{k_0},m}(D_1,d),\ \mathcal{M}^e_m := \mathcal{M}^{V_e}_{\chi_e,m,(1^d),m}(E,d), \\
\mathcal{M}^2_m :=& \mathcal{M}^{V_2}_{\chi_2,m,(1^d),m^{k_0+1}\dots,m^k}(D_2,d).
\end{aligned}$$

To write $\sum_{f_1 \in \mathcal{M}^1_m} (-1)^{p(f_1)}$ as contribution to the spin Hurwitz number $H^{(h_1,p_1)}_{m^{k+1},m^1,\dots,m^{k_0},m}$ we need to correct by a factor of $d!m! \prod_{i=1}^{k_0} m^i!$, analogously for $\sum_{f_2 \in \mathcal{M}^2_m} (-1)^{p(f_2)}$.

$$\sum_{f=(f_2,f_e,f_1)\in\mathcal{M}_{m,0}} (-1)^{p(f_1)+p(f_2)} = \frac{(d!)^3 m! \prod_{i=1}^k m^i!}{|m|} H^{(h_1,p_1)}_{m^1,\dots,m^{k_0},m} H^{(h_2,p_1)}_{m,m^{k_0+1},\dots,m^k}.$$

Substitute back in equation 3 and the result follows.
Lee and Parker use theorem 4.19 to compute the degree $d = 3$ and 4 spin Hurwitz numbers for every genus.

Theorem 4.28 (Proposition 7.1 [Lee13]). The degree 3 spin Hurwitz numbers are

$$H^{(h,\pm)}_{(3)^k} = 3^{2h-2}((-1)^k 2^{k+h-1} \pm 1).$$

Theorem 4.29 (Theorem 11.1. [LP13]). The degree 4 spin Hurwitz numbers are

$$H^{(h,\pm)}_{(31)^k} = (3!)^{2h-2} 2^k (\pm 2^{k+h-1} + (-1)^k).$$

We want to investigate these from the tropical point of view.

5 Tropical spin Hurwitz numbers

This section develops a tropical counterpart of spin Hurwitz numbers. Similarly to tropical Hurwitz numbers, *tropical spin Hurwitz numbers* are a weighted count of *tropical spin covers*, i.e. tropical covers that, in addition, carry a certain parity.
In view of our previous approach to tropical geometry it is natural to iterate the degeneration procedure from section 4.4 to get tropical covers as maps between the dual graphs of source and target curve. We need to endow these with an additional structure that encodes the parity of the "dual" covering maps. For degree 3 and 4 this can be done in a unique way, which implies that we can keep the usual definition of multiplicity given in (3.13). For higher degrees, however, we encounter additional problems, that have to do with the fact, that the process of tropicalization "looses" a lot of information.

The uniqueness problem. Recall the connection between tropical curves and algebraic degeneration: A tropical Hurwitz cover $\pi : \Gamma \to B$ can be interpreted as the natural graph theoretic map between the dual graphs of source and target curve of an algebraic cover whose target curve is a maximal stable degeneration. The multiplicity of π is designed to reflect the weighted count of Hurwitz maps, i.e.

$$mult(\pi) = \sum_f \frac{1}{|Aut(f)|},$$

where the sum runs over isomorphism classes of maps that contribute to $H_{g \xrightarrow{d} h}(m^1, ..., m^k)$ and "degenerate" to π. The contribution of these maps to the spin Hurwitz number $H^{(h,p)}_{m^1,...,m^k}$, however, is

$$\sum_f \frac{(-1)^{p(f)}}{|Aut(f)|}.$$

This is a problem since the parity $p(f)$ is in general not the same for all f, even if after degeneration the domain curves have the same associated dual graphs. Therefore, we cannot always assign a unique parity to π while keeping the original multiplicity. A tropical cover π just encompasses the data of "too many" covering maps in the algebraic world.
But still, do we have cases where the usual multiplicity is good enough? And, if yes, how do we track them down? Viewing tropical covers as shadows of degenerate Hurwitz covers allows us to reduce the uniqueness problem to smaller entities: the analysis of spin Hurwitz covers that contribute to numbers of the form $H^{(h,p)}_{m^1,m^2,m^3}$ and their tropical counterpart under the duality described in definition 3.23. We explain the case $(h,p) = (0,0) =: (0,+)$ in detail before we turn to a more general setting in subsection 5.1.1 and 5.1.2. Let $\pi : \Gamma \to \mathbb{TP}^1$ be a tropical Hurwitz cover as in definition 3.20 of degree d with odd ramification only. Let v' be a branch point of π and for $i = 1, ..., t$ let v_i be a ramification point of Γ with $\pi(v_i) = v'$ whose ramification profile is given by the collection of weights of ends/edges mapping to the end/edges adjacent to v' (e.g. as in figure 22 b)).
Locally, π is dual to an algebraic Hurwitz cover $f : C \to \mathbb{P}^1$ (e.g. as in figure 22 a)) with

- $C = \bigcup_{i=1}^{t} C_i$ is the disjoint union of smooth curves C_i with $g(C_i) = g(v_i)$.
- f is a collection of Hurwitz covers $f = (f_1, ..., f_t)$, such that f_i counts towards the local Hurwitz number of Γ at v_i denoted by H_{v_i}.

- parity $p(f)$ given by $\sum_{i=1}^{t} p(f_i) \mod 2$ where $p(f_i)$ is as in definition 4.8 (specifying the theta characteristic on the target curve is not necessary since $\mathbb{P}^1$ admits only one).

In other words f counts, as a Hurwitz cover, towards a disconnected Hurwitz number of the form $H_{\sum_{i=1}^{t} g(v_i) \xrightarrow{d} 0}(m^1, m^2, m^3)$ and, as spin Hurwitz cover, towards $H^{(0,+)}_{m^1,m^2,m^3}$, where for $i = 1,2,3$ the partition m^i is odd.
Globally, a degenerate Hurwitz cover f_0 dual to π is a tuple of local Hurwitz covers f (with appropriate identifications on source and target curves) whose parity is given by the sum of local parities $p(f)$. Asking whether the parity of f_0 is uniquely determined by π, boils down to asking whether the parities of these local Hurwitz covers f (and their components f_i) are unique. Figure 22 illustrates this duality.

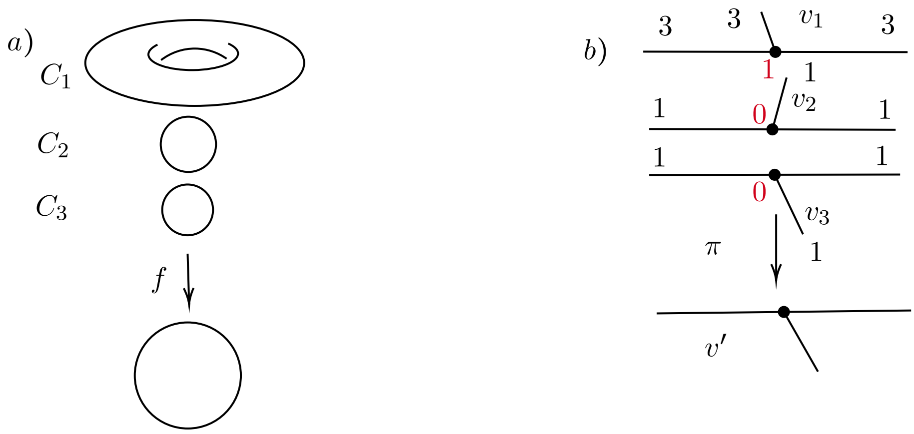


Figure 22: Illustration of the duality: the red numbers denote the genus of the corresponding vertex, the black numbers are edge weights.

Lemma 5.1. Let $f : C \to \mathbb{P}^1$ be a Hurwitz map satisfying the discrete data (h, d, m^1, m^2, m^3), where m^i is an odd partition of d for $i = 1,2,3$.

1. If $g(C) = 0$, then $p(f) = 0$.
2. If $(h, d, m^1, m^2, m^3) = (0, 3, (3)^3)$, then $p(f) = 1$.
3. If $(h, d, m^1, m^2, m^3) = (0, 4, (3,1)^3)$ and C is connected $p(f) = 0$, else $p(f) = 1$.

Proof. For case 1 recall that a curve of genus 0 has modulo equivalence only one theta characteristic given by $-P$ for any point $P \in C$. Since $\tilde{L}_f$, the twisted pullback of $-P$ along f is also a theta characteristic we have $p(f) = 0$ as in example 4.2. Case 2 follows from example 4.10. If $d = 4$, the Riemann Hurwitz formula yields $\chi(C) = 2$. The curve C is either connected and of genus 0 or the disjoint union of a rational curve C_0 and an elliptic curve C_1 ([LP13]). If the first holds, we can conclude $p(f) = 0$ by case 1. Else, the map f is a pair of maps (f_1, f_2), where

- f_1 satisfies the discrete data $(0, 1, (1)^3)$ and by case 1 $p(f_1) = 0$.
- f_2 satisfies the discrete data $(0, 3, (3)^3)$ and by case 2 $p(f_2) = 1$.

In total we get $p(f) = 1$. This proves the claim. □

Corollary 5.2. Let $m = (3)$ or $m = (3,1)$, π be a tropical 3-cycle covering of degree 3 or 4 with odd edge weights only. Then for any families of Hurwitz covers $\mathcal{F}, \tilde{\mathcal{F}}$ (as in section 4.4) with special fibres f_0, $\tilde{f}_0$ whose dual map is π we have $p(f_0) = p(\tilde{f}_0)$.

Proof. Let π be such a tropical Hurwitz cover with dual algebraic cover f_0. By lemma 5.1 the component maps of f_0 denoted f_v for $v \in V(\Gamma)$ have a parity which is uniquely determined by $g(v)$. Indeed, if v is a genus 0 vertex, we have $p(f) = 0$. Else $g(v) = 1$ and v is 2-valent with adjacent edges of weight 3. By lemma 5.1 case 2 and 3, $p(f) = 1$. In total, $p(f_0)$ is uniquely determined by π. □

5.1 Tropical spin covers and tropical spin Hurwitz numbers

The even case. Fix integers k, h, g, d and odd partitions $m^1, ..., m^k$ of d satisfying the Riemann-Hurwitz formula. We consider a maximally degenerate curve B of genus h together with a set of k ends, our tropical target curve, and a parity function

$$s_B : V(B) \to \mathbb{Z}/2\mathbb{Z}, v \mapsto 0.$$

Definition 5.3. Let $\pi : \Gamma \to B$ be a tropical Hurwitz covering of B (definition 3.20), such that the edge weights of Γ are odd. An *admissible* parity function on Γ is a function $s : V(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$, such that for all $v \in V(\Gamma)$ there exists a spin Hurwitz Cover f (with connected domain curve) contributing to the local Hurwitz number H_v associated to v of the same parity, i.e. $p(f) = s(v)$.

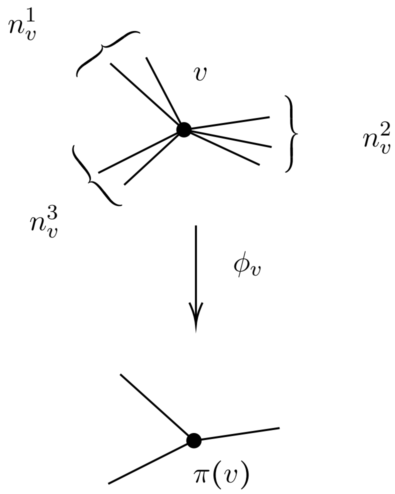
Definition 5.4. A *tropical spin Hurwitz cover* (π, s) of (B, s_B) is a tropical Hurwitz cover $\pi : \Gamma \to B$ as in definition 3.20 together with an admissible parity function s on Γ. Its parity $p(\pi, s)$ is given by

$$p(\pi, s) := \sum_{v \in V(\Gamma)} s(v) mod\, 2.$$

For each vertex $v \in V(\Gamma)$ let $H_v = H_{g(v) \xrightarrow{d(v)} 0}(n_1^v, n_2^v, n_3^v)$ be its local Hurwitz number. Consider a partition of $\mathcal{M}$, the set of isomorphism classes of maps contributing to the *connected* spin Hurwitz number $H^{(0,+,c)}_{n_1^v, n_2^v, n_3^v}$, into the spaces of maps with even and odd parity, i.e. $\mathcal{M} = \mathcal{M}^0 \cup \mathcal{M}^1$, and write

$$H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v} = (H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v})_0 - (H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v})_1, \text{ where}$$
$$(H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v})_1 := \sum_{f \in \mathcal{M}^1} \frac{1}{|Aut(f)|} \text{ and } (H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v})_0 := \sum_{f \in \mathcal{M}^0} \frac{1}{|Aut(f)|}.$$

Definition 5.5. Analogous to definition 3.20, we can define the *local spin Hurwitz number* $H^{(0,+)}((\pi, s), v) := (H^{(0,+,c)}_{n_1^v,n_2^v,n_3^v})_{s(v)}$ of a tropical spin Hurwitz cover $(\pi, s) : (\Gamma, s) \to (B, s_B)$ at a fixed vertex v of Γ to be the number associated to the open cover ϕ_v from the star graph with vertex v of genus $g(v)$ and ends labelled by the partitions n_1^v, n_2^v, n_3^v to its 3-valent image.



Note that the numbers $H^{(0,+)}((\pi,s),v)$ depend on the parity function on Γ.

Remark 5.6. The algebraic origin of definition 5.4 was essentially given in the preceding paragraph: it is customized to encode the data of a classical spin Hurwitz number and allows for the reconstruction of spin Hurwitz covers from it. In this context, note that requiring edge weights to be odd is a *realizability condition.* It guarantees the existence of spin Hurwitz cover dual to (π,s).

Definition 5.7. An *isomorphism of tropical spin Hurwitz covers* $(\pi_1,s):(\Gamma_1,s_1)\to(B,s_B)$ and $(\pi_2,s_2):(\Gamma_2,s_2)\to(B,s_B)$ is an isomorphism $\phi:\Gamma_1\to\Gamma_2$ of tropical curves such that $\pi_1=\pi_2\circ\phi$ and $s_1(v_1)=s_2(\phi(v_1))$ for all $v_1\in V(\Gamma_1)$.

The multiplicity of a tropical spin Hurwitz cover should account for the number of ways in which it may be promoted to a degenerate spin Hurwitz cover of nodal curves with same local parities.

Definition 5.8. To a tropical spin Hurwitz cover (π,s) we assign the following multiplicity:

$$mult(\pi,s):=\frac{1}{|Aut(\pi,s)|}\prod_{e\in E(\Gamma)\text{bounded}}\omega(e)\prod_{v\in V(\Gamma)}n_1^v!\,n_2^v!\,n_3^v!\,H^{(0,+)}((\pi,s),v).$$

Note that multiplicity depends on the parity function on Γ. We have all the ingredients we need together and give the following:

Definition 5.9. Fix integers k,h,g,d and odd partitions $m^1,\dots,m^k$ of d satisfying the Riemann-Hurwitz formula. Fix a target (B,s_B) of genus h as above. The *(even) tropical spin Hurwitz number* $\mathbb{T}H^{(h,+)}_{m^1,\dots,m^k}$ is a weighted count of tropical spin Hurwitz covers $(\pi,s):(\Gamma,s)\to(B,s_B)$ as in definition 5.4:

$$\mathbb{T}H^{(h,+)}_{m^1,\dots,m^k}=\sum_{(\pi,s)\in\mathbb{T}\mathcal{S}}(-1)^{p(\pi,s)}\cdot mult(\pi,s),\text{where}$$

$$\mathbb{T}\mathcal{S}:=\{(\pi,s):(\Gamma,s)\to(B,s_B):\prod_{e\in E(\Gamma)}\omega(e)\text{ is odd.}\}.$$

The set $\mathbb{T}\mathcal{S}$ is the set of relevant spin Hurwitz covers.

Remark 5.10. We wish to mention that our definition of $p(\pi,s)$ is compatible with the parity notion of tropical spin curves in Definition 2.4.2 ([CMP20]). The corresponding tropical spin curve is given by (Γ,P,s) with $P=\varnothing$ and parity function $s:V(\Gamma/P)=V(\Gamma)\to\mathbb{Z}/2\mathbb{Z},v\mapsto s(v)$ (the underlying graph is a spin graph as in definition 4.18). It is obvious that this is indeed a tropical spin curve, since s satisfies the requirement, $s(v)=0$ for a genus 0 vertex v, and P is a cycle. Omitting $P=\varnothing$, we refer to (B,s_B) and (Γ,s) as tropical spin curves and notice that the target (B,s_B) is even (Definition 2.1.1. [CMP20]).

The odd case. As before fix integers k, h, g, d where $h \geqslant 1$ and odd partitions $m^1, ..., m^k$ of d satisfying the Riemann-Hurwitz formula. We consider a target curve B that is maximally degenerate except for a 1-valent vertex v' of genus 1 together with an *odd* parity function

$$s_B : V(B) \to \mathbb{Z}/2\mathbb{Z}, v \mapsto \begin{cases} 0, & g(v) = 0 \\ 1, & g(v) = 1 \end{cases}.$$

We call such a curve *almost maximally degenerate.*

Remark 5.11. Note that in light of the degeneration formulas (theorem 4.19) an almost maximally degenerate base curve is the equivalent of a maximally degenerate one (for base curves with odd theta characteristics) in the spin world.
The curve B is obtained from a maximally degenerate curve B' by contracting the only cycle as shown in figure 23 for a base curve of genus 1.

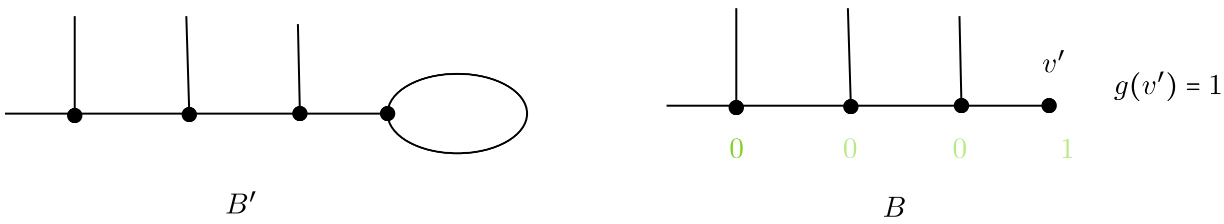


Figure 23: Base curve B with parity function (in green) obtained by contracting the cycle in B'.

Definition 5.12. Let $\pi : \Gamma \to B$ be a tropical cover of B, such that the edge weights of Γ are odd and the Riemann-Hurwitz condition is satisfied at each vertex. An *admissible* parity function on Γ is a function $s : V(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$ such that:

- $s : V(\Gamma) \smallsetminus \pi^{-1}(v') \to \mathbb{Z}/2\mathbb{Z}$ is an admissible parity function on $\Gamma \smallsetminus \pi^{-1}(v')$,
- for $v \in \pi^{-1}(v')$ there exists a spin Hurwitz Cover f (with connected domain curve) that counts towards $H_{n^v}^{(1,-)}$ of the same parity, i.e. $p(f) = s(v)$,

where v' denotes the genus one vertex of B. We call the pair (π, s) a *tropical spin Hurwitz cover* of (B, s_B) of parity $p(\pi, s) := \sum_{v \in V(\Gamma)} s(v) mod\, 2$. The notion of *isomorphisms* of tropical spin Hurwitz covers is analogous to the one in 5.7.
For fixed vertex $v \in V(\Gamma)$ we define the *local spin Hurwitz number* of v as in definition 5.5, if $v \in V(\Gamma) \smallsetminus \pi^{-1}(v')$, and write $H^{(0,+)}((\pi, s), v)$ in this case, and as $H^{(1,-)}((\pi, s), v) := (H_{n^v}^{(1,-,c)})_{s(v)}$ else, where

$$H_{n^v}^{(1,-,c)} = (H_{n^v}^{(1,-,c)})_0 - (H_{n^v}^{(1,-,c)})_1.$$

The multiplicity associated to (π, s) is given by $mult(\pi, s) :=$

$$\frac{1}{|Aut(\pi, s)|} \prod_{e \in E(\Gamma) \text{bounded}} \omega(e) \prod_{v \in V(\Gamma) \smallsetminus \pi^{-1}(v')} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi, s), v) \prod_{v \in \pi^{-1}(v')} n^v! H^{(1,-)}((\pi, s), v).$$

Definition 5.13. Fix integers k, h, g, d where $h \geqslant 1$ and odd partitions $m^1, ..., m^k$ of d satisfying the Riemann-Hurwitz formula. Fix a target (B, s_B) of genus h as above. The *(odd) tropical* *spin Hurwitz number* $\mathbb{T}H^{(h,-)}_{m^1,\dots,m^k}$ is a weighted count of tropical spin Hurwitz covers (π, s) as in 5.12:

$$\mathbb{T}H^{(h,-)}_{m^1,\dots,m^k} = \sum_{(\pi,s)\in\mathbb{T}\mathcal{S}} (-1)^{p(\pi,s)} \cdot mult(\pi, s), \text{where}$$
$$\mathbb{T}\mathcal{S} := \{(\pi, s) : (\Gamma, s) \to (B, s_B) : \prod_{e\in E(\Gamma)} \omega(e) \text{ is odd.}\}.$$

The set $\mathbb{T}\mathcal{S}$ is the set of relevant spin Hurwitz covers.

5.1.1 Degree 3 and 4 with base $\mathbb{TP}^1$ and $\mathbb{TE}$

As a first step, we want to understand degree 3 or 4 spin Hurwitz covers of $\mathbb{P}^1$ or of an elliptic curve E with even theta characteristic tropically. Restricting to the cases of degree 3 or 4 has two advantages. An admissible parity function on the cover curve is unique and the multiplicity coincides with the usual definition for tropical Hurwitz covers. Hence, we achieve a complete combinatorial treatment of the counting problem 5.9.
For the remainder of this subsection we fix the tropical target curve B to be either the tropical line, i.e. $B = \mathbb{TP}^1$, or a tropical elliptic curve, $B = \mathbb{T}E$ (definitions 3.17 and 3.16), and a parity function

$$s_B : V(B) \to \mathbb{Z}/2\mathbb{Z}, v \mapsto 0.$$

Next we determine the parity of a cover of (B, s_B).

Lemma 5.14. Let $(\pi, s) : (\Gamma, s) \to (B, s_B)$ be a tropical spin Hurwitz cover of degree $d \in \{3, 4\}$ of (B, s_B), i.e. π is a 3-cycle covering of $B = \mathbb{TP}^1$ or $B = \mathbb{T}E$ such that the edge weights of Γ are odd. Then an admissible parity function on Γ is unique and given by

$$s(v) := \begin{cases} 0, & g(v) = 0 \\ 1, & g(v) = 1 \end{cases} \text{ where}$$
$$p(\pi) := p(\pi, s) = \sum_{v\in V(\Gamma)} s(v) \; mod \; 2$$

is the parity of π (equivalently of Γ). In particular, both a single edge and a single circle have parity 0.

Proof. This follows directly from corollary 5.2. □

An easy consequence is

Lemma 5.15. Let $(\pi, s) : (\Gamma, s) \to (B, s_B)$ be a tropical spin Hurwitz as in lemma 5.14. The multiplicity $mult(\pi, s)$ of (π, s) defined in 5.8 is just given by $mult(\pi)$, where $mult(\pi)$ is the usual multiplicity of π considered as a tropical Hurwitz cover.

Proof. Since an admissible parity function Γ is unique, we have that $H_v = H^{(0,+)}((\pi, s), v)$ for each vertex $v \in V(\Gamma)$. Hence, definition 3.21 and definition 5.8 agree. □

Our goal is to count tropical spin Hurwitz covers of $\mathbb{TP}^1$ or $\mathbb{T}E$. We need the following input data: Fix positive integers k and $h = 0$ or $h = 1$ together with a choice of $d = 3$ or $d = 4$ with (the only) corresponding odd partitions $m = (3)$ or $m = (3,1)$ of d. The tropical spin Hurwitz number $\mathbb{T}H^{(h,+)}_{m^k}$ for the discrete data (h, d, m^k) simplifies to:

$$\mathbb{T}H^{(h,+)}_{m^k} = \sum_{\pi \in M_k} (-1)^{p(\pi)} \cdot mult(\pi), \text{where we set}$$
$$M_k := \{(\pi, s) : (\Gamma, s) \to (B, s_B) : \prod_{e \in E(\Gamma)} \omega(e) \text{ is odd.}\}$$

to be the set of relevant spin Hurwitz covers. Moreover, for any choice of (h, d, m^k) we implicitly require $g(\Gamma)$ to satisfy the Riemann-Hurwitz formula.

Example 5.16. Figure 5.16 illustrates the computation of $\mathbb{T}H^{(0,+)}_{(3)^3} = -\frac{1}{3}$ and $\mathbb{T}H^{(0,+)}_{(3)^4} = \frac{1}{3} + \frac{2}{3} = 1$.

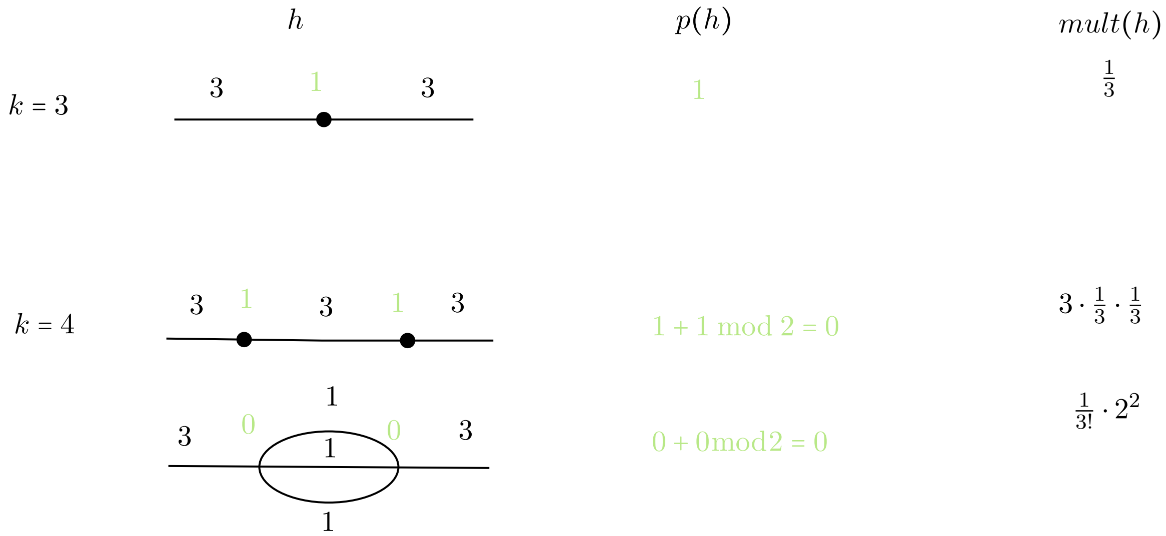


Figure 24: Tropical spin curves with parity function in green that contribute to $\mathbb{T}H^{(0,+)}_{(3)^3} = \frac{-1}{3}$ and $\mathbb{T}H^{(0,+)}_{(3)^4} = \frac{1}{3} + \frac{2}{3} = 1$.

We have a count analogous to the one in [LP13] in the tropical world.

Proposition 5.17. For $k \in \mathbb{N}$ we have $\mathbb{T}H^{(0,+)}_{(3)^k} = \frac{1}{9}((-1)^k 2^{k-1} + 1)$.

Proof. We know that any tropical Hurwitz spin cover $\pi : (\Gamma, s) \to (\mathbb{TP}^1, s_{\mathbb{P}^1}) \in M_k$ (which counts with non-zero multiplicity) is an arbitrary combination of the two building blocks, double Wiener and 2-valent genus 1 vertices (see example 3.15) such that resulting graph has $k-2$ inner vertices. In particular, genus 0 vertices occur in pairs only. First, let us analyse the parity of such a cover. We claim:

$$p(\pi) = k \mod 2 \text{ for } k > 3.$$

Indeed, by definition, we have $p(\pi) = \sum_{v \in V(\Gamma)} p(v) \; mod \; 2 = |\{v : g(v) = 1\}| mod \; 2$. Since Γ has $k-2$ vertices we obtain:

$$\underbrace{k-2}_{=k \mod 2} = |\{v : g(v) = 1\}| + \underbrace{|\{v : g(v) = 0\}|}_{=0 \mod 2}.$$

Thus, k is even if and only if the right hand side of the equation above is. This, on the other hand, is the case, if and only if $|\{v : g(v) = 1\}|$ is even, i.e. $p(\pi) = 0 \mod 2$. This proves the claim.
Note that this implies that modulo sign the degree 3 spin Hurwitz number is equal to the usual Hurwitz number. Therefore, it suffices to show that

$$\mathbb{T}H_{g \xrightarrow{3} 0}((3)^k) = (-1)^k \cdot \frac{1}{9}((-1)^k 2^{k-1} + 1) = \frac{1}{9}(2^{k-1} + (-1)^k)$$

holds. Next, we look at ways to obtain a tropical cover $\pi \in M_k{}^3$ from a cover $\tilde{\pi} : \tilde{\Gamma} \to \mathbb{TP}^1$ in M_{k-1}. Label the vertices of $\tilde{\Gamma}$ from 1 to $k-1$. If vertex $k-1$ has genus 1, we have a choice to either attach an additional genus 1 vertex with an end of weight 3 to vertex $k-1$ or to split the end of weight 3 into 3 strands of weight 1. If vertex $k-1$ of $\tilde{\Gamma}$ has genus 0, we can only attach.

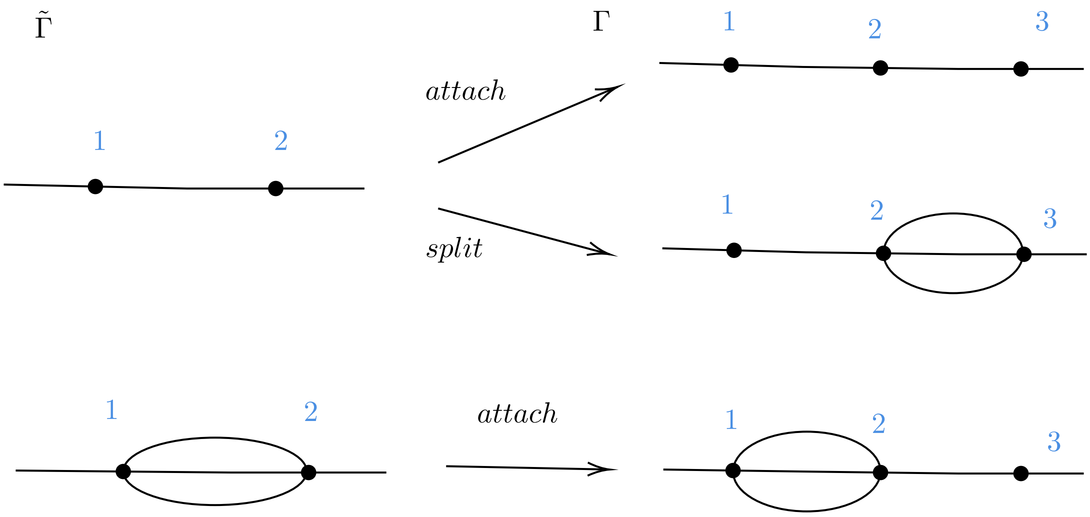


Figure 25: Parity, edge weights and ends are omitted to emphasize the structure of the recursion. Vertex labels are drawn in blue.

We obtain all degree 3 covers that contribute to $\mathbb{T}H^{(0,+)}_{(3)^k}$ from covers that contribute to $\mathbb{T}H^{(0,+)}_{(3)^{k-1}}$ in this way: For $k \geqslant 3$ we have $M_k := M_k^1 \cup M_k^2$, where

$$M_k^1 := \{h : \Gamma \to \mathbb{TP}^1 : \text{ vertex } k \text{ has genus } 1\ \} \text{ and } M_k^2 := \{h : \Gamma \to \mathbb{TP}^1 : \text{ vertex } k \text{ has genus } 0\ \}.$$

In particular, we see $|M_k^1| = |M_{k-1}|$ since attaching a genus 1 vertex is always possible, and $|M_k^2| = |M_{k-1}^1|$.
We have good feeling how the recursion works on the structure side, we need to analyse how the multiplicity is affected. Start with a cover $\pi \in M_k^1$. Then there exists a cover $\tilde{\Gamma} \in M_{k-1}$,[4] such that Γ is obtained from $\tilde{\Gamma}$ by attaching a genus 1 vertex. Hence, we have

$$\begin{aligned} mult(\Gamma) &= mult(\tilde{\Gamma}) \cdot \underbrace{3}_{\text{edge contribution}} \cdot \underbrace{\frac{(3-1)(3-2)}{3!}}_{\text{vertex contribution}} \\ &= mult(\tilde{\Gamma}). \end{aligned}$$

[3]We omit s to keep notation shorter.
[4]We use the base curve $\tilde{\Gamma}$, when referring to the tropical cover $\tilde{\pi} : \tilde{\Gamma} \to \mathbb{TP}^1$ to keep notation shorter.

Otherwise, let $\Gamma \in M_k^2$ be obtained from $\tilde{\Gamma} \in M_{k-1}^1$ by the splitting procedure, then

$$\begin{aligned} mult(\Gamma) &= \frac{1}{2} mult(\tilde{\Gamma}) \cdot 2 \cdot 2 \\ &= 2 mult(\tilde{\Gamma}). \end{aligned}$$

The factor $\frac{1}{2}$ accounts for the new double Wiener and the two factors 2 are contributions from the new genus 0 vertices.

We are now ready to prove proposition 5.17, i.e.

$$\mathbb{T}H_{g \xrightarrow{3} 0}((3)^k) = \frac{1}{9}(2^{k-1} + (-1)^k) \tag{5}$$

holds. Figure 5.16 shows (5) for $k = 3$ and $k = 4$. Let $k \in \mathbb{N}$ and suppose that (5) holds for all $k' < k$. Then

$$\begin{aligned} \mathbb{T}H_{g \xrightarrow{3} 0}((3)^k) &= \sum_{\Gamma \in M_k} mult(\Gamma) = \sum_{\Gamma \in M_k^1} mult(\Gamma) + \sum_{\Gamma \in M_k^2} mult(\Gamma) \\ &= \sum_{\tilde{\Gamma} \in M_{k-1}} mult(\tilde{\Gamma}) + \sum_{\tilde{\Gamma} \in M_{k-1}^1} 2mult(\tilde{\Gamma}) \\ &= \sum_{\tilde{\Gamma} \in M_{k-1}} mult(\tilde{\Gamma}) + \sum_{\Gamma' \in M_{k-2}} 2mult(\Gamma') \\ &= \mathbb{T}H_{k-3 \xrightarrow{3} 0}((3)^{k-1}) + 2\mathbb{T}H_{k-4 \xrightarrow{3} 0}((3)^{k-2}) \\ &= \frac{1}{9}(2^{k-2} + (-1)^{k-1}) + \frac{2}{9}(2^{k-3} + (-1)^{k-2}) \\ &= \frac{1}{9}(2 \cdot 2^{k-2} + (-1)(-1)^{k-2} + 2 \cdot (-1)^{k-2}) \\ &= \frac{1}{9}(2^{k-1} + (-1)^k). \end{aligned}$$

Proposition 5.17 follows by induction. □

Proposition 5.18. For $k \in \mathbb{N}$ we have $\mathbb{T}H^{(1,+)}_{(3)^k} = (-1)^k 2^k + 1$.

Proof. The case $k = 0$ is shown in figure 26. Let $k \geqslant 1$. We count tropical spin Hurwitz coverings of degree 3 and genus $k + 1$ of a tropical elliptic curve $\mathbb{TE}$ with k labelled vertices, $v_1', ..., v_k'$, by counting coverings of $\mathbb{TP}^1$ obtained after cutting an edge.

Cutting procedure. For (π, s) relevant to $\mathbb{T}H^{(1,+)}_{(3)^k}$, consider the labelling of the underlying curve Γ induced by the labelling of $\mathbb{TE}$, i.e. $v_i := \pi^{-1}(v_i')$ for $i = 1, ..., k$. Note that there is no ambiguity since $\pi^{-1}(v_i')$ contains exactly one vertex. Cut the edge(s) connecting v_k and v_1 (see figure 27 for reference). The straightened curve $\tilde{\Gamma}$ has either

- two ends of weight 3 (type 1).
- two tines 3 ends of weight 1 (type 2).

We interpret $\tilde{\Gamma}$ as a tropical spin Hurwitz cover of $\mathbb{TP}^1$ with ramification profile (3) or (1,1,1) over $\pm\infty$ depending on whether $\tilde{\Gamma}$ is of type 1 or 2. Following this procedure we obtain all degree 3 covers, that contribute to $\mathbb{T}H^{(0,+)}_{(3)^{k+2}}$ (denote their set by M_1) and $\mathbb{T}H^{(0,+)}_{(3)^k,(1^3)^2}$ (denote their set by M_2), respectively. Hence, cutting defines a surjection $c : M_k \to M_1 \cup M_2$. In fact, c is also injective with inverse given by the unique possibility of gluing the ends over $\pm\infty$.

How does the multiplicity of a cover change under the cutting procedure? Graphs of type 1 loose an edge of weight 3. Thus we have $mult(\tilde{\Gamma}) = \frac{1}{3}mult(\Gamma)$. For graphs of type 2 we replace a double Wiener by two balanced double forks. This yields $mult(\tilde{\Gamma}) = \frac{1}{3!}mult(\Gamma)$. In total

$$\begin{aligned}\mathbb{T}H^{(1,+)}_{(3)^k} &= 3\,\mathbb{T}H^{(0,+)}_{(3)^{k+2}} + 3!\,\mathbb{T}H^{(0,+)}_{(3)^k,(1^3)^2} \\ &= 3\,\mathbb{T}H^{(0,+)}_{(3)^{k+2}} + 3!\,\mathbb{T}H^{(0,+)}_{(3)^k} \\ &= 3\,\frac{-1}{9}((-1)^{k+1}2^{k+1} - 1) + 3!\,\frac{-1}{9}((-1)^{k-1}2^{k-1} - 1) \\ &= (-1)^k 2^k + 1\end{aligned}$$

where we implicitly used that the parity of a cover remains unaffected by the cutting procedure.

□

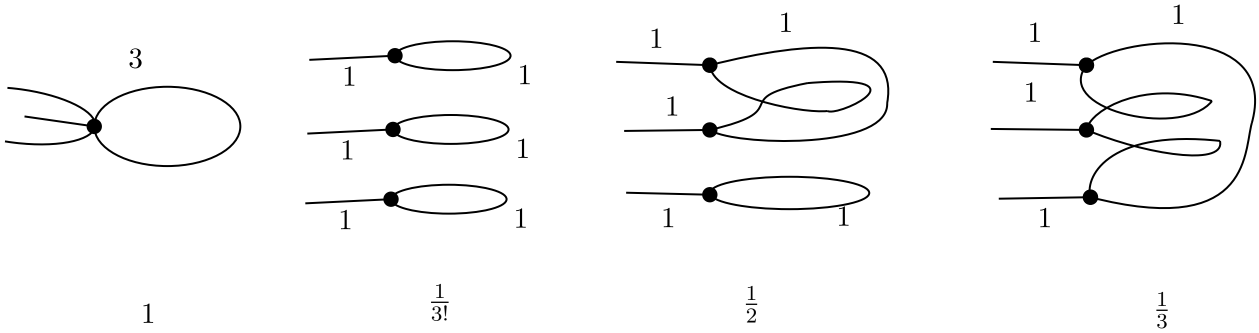


Figure 26: Computation of $\mathbb{T}H^{(1,+)}_{(1,1,1)} = 2$

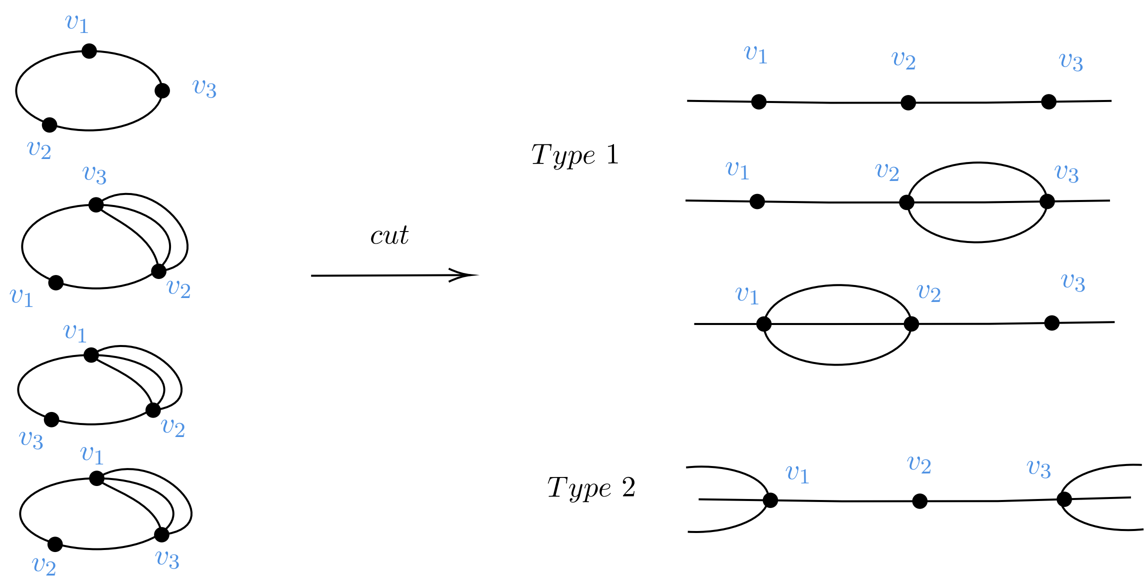


Figure 27: Graphs obtained after cutting all tropical degree 3 coverings of $\mathbb{T}E$ with $k = 3$ branch points along the edge $v_3 - v_1$.

Proposition 5.19. For degree 4 spin tropical Hurwitz numbers with base $\mathbb{TP}^1$ and $k \in \mathbb{N}$ we have the equality $\mathbb{T}H^{(0,+)}_{(31)^k} = \frac{-1}{3\cdot 3!}((-1)^{k-1}2^{k-1} - 4^{k-1})$.

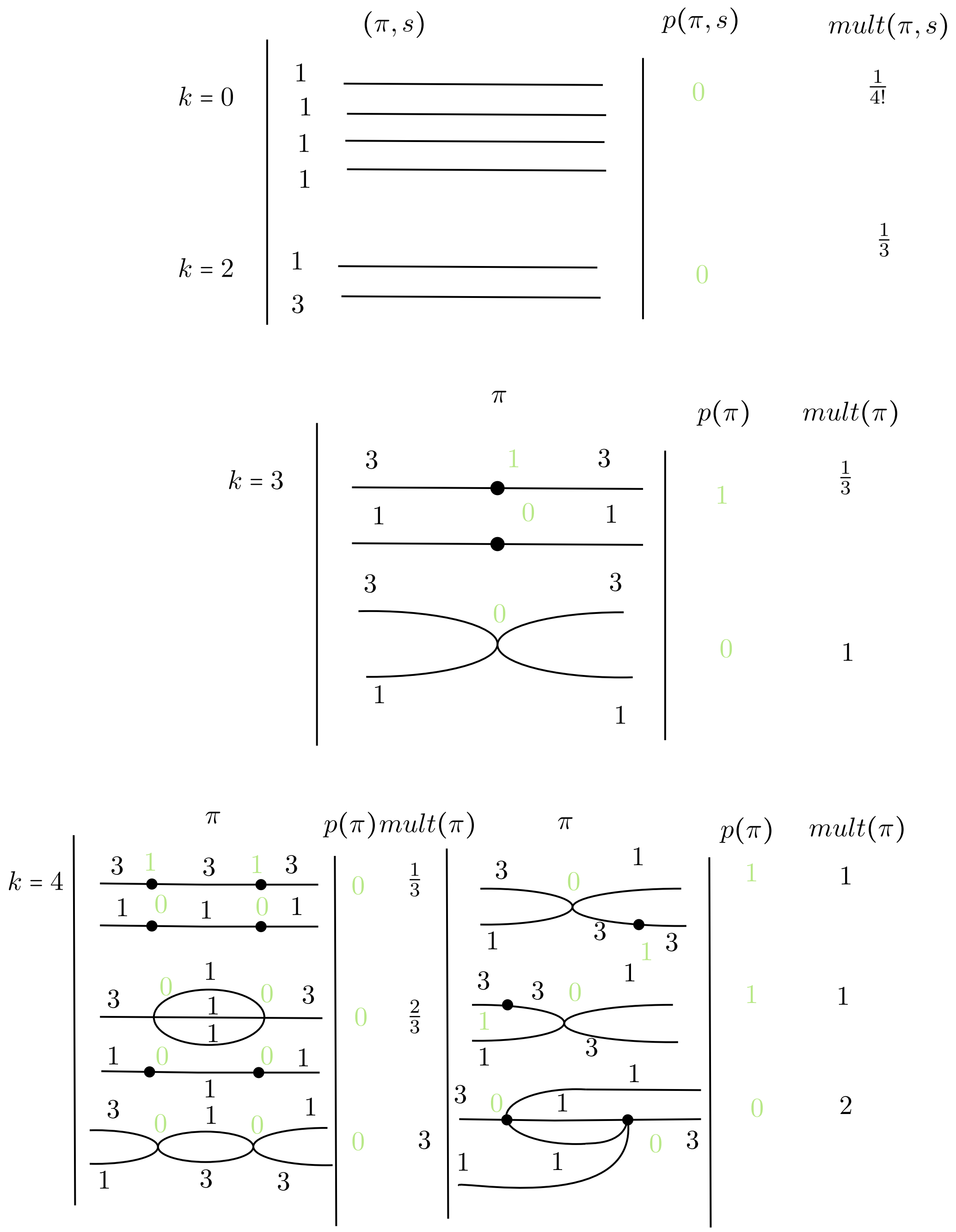


Figure 28: Tropical spin covers that count towards $\mathbb{T}H^{(0,+)}_{(31)^k}$ for $k = 0, 2, 3, 4$.

We see:

$$\mathbb{T}H^{(0,+)}_{(31)^0} = \frac{1}{4!}, \ \mathbb{T}H^{(0,+)}_{(31)^2} = \frac{1}{3}, \ \mathbb{T}H^{(0,+)}_{(31)^3} = 1 - \frac{1}{3} = \frac{2}{3} \text{ and } \mathbb{T}H^{(0,+)}_{(31)^4} = \frac{1}{3} + \frac{2}{3} + 3 - 1 - 1 + 2 = 4.$$

Proof. We prove the statement by induction on the number of branch points k. Proposition 5.19 holds for $k \leqslant 4$ (see figure 28 for $k \neq 1$). If $k = 1$, $\mathbb{T}H^{(0,+)}_{(31)^1} = 0$ is obvious since $M_1 = \varnothing$. Let $k > 5$ and suppose proposition 5.19 holds for all $\tilde{k} < k$. Let us analyse the construction of a tropical spin cover that contributes to $\mathbb{T}H^{(0,+)}_{(31)^k}$: Start with two strands over $-\infty$, one of weight 3 and one of weight 1. There are three possibilities to continue (since joining 2 strands of weight 1 creates a vertex of multiplicity 0):

1. Join both strands to form a butterfly vertex of genus 0.
2. Create a genus 1 vertex on the strand of weight 3.
3. Split the strand of weight 3.

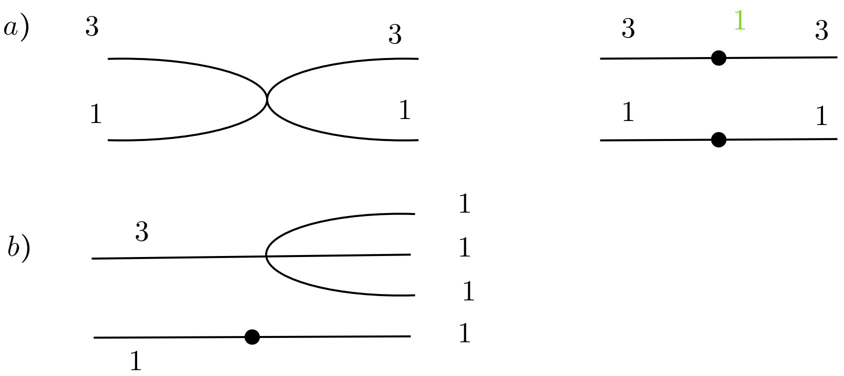
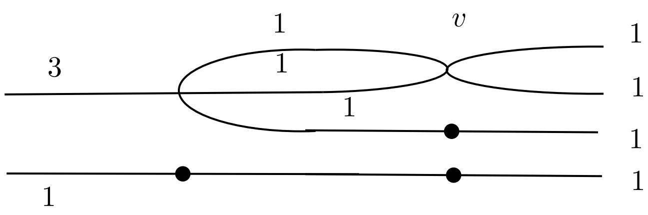


Figure 29: a) shows possibility 1 on the left, 2 on the right and b) possibility 3.

Note, choosing either possibility 1 or 2 leads us back to the starting position, one strand of weight 3 and one of weight 1: Thus, all possibilities to complete this construction are given by gluing the source curve $\tilde{\Gamma}$ of a tropical spin Hurwitz cover $(\tilde{\pi}, \tilde{s})$ that counts towards $\mathbb{T}H^{(0,+)}_{(31)^{k-1}}$. Denote by $\pi : (\Gamma, s) \to \mathbb{TP}^1$ the cover obtained in this way. We have:

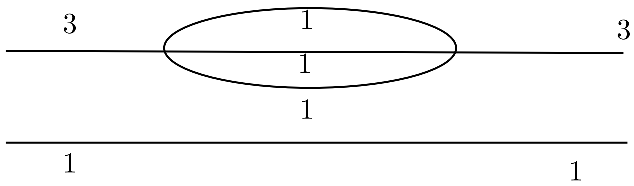
$$p(\pi) = p(\tilde{\pi}) \text{ and } mult(\pi) = 3mult(\tilde{\pi}) \text{ for possibility 1 and}$$
$$p(\pi) = (1 + p(\tilde{\pi})) \mod 2 \text{ and } mult(\pi) = \frac{1}{3} \cdot 3 \cdot mult(\tilde{\pi}) \text{ for possibility 2,}$$

since gluing a genus 0 vertex to $\tilde{\Gamma}$ (possibility 1) creates a bounded edge of weight 3, but leaves the parity unchanged. For possibility 2 we have to take into account that, in addition to the edge of weight 3, a vertex of genus 1 changes the parity and has weight $\frac{1}{3}$.
Choosing possibility 3 leaves us with 4 strands of weight 1. Joining is the only option. Note that joining only two creates a vertex of multiplicity 0:

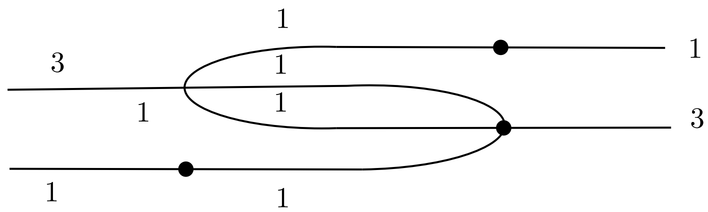


Thus, we are forced to join exactly three, which puts us back to our starting point. Like before all possibilities to complete this construction are given by gluing the source curve $\tilde{\Gamma}$ of a tropical spin Hurwitz cover $(\tilde{\pi}, \tilde{s})$ that counts towards $\mathbb{T}H^{(0,+)}_{(31)^{k-2}}$. Denote by $\pi : (\Gamma, s) \to \mathbb{TP}^1$ the cover obtained in this way. The process of splitting the strand of weight 3 and joining three strands of weight 1 only creates genus 0 vertices. Thus, we have $p(\pi) = p(\tilde{\pi})$. To compute $mult(\pi)$ we have to distinguish between two cases shown below.

Case 1 :



Case 2 :



We obtain

$$mult(\pi) = \frac{3 \cdot 2^2}{3!} mult(\tilde{\pi}) \text{ in case 1 and } mult(\pi) = \frac{3 \cdot 2^2}{2} mult(\tilde{\pi}) \text{in case 2,}$$

where the factor of $\frac{1}{2}$ $(\frac{1}{3!})$ comes from the single (double) Wiener. In total, we have

$$\begin{aligned}
\mathbb{T}H^{(0,+)}_{(31)^k} &= 3\,\mathbb{T}H^{(0,+)}_{(31)^{k-1}} - \mathbb{T}H^{(0,+)}_{(31)^{k-1}} + 2\,\mathbb{T}H^{(0,+)}_{(31)^{k-2}} + 2 \cdot 3\,\mathbb{T}H^{(0,+)}_{(31)^{k-2}} \\
&= 2\mathbb{T}H^{(0,+)}_{(31)^{k-1}} + 8\mathbb{T}H^{(0,+)}_{(31)^{k-2}} \\
&= \frac{-1}{3 \cdot 3!}(2((-1)^{k-2}2^{k-2} - 4^{k-2}) + 8((-1)^{k-3}2^{k-3} - 4^{k-3})) \\
&= \frac{-1}{3 \cdot 3!}((-1)^{k-2}2^{k-1} + \underbrace{2(-1)^{k-3}}_{=-2(-1)^{k-2}} - 24^{k-2} - 24^{k-2}) \\
&= \frac{-1}{3 \cdot 3!}((-1)^{k-1}2^{k-1} - 4^{k-1}).
\end{aligned}$$

□

Proposition 5.20. For degree 4 spin tropical Hurwitz numbers with base $\mathbb{T}E$ and $k \in \mathbb{N}$ we have the equality $\mathbb{T}H^{(1,+)}_{(31)^k} = (-1)^k 2^k + 4^k$.

Proof. The strategy of proof is analogous to the one of proposition 5.18. We label the vertices of $\mathbb{TE}$ by $v'_1, ..., v'_k$. For each (π, s) relevant to $\mathbb{T}H^{(1,+)}_{(31)^k}$, consider the curve $\tilde{\Gamma}$ obtained after cutting the edge(s) connecting $\pi^{-1}(v'_k)$ and $\pi^{-1}(v'_1)$. Then $\tilde{\Gamma}$ has

- two ends of weight 3 and 1 (type 1).
- four ends of weight 1 (type 2).

We interpret $\tilde{\Gamma}$ as a tropical spin Hurwitz cover of $\mathbb{TP}^1$ with ramification profile (31) or (1111) over $\pm\infty$ depending on whether $\tilde{\Gamma}$ is of type 1 or 2. This procedure yields all degree 4 covers that contribute to $\mathbb{T}H^{(0,+)}_{(31)^{k+2}}$ (denote their set by M_1) and $\mathbb{T}H^{(0,+)}_{(31)^k,(1^4)^2}$ (denote their set by M_2), respectively. Hence, cutting defines a surjection $c : M_k \to M_1 \cup M_2$. However, c is only injective when restricted to M_1. On M_2 c is 2-to-1. Indeed, for each $\tilde{\Gamma} \in M_2$ we have exactly two possibilities of gluing the ends over $\pm\infty$ (figure 30) and thus creating either a single or a double Wiener. The multiplicity changes accordingly. If $\tilde{\Gamma} := c(\Gamma) \in M_1$, then $mult(\tilde{\Gamma}) =$

$\frac{1}{3}mult(\Gamma)$. If $\tilde{\Gamma} := c(\Gamma) \in M_2$, we either replace a double Wiener by two balanced double forks, i.e. $mult(\tilde{\Gamma}) = \frac{1}{3!}mult(\Gamma)$, or we replace a single Wiener by two balanced double forks, hence $mult(\tilde{\Gamma}) = \frac{1}{33!}mult(\Gamma)$. In total

$$\begin{aligned}
\mathbb{T}H^{(1,+)}_{(31)^k} &= 3\,\mathbb{T}H^{(0,+)}_{(31)^{k+2}} + (3+1)3!\,\mathbb{T}H^{(0,+)}_{(3)^k,(1^4)^2} \\
&= 3\,\mathbb{T}H^{(0,+)}_{(31)^{k+2}} + 4!\,\mathbb{T}H^{(0,+)}_{(31)^k} \\
&= 3\,\frac{-1}{3\cdot 3!}((-1)^{k+1}2^{k+1} - 4^{k+1}) + 4!\,\frac{-1}{3\cdot 3!}((-1)^{k-1}2^{k-1} - 4^{k-1}) \\
&= \frac{1}{3!}(-1)^{k+2}2^{k+1} - \frac{1}{3!}4^{k+1} + \frac{2}{3}(-1)^k 2^k - \frac{1}{3}4^k \\
&= \frac{1}{3}(-1)^k 2^k - \frac{2}{3}4^k + \frac{2}{3}(-1)^k 2^k - \frac{1}{3}4^k \\
&= (-1)^k 2^k + 4^k
\end{aligned}$$

where we implicitly used that the parity of a cover remains unaffected by the cutting procedure.

□

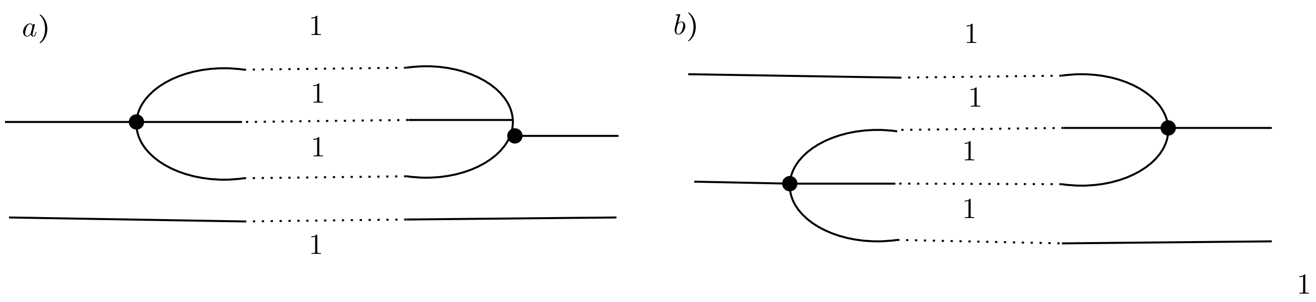


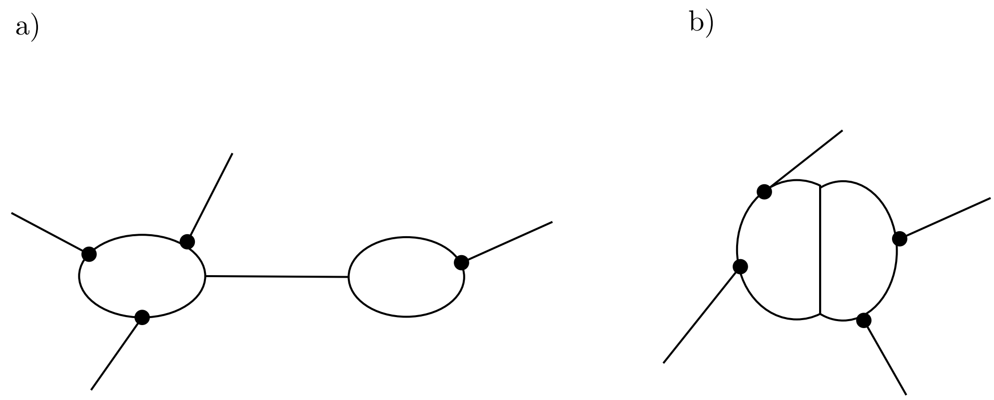
Figure 30: Two ways to glue ends of a curve $\tilde{\Gamma} \in M_2$.

5.1.2 Degree 3 with base of arbitrary genus

We enrich the results from the previous subsection by allowing a base curve of arbitrary genus for the case of degree 3. Fix positive integers k and h. We consider a maximally degenerate curve B of genus h together with a set of k ends and parity function

$$s_B : V(B) \to \mathbb{Z}/2\mathbb{Z}, v \mapsto 0.$$

Example 5.21. Two possible base curves of genus 2.



Determining coverings of a curve B of arbitrary genus with a specified ramification behaviour is not easy. The degeneration perspective offers a structured way to reconstruct such a cover Γ from looking at all possible local Hurwitz numbers.

Lemma 5.22. All non-zero local Hurwitz numbers of a Hurwitz cover $\pi : \Gamma \to B$ of degree 3 with odd edge weights only, where B is a curve of arbitrary genus, are listed below.

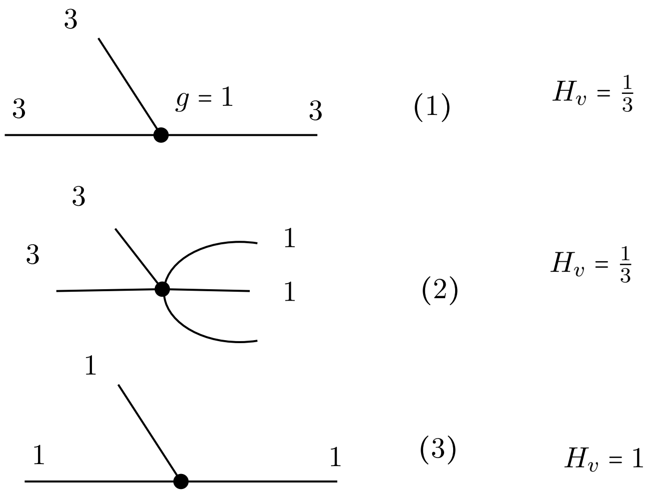


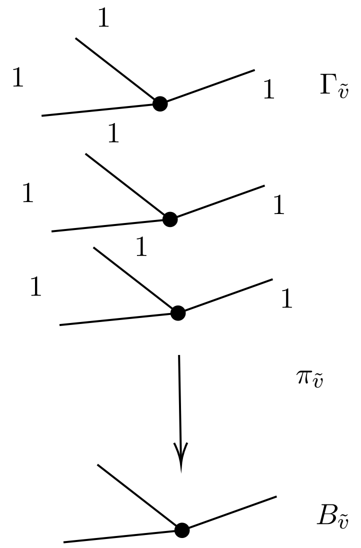
Figure 31: Vertices with corresponding local Hurwitz numbers.

Hence, we have that

$$mult(\pi) = \frac{1}{|Aut(\Gamma)|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \frac{1}{3}^{|\{v: g(v)=1\}|} 2^I,$$

where I is the set of vertices as in figure 31 (2).

Proof. For $\tilde{v} \in V(B)$ consider the local open cover $\pi_{\tilde{v}} : \Gamma_{\tilde{v}} \to B_{\tilde{v}}$ where $B_{\tilde{v}}$ (respectively $\Gamma_{\tilde{v}}$) is the star graph (respectively the possibly disjoint union of star graphs) obtained by cutting the edges adjacent to $\tilde{v}$ (adjacent to vertices $v \in \pi^{-1}(\tilde{v})$). Since we require edge weights of Γ to be odd $\pi_{\tilde{v}}$ is dual to a Hurwitz cover $f : C \to \mathbb{P}^1$ that contributes to one of the following (possibly disconnected) Hurwitz numbers whose values can be computed by counting monodromy representation: $H^{\bullet}{}_{\underset{1\to 0}{3}}((3)^3) = \frac{1}{3}$, $H^{\bullet}{}_{\underset{0\to 0}{3}}((3)^2, (1,1,1)) = \frac{1}{3}$, $H^{\bullet}{}_{\underset{1\to 0}{3}}((1,1,1)^2, (3)) = 0$ and $H^{\bullet}{}_{\underset{1\to 0}{3}}((1,1,1)^3) = \frac{1}{3!}$. If f contributes to one of the first two, C is connected and $\Gamma_{\tilde{v}}$ is as in figure 31 $(1)-(2)$. If f contributes to $H^{\bullet}{}_{\underset{1\to 0}{3}}((1,1,1)^3)$ a computation of the Euler characteristic $\chi(C) = 6$ yields that C is a disjoint union of three rational curves (Lemma 7.2. [Lee13]). Hence, $\Gamma_{\tilde{v}}$ is a disjoint union of three (open) tropical curves as in figure 31 (3).



The statement about the multiplicity follows from multiplying the local Hurwitz numbers with the automorphisms of the three local partitions, i.e. 1 for the first and third and 3! for the second number, and substituting in definition 5.8. □

Lemma 5.23. Let $(\pi, s) : \Gamma \to B$ be a tropical spin Hurwitz cover of degree 3 of (B, s_B), i.e. π is as in lemma 5.22. Then there exists only one admissible parity function on Γ given by

$$s(v) := \begin{cases} 0, & g(v) = 0 \\ 1, & g(v) = 1 \end{cases} \text{ with}$$
$$p(\pi, s) = \sum_{v \in V(\Gamma)} s(v) \, mod \, 2$$

and thus $mult(\pi, s) = mult(\pi)$ where $mult(\pi)$ is the multiplicity of π given in lemma 5.22.

Proof. The statement about the parity follows from the analogous statement for a cover of $\mathbb{TP}^1$ and $mult(\pi, s) = mult(\pi)$ is a direct consequence.

□

Proposition 5.24. For $k, h \in \mathbb{N}$ we have $\mathbb{T}H^{(h,+)}_{(3)^k} = 3^{2h-2}((-1)^k 2^{k+h-1} + 1)$.

As usual, we choose the genus g of the source curve such that the Riemann-Hurwitz formula is satisfies, i.e. $g = 3h + k - 2$. The proof will be a double induction on h and k. We do the base case separately:

Proposition 5.25. The tropical spin Hurwitz numbers counting covers with base of genus 2 and k almost simple ramifications satisfy: $\mathbb{T}H^{(2,+)}_{(3)^k} = 9((-1)^k 2^{k+1} + 1)$.

The proof is similar to the one of proposition 5.18. We want to cut the target curve B (see figure 32) along the central edge and count covers of elliptic curves instead. Relating the multiplicities of the cut covers to the glued one is more involved since the automorphism group of the cover can be more complicated. For this reason we introduce a labelling and count labelled covers instead.

Proof. We construct a tropical analogue of the algebraic degeneration by a simple cutting procedure and prove a tropical degeneration formula for the numbers $\mathbb{T}H^{(2,+)}_{(3)^k}$:

$$\mathbb{T}H^{(2,+)}_{(3)^k} = 3!\mathbb{T}H^{(1,+)}_{(3)^{k-1}}\mathbb{T}H^{(1,+)}_{(3)^1} + 3\mathbb{T}H^{(1,+)}_{(3)^k}\mathbb{T}H^{(1,+)}_{(3)^2} \text{ for } k > 0.$$
$$\mathbb{T}H^{(2,+)}_{(3)^0} = 3!\mathbb{T}H^{(1,+)}_{(3)^0}\mathbb{T}H^{(1,+)}_{(3)^0} + 3\mathbb{T}H^{(1,+)}_{(3)^1}\mathbb{T}H^{(1,+)}_{(3)^1} \text{ for } k = 0.$$

The results from subsection 5.1.1 then yield proposition 5.25.
1. Labelling. We consider the following labelled base curves: If $k = 0$, the curve B is a maximally degenerate genus 2 curve as in figure 32 a). If $k \geqslant 1$, we require the k ends to be distributed as in figure 32 b).

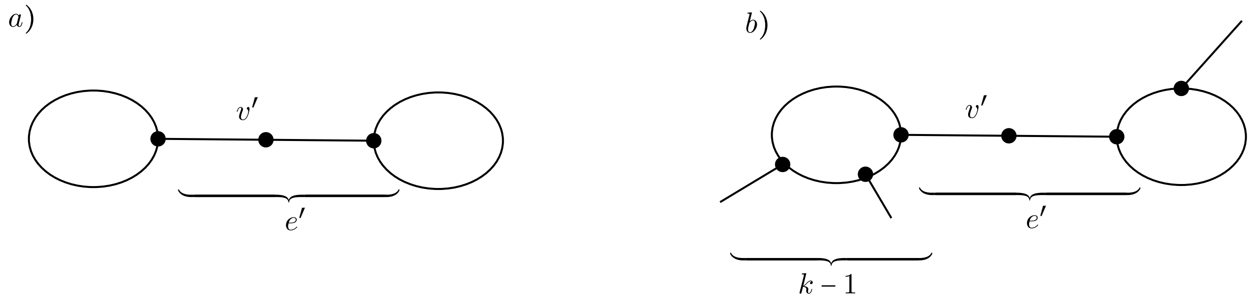


Figure 32: Labelled base curve of genus 2.

Let $v' \in e'$ be a point on the central edge. For a degree 3 spin Hurwitz cover $(\pi, s) : (\Gamma, s) \to (B, s_B)$ (Γ is of genus $k+4$) we consider the cover (π^l, s^l), where the preimages of v' are labelled. Let m be the partition of 3 that corresponds to $\pi^{-1}(e')$. We have two possible cases:

1. If $m = (3)$, labelled and unlabelled coverings are in one-to-one correspondence.
2. If $m = (1,1,1)$, an unlabelled cover (π, s) yields $\frac{|Aut(1,1,1)|}{|Aut(G)|}$ non-isomorphic labelled ones where $G := Aut(\pi, s)/Aut(\pi^l, s^l)$ is the quotient group and (π^l, s^l) fixed.

If we count labelled covers instead of unlabelled ones, we have to divide by the number of labellings each cover (π, s) induces, i.e. by $\frac{|Aut(1,1,1)|}{|Aut(G)|}$, if $m = (1,1,1)$, and by 1, if $m = (3)$:

$$\begin{aligned}
\mathbb{T}H^{(2,+)}_{(3)^k} &= \sum_{(\pi,s)} (-1)^{p(\pi,s)} mult(\pi, s) \\
&= \sum_{(\pi,s)} (-1)^{p(\pi,s)} \frac{1}{|Aut(\pi, s)|} \prod_{e \in E(\Gamma)} \omega(e) \prod_{v \in V(\Gamma)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi, s), v) \\
&= \sum_{\substack{(\pi^l,s^l) \\ m=(1,1,1)}} \frac{(-1)^{p(\pi^l,s^l)}}{|Aut(1,1,1)||Aut(\pi^l, s^l)|} \prod_{e \in E(\Gamma^l)} \omega(e) \prod_{v \in V(\Gamma^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi, s), v) \\
&+ \sum_{\substack{(\pi^l,s^l) \\ m=(3)}} (-1)^{p(\pi^l,s^l)} \frac{1}{|Aut(\pi^l, s^l)|} \prod_{e \in E(\Gamma^l)} \omega(e) \prod_{v \in V(\Gamma^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi, s), v),
\end{aligned}$$

where the product $\prod_{e \in E(\Gamma)} \omega(e)$ goes only over bounded edges and $p(\pi^l, s^l) = p(\pi, s)$ since labelling does not affect the parity function. Anticipating the cutting procedure we rearrange the factors as follows: Factors that contribute to the left part of Γ^l (to the right part, respectively) get a subscript 1 (respectively 2)(see figure 33).

$$\begin{aligned}
\mathbb{T}H^{(2,+)}_{(3)^k} = & \sum_{\substack{(\pi^l,s^l)\\ m=(1,1,1)}} \frac{1}{|Aut(1,1,1)| \cdot |Aut(\pi^l,s^l)|} \cdot \\
& (-1)^{\sum_{v\in V(\Gamma^l_1)} s^l(v)} \prod_{e\in E(\Gamma^l_1)\text{bounded}} \omega(e) \prod_{v\in V(\Gamma^l_1)} n^v_1! n^v_2! n^v_3! H^{(0,+)}((\pi,s),v) \cdot \\
& (-1)^{\sum_{v\in V(\Gamma^l_2)} s^l(v)} \prod_{e\in E(\Gamma^l_2)\text{bounded}} \omega(e) \prod_{v\in V(\Gamma^l_2)} n^v_1! n^v_2! n^v_3! H^{(0,+)}((\pi,s),v) + \\
& \sum_{\substack{(\pi^l,s^l)\\ m=(3)}} \frac{3}{|Aut(\pi^l,s^l)|} (-1)^{\sum_{v\in V(\Gamma^l_1)} s^l(v)} \prod_{e\in E(\Gamma^l_1)\text{bounded}} \omega(e) \prod_{v\in V(\Gamma^l_1)} n^v_1! n^v_2! n^v_3! H^{(0,+)}((\pi,s),v) \cdot \\
& (-1)^{\sum_{v\in V(\Gamma^l_2)} s^l(v)} \prod_{e\in E(\Gamma^l_2)\text{bounded}} \omega(e) \prod_{v\in V(\Gamma^l_2)} n^v_1! n^v_2! n^v_3! H^{(0,+)}((\pi,s),v),
\end{aligned}$$

where we used that $p(\pi^l,s^l) = \sum_{v\in V(\Gamma^l)} s^l(v) \mod 2$ and thus

$$(-1)^{p(\pi^l,s^l)} = (-1)^{\sum_{v\in V(\Gamma^l_1)} s^l(v)} \cdot (-1)^{\sum_{v\in V(\Gamma^l_2)} s^l(v)}.$$

The factor 3 in line 4 accounts for the bounded edge of weight 3 that connects Γ^l_1 and Γ^l_2.
2. Cutting procedure. Cutting a cover (π^l,s^l) at v' and $(\pi^l)^{-1}(v')$ creates a pair

$$[(\pi^l_1,s^l_1) : (\Gamma^l_1,s^l_1) \to (B_1,s_{B,1}), (\pi^l_2,s^l_2) : (\Gamma^l_2,s^l_2) \to (B_2,s_{B,2})]$$

of covers of $\mathbb{TE}$ where the labelling over the newly created ends is the one inherited from (π^l,s^l) (figure 33) and the parity function on Γ^l_i is just the restriction of s^l to $V(\Gamma^l_i)$ for $i = 1,2$. In fact, we have a bijection between covers (π^l,s^l) and pairs of covers $[(\pi^l_1,s^l_1),(\pi^l_2,s^l_2)]$ with additional labelling that count towards

- <u>Case $k > 0$:</u> $\mathbb{T}H^{(1,+)}_{(3)^k}$ and $\mathbb{T}H^{(1,+)}_{(3)^2}$, if $m = (3)$, and towards $\mathbb{T}H^{(1,+)}_{(3)^{k-1},(1,1,1)}$ and $\mathbb{T}H^{(1,+)}_{(3)^1,(1,1,1)}$, if $m = (1,1,1)$.
- <u>Case $k = 0$:</u> $\mathbb{T}H^{(1,+)}_{(3)}$ and $\mathbb{T}H^{(1,+)}_{(3)}$, if $m = (3)$, and towards $\mathbb{T}H^{(1,+)}_{(1,1,1)}$ and $\mathbb{T}H^{(1,+)}_{(1,1,1)}$, if $m = (1,1,1)$.

The bijective correspondence is clear for $m = (3)$ since gluing two covers along a single edge is unique irrespective of the labelling. If $m = (1,1,1)$, we require π^l_1 and π^l_2 to be glued along matching labels (figure 34), which guarantees uniqueness in this case.

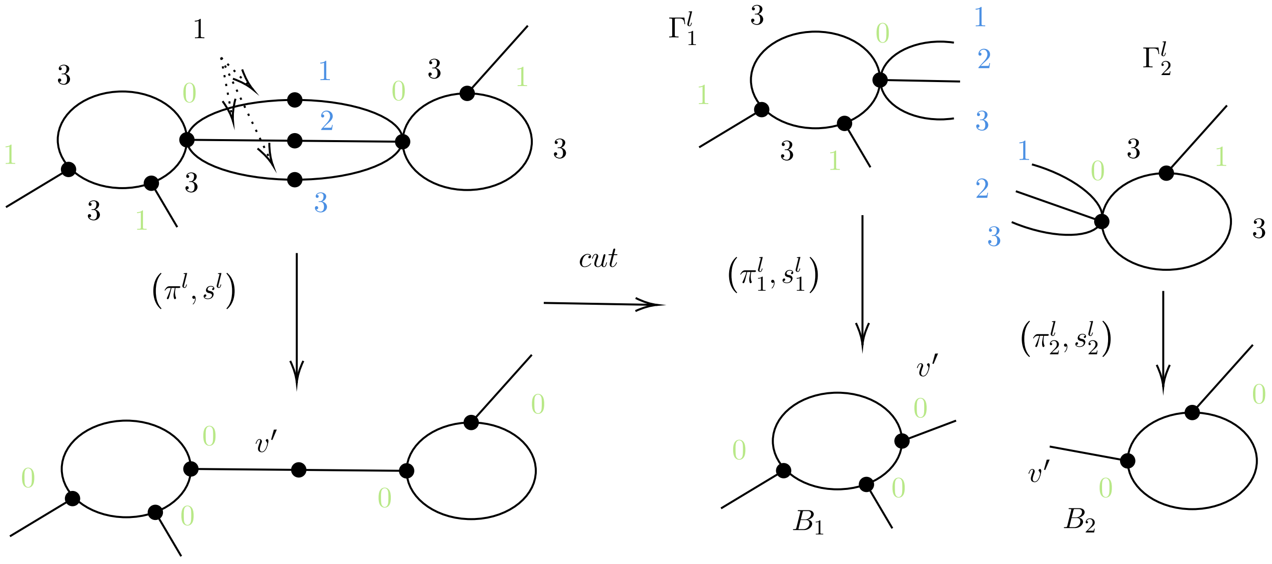


Figure 33: Cutting procedure for $m = (1,1,1)$ with labelling in blue.

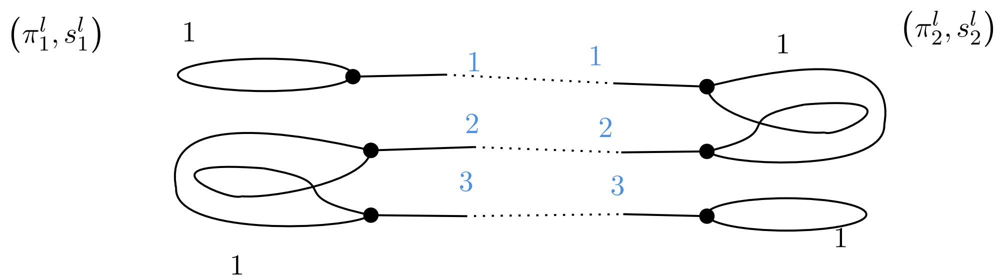


Figure 34: Gluing of two labelled spin Hurwitz covers (case $k = 0$).

3. Relating multiplicities. The bijection justifies summing over pairs $[(\pi_1^l, s_1^l), (\pi_2^l, s_2^l)]$ instead. Moreover, we have $|Aut(\pi^l, s^l)| = |Aut(\pi_1^l, s_1^l)| \cdot |Aut(\pi_2^l, s_2^l)|$ since the labelling makes edges in $\pi^{-1}(e')$ distinguishable. Recognizing

$$(-1)^{\sum_{v \in V(\Gamma_i)} s(v)} = (-1)^{p(\pi_i^l, s_i^l)} \text{ for } i = 1,2$$

yields:

$$\begin{aligned}
\mathbb{T}H^{(2,+)}_{(3)^k} = &\sum_{\substack{[(\pi_1^l,s_1^l),(\pi_2^l,s_2^l)] \\ m=(1,1,1)}} \frac{1}{3!} \cdot \frac{(-1)^{p(\pi_1^l,s_1^l)}}{|Aut(\pi_1^l, s_1^l)|} \prod_{e \in E(\Gamma_1^l)} \omega(e) \prod_{v \in V(\Gamma_1^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v) \cdot \\
&\frac{(-1)^{p(\pi_2^l,s_2^l)}}{|Aut(\pi_2^l, s_2^l)|} \prod_{e \in E(\Gamma_2^l)} \omega(e) \prod_{v \in V(\Gamma_2^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v) + \\
&\sum_{\substack{[(\pi_1^l,s_1^l),(\pi_2^l,s_2^l)] \\ m=(3)}} 3 \cdot \frac{(-1)^{p(\pi_1^l,s_1^l)}}{|Aut(\pi_1^l, s_1^l)|} \prod_{e \in E(\Gamma_1^l)} \omega(e) \prod_{v \in V(\Gamma_1^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v) \cdot \\
&\frac{(-1)^{p(\pi_2^l,s_2^l)}}{|Aut(\pi_2^l, s_2^l)|} \prod_{e \in E(\Gamma_2^l)} \omega(e) \prod_{v \in V(\Gamma_2^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v),
\end{aligned} \tag{A}$$

where products of the type $\prod_{e\in E(\Gamma)} \omega(e)$ go over bounded edges only. We rearrange equation (A) to sum over $p(\pi_1^l, s_1^l)$, respectively (π_2^l, s_2^l), separately and get

$$\frac{1}{3!}\cdot \underbrace{\Big(\sum_{\substack{(\pi_1^l,s_1^l)\\ m=(1,1,1)}} \frac{(-1)^{p(\pi_1^l,s_1^l)}}{|Aut(\pi_1^l,s_1^l)|} \prod_{e\in E(\Gamma_1^l)} \omega(e) \prod_{v\in V(\Gamma_1^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v)\Big)}_{=3!\mathbb{T}H^{(1,+)}_{(3)^{k-1},(1,1,1)} \ (3!\mathbb{T}H^{(1,+)}_{(1,1,1)} \text{ for } k=0)} \cdot$$

$$\underbrace{\Big(\sum_{\substack{(\pi_2^l,s_2^l)\\ m=(1,1,1)}} \frac{(-1)^{p(\pi_2^l,s_2^l)}}{|Aut(\pi_2^l,s_2^l)|} \prod_{e\in E(\Gamma_2^l)} \omega(e) \prod_{v\in V(\Gamma_2^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v)\Big)}_{=3!\mathbb{T}H^{(1,+)}_{(3),(1,1,1)} \ (3!\mathbb{T}H^{(1,+)}_{(1,1,1)} \text{ for } k=0)}$$

for the first summand in equation (A) and

$$3\cdot \underbrace{\Big(\sum_{\substack{(\pi_1^l,s_1^l)\\ m=(3)}} \frac{(-1)^{p(\pi_1^l,s_1^l)}}{|Aut(\pi_1^l,s_1^l)|} \prod_{e\in E(\Gamma_1^l)} \omega(e) \prod_{v\in V(\Gamma_1^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v)\Big)}_{=\mathbb{T}H^{(1,+)}_{(3)^k} \ (\mathbb{T}H^{(1,+)}_{(3)} \text{ for } k=0)} \cdot$$

$$\underbrace{\Big(\sum_{\substack{(\pi_2^l,s_2^l)\\ m=(3)}} \frac{(-1)^{p(\pi_2^l,s_2^l)}}{|Aut(\pi_2^l,s_2^l)|} \prod_{e\in E(\Gamma_2^l)} \omega(e) \prod_{v\in V(\Gamma_2^l)} n_1^v! n_2^v! n_3^v! H^{(0,+)}((\pi,s),v)\Big)}_{=\mathbb{T}H^{(1,+)}_{(3)^2} \ (\mathbb{T}H^{(1,+)}_{(3)} \text{ for } k=0)}$$

for the second summand in equation (A). The equality with the respective spin Hurwitz numbers holds since forgetting the additional labelling yields a factor of 3! for the case $m = (1,1,1)$ and a factor 1 for the case $m = (3)$. In total we have

$$\begin{aligned}\mathbb{T}H^{(2,+)}_{(3)^k} &= 3!\mathbb{T}H^{(1,+)}_{(3)^{k-1}}\mathbb{T}H^{(1,+)}_{(3)^1} + 3\mathbb{T}H^{(1,+)}_{(3)^k}\mathbb{T}H^{(1,+)}_{(3)^2}.\\ (&= 3!\mathbb{T}H^{(1,+)}_{(1,1,1)}\mathbb{T}H^{(1,+)}_{(1,1,1)} + 3\mathbb{T}H^{(1,+)}_{(3)}\mathbb{T}H^{(1,+)}_{(3)} \text{ for } k=0)\end{aligned}$$

Together with proposition 5.18 the result follows: We only present the computation for $k > 1$. The case $k = 0$ is analogous and will be left to the reader:

$$\begin{aligned}\mathbb{T}H^{(2,+)}_{(3)^k} &= 3!\mathbb{T}H^{(1,+)}_{(3)^{k-1}}\mathbb{T}H^{(1,+)}_{(3)^1} + 3\mathbb{T}H^{(1,+)}_{(3)^k}\mathbb{T}H^{(1,+)}_{(3)^2}\\ &= (3!)\cdot((-1)^{k-1}2^{k-1}+1)(-1) + 3((-1)^k 2^k + 1)5\\ &= 3((-1)^k 2^k - 2) + 3((-1)^k 2^k + 1)5\\ &= 18(-1)^k 2^k + 9 = 9((-1)^k 2^{k+1} + 1).\end{aligned}$$

□

Remark 5.26. Choosing B to be a dumbbell graph (example 5.21 a) instead of b)) allows us to construct a tropical analogue of the algebraic degeneration by a simple cutting procedure. Indeed, the central edge enables us to separate the count of coverings of B into the count of covers of two elliptic curves.

Notice that the proof of proposition 5.25 simplifies for $k \geqslant 2$. Due to the simple structure of the automorphism group of a cover Γ (relevant to the count of proposition 5.25), there is always a unique way of gluing a compatible pair (π_1, π_2) to recover Γ. Even if $m = (1,1,1)$, a labelling is not necessary: Automorphisms of Γ that permute the three edges of weight 1 that cover e' only come from double Wiener or *bifurcated double Wiener* and contribute to $Aut(\Gamma)$ with a factor of 3! for each. The multiplicity of the resulting cover is then given by $3! \cdot mult(\pi_1) \cdot mult(\pi_2)$.

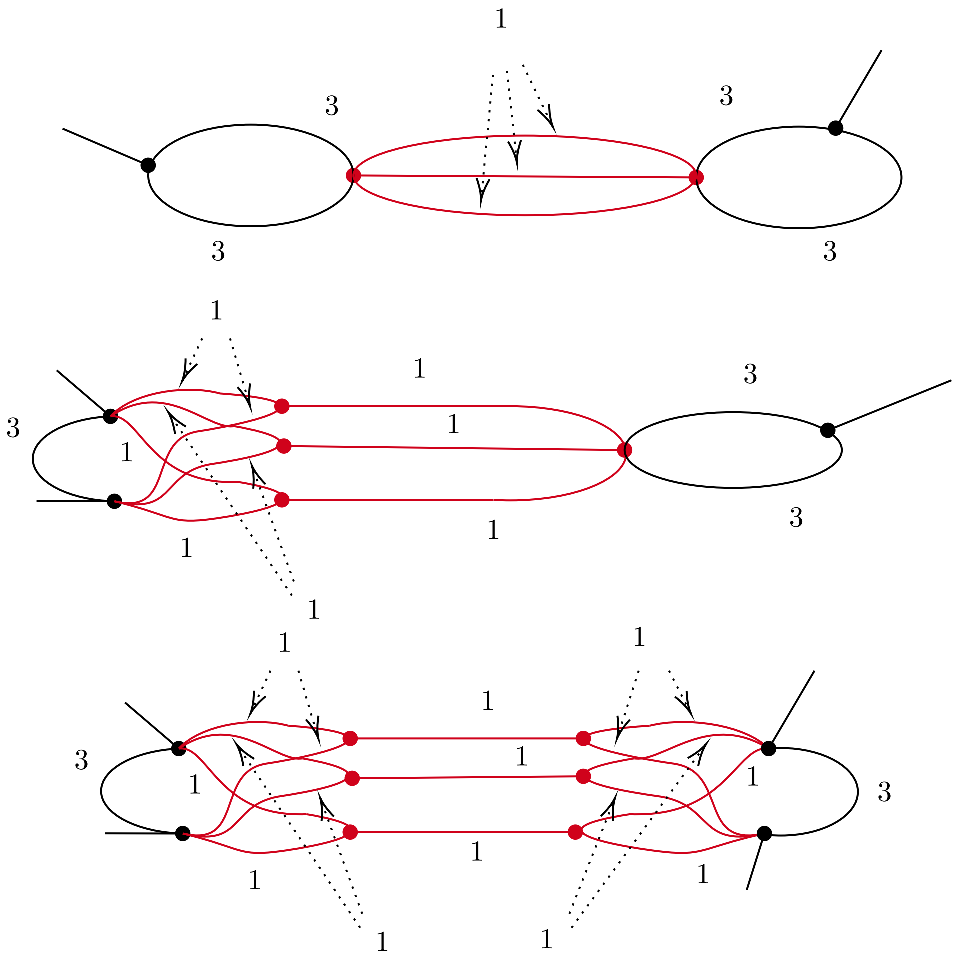


Figure 35: A double Wiener at the top and all possibilities for a bifurcated double Wiener (or its reflections, with the three bifurcations on the right) below.

Proof proposition 5.24. Fix a maximally degenerate genus h curve B as in figure 36, that is B is an ant graph consisting of h circles connected by $h-1$ edges. If $k \geqslant 1$, we distribute k ends onto the circles of B such that the last $h-1$ have $k-1$ ends attached and the first only one.

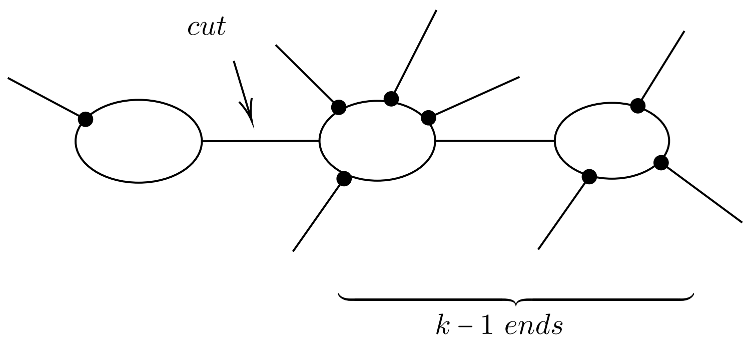


Figure 36: Ant graph of genus 3.

We prove proposition 5.24 by applying induction on h. The base case $h = 2$ has been proven in proposition 5.25. Now, suppose that proposition 5.24 holds for $h-1$ and all k, we have to show $\mathbb{T}H^{(h,+)}_{(3)^k} = 3^{2h-2}((-1)^k 2^{k+h-1} + 1)$ for all k. As in the proof of proposition 5.25 we can count pairs of tropical spin Hurwitz covers of a curve of genus $h-1$ and an elliptic curve instead. To every $(\pi, s) : (\Gamma, s) \to (B, s_B)$ we associate a pair $[(\pi_1, s_1), (\pi_2, s_2)]$ by simultaneously cutting B along the first intermediate edge and Γ along its preimage. With the same arguments we can show that analogous degeneration formulas hold. The rest follows by induction:

$$\begin{aligned}\mathbb{T}H^{(h,+)}_{(3)^k} &= (3!)\mathbb{T}H^{(h-1,+)}_{(3)^{k-1}}\mathbb{T}H^{(1,+)}_{(3)^1} + 3\mathbb{T}H^{(h-1,+)}_{(3)^k}\mathbb{T}H^{(1,+)}_{(3)^2}\\ &= (3!)\cdot 3^{2(h-1)-2}((-1)^{k-1}2^{k-1+h-1-1}+1)(-1) + 3\cdot 3^{2(h-1)-2}((-1)^k 2^{k+h-1-1}+1)5\\ &= 3^{2h-3}((-1)^k 2^{k+h-2} - 2) + 5\cdot 3^{2h-3}((-1)^k 2^{k+h-2}+1)\\ &= 6\cdot 3^{2h-3}(-1)^k 2^{k+h-2} - 2\cdot 3^{2h-3} + 5\cdot 3^{2h-3} = 3^{2h-2}((-1)^k 2^{k+h-1}+1),\end{aligned}$$

for $k \geqslant 1$ and

$$\mathbb{T}H^{(h,+)}_{(3)^0} = (3!)\mathbb{T}H^{(h-1,+)}_{(3)^0}\mathbb{T}H^{(1,+)}_{(3)^0} + 3\mathbb{T}H^{(h-1,+)}_{(3)^1}\mathbb{T}H^{(1,+)}_{(3)^1} = 3^{2h-2}(2^{h-1}+1),$$

for $k = 0$. □

5.1.3 Degree 3 with genus 1 base and odd theta characteristic.

We let the target curve B be a straight line with k vertices and k ends such that the first $k-1$ are tri-valent genus 0 vertices and the last one, v', has genus 1 (figure 23) with parity function

$$s_B : V(B) \to \mathbb{Z}/2\mathbb{Z}, v \mapsto \begin{cases} 0, & g(v) = 0 \\ 1, & g(v) = 1 \end{cases}.$$

Describtion of tropical spin covers of (B, s_B). Let $\pi : \Gamma \to B$ be a tropical cover such that the edge weights of Γ are odd. By simultaneously cutting B along the bounded edge that is adjacent to v' and Γ along its preimage, we obtain a tropical Hurwitz cover of $\mathbb{TP}^1$ together with a (possibly disconnected) cover of the one-valent star graph B'_v with genus 1 vertex v'. Both coverings can be analysed separately. We endow the first with a parity function as in definition 5.3. The second can be reconstructed from local Hurwitz numbers (as in subsection 5.1.2) of the form $H_v = H_{g(v)\xrightarrow{d(v)}1}(n_v)$ where n_v is an odd partition of $d(v)$. Assigning parities involves the computation of the corresponding algebraic spin Hurwitz numbers.

Lemma 5.27. Let $\pi : \Gamma \to B$ be a tropical Hurwitz cover of degree 3 such that the edge weights of Γ are odd and B is as above. All covers $\pi_{v'} : \Gamma_{v'} \to B'_v$ are listed below:

1. $\Gamma_{v'}$ is a vertex of genus 2 with one end of weight 3 (figure 37, type 1),
2. $\Gamma_{v'}$ is a vertex of genus 1 with three ends of weight 1 (figure 37, type 2),
3. $\Gamma_{v'}$ is a disjoint union of a vertex of genus 1 with two ends of weight 1 and a vertex of genus 1 with one end of weight 1 (figure 37, type 3),
4. $\Gamma_{v'}$ is a disjoint union of three vertices of genus 1 each with an edge of weight 1 (figure 37, type 4).

Then Γ is obtained by gluing $\Gamma_{v'}$ to a tropical 3-cycle cover $\tilde{\Gamma}$ as in definition 3.10 whose ramification profile over the left end is (3) and over the right end is either (3) (if $\Gamma_{v'}$ is of type 1) or $(1,1,1)$ (for type 2-4).

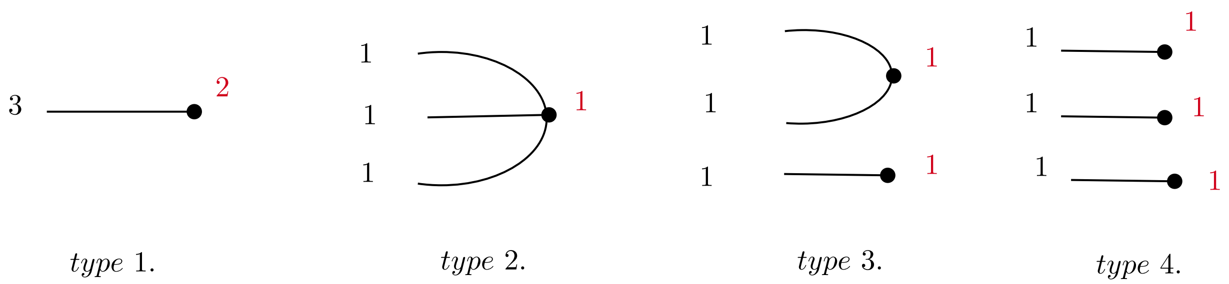


Figure 37: All combinatorial types for $\Gamma_{v'}$ numbered according to lemma 5.27 with genus function in red.

Proof. For the first part (statement 1-4) we compute all Hurwitz numbers of the form $H^{\bullet}_{g\xrightarrow{3}1}(n)$ where n is an odd partition of 3 and g depends on n. There are two possibilities $n = (3)$ and $n = (1,1,1)$ with corresponding Hurwitz numbers:

$$H^{\bullet}_{1\xrightarrow{3}1}((1,1,1)) \text{ and } H^{\bullet}_{2\xrightarrow{3}1}((3)).$$

First, note that a Hurwitz cover that contributes to $H^{\bullet}_{2\xrightarrow{3}1}((3))$ has a connected source curve, hence $H^{\bullet}_{2\xrightarrow{3}1}((3)) = H_{2\xrightarrow{3}1}((3))$ and we have only one tropical picture (type 1). For $H^{\bullet}_{1\xrightarrow{3}1}((1,1,1))$ the source curve may be disconnected. This gives rise to the remaining three possibilities (type 2-4).
We compute $H_{2\xrightarrow{3}1}((3))$ tropically (see figure 38):

$$H_{2\xrightarrow{3}1}((3)) = 1 + 2 = 3.$$

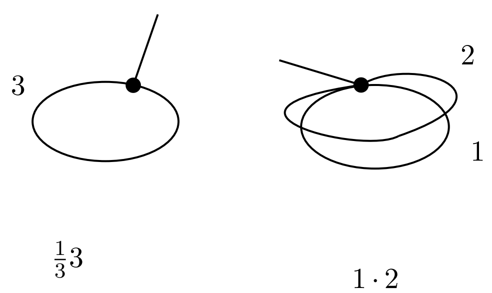


Figure 38: Computation of $\mathbb{T}H_{2\xrightarrow{3}1}((3))$.

For $H^{\bullet}_{1\xrightarrow{3}1}((1,1,1))$ we count monodromy representations, i.e. tuple $(\sigma_1,\sigma_2) \in {S_3}^2$ such that ${\sigma_1}{\sigma_2}{\sigma_1}^{-1}{\sigma_2}^{-1} = id$. We have

$$H^{\bullet}_{1\xrightarrow{3}1}((1,1,1)) = \frac{1}{3!} + \frac{9}{3!} + \frac{8}{3!} = 3,$$

where we arranged the sum to match the tropical pictures: The first summand corresponds to the tuple (id, id) (figure 37, type 4), the second to tuples where either σ_1 or σ_2 or both are transpositions (figure 37, type 3) and the third to tuples where either σ_1 or σ_2 or both are 3-cycles (figure 37, type 2).
The last statement about Γ is clear from the discussion above lemma 5.27.

□

Remark 5.28. Alternatively, a tropical cover as in lemma 5.27 can be obtained from a tropical Hurwitz cover of B' by simultaneously contracting the cycle on the base curve and its preimage (figure 39).

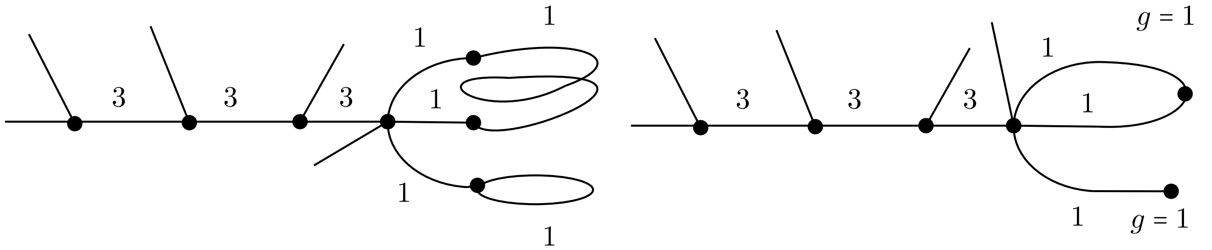


Figure 39: A tropical cover of B' before contraction on the left, after contraction on the right. Only bounded edges are labelled with weights.

Lemma 5.29. A cover $\pi : \Gamma \to B$ as in lemma 5.27 defines a unique admissible parity function on Γ:

$$s(v) := \begin{cases} 0, & g(v) = 0 \\ 1, & g(v) \geqslant 1 \end{cases} \text{ with } p(\pi, s) = \sum_{v \in V(\Gamma)} s(v) \, mod \, 2.$$

Proof. Let s be an admissible parity function. From section 5.1.1 we know that the restriction of s on $V(\Gamma) \smallsetminus \pi^{-1}(v')$ is unique. Consider the (possibly disconnected) open cover $\pi_{v'}$ from $\Gamma_{v'}$ to $B_{v'}$ where $\Gamma_{v'}$ is as in figure 37 type 1-4. Let $f : C \to E$ be a Hurwitz cover dual to $\pi_{v'}$ where E is a smooth curve of genus 1. We endow E with the only odd theta characteristic[5], the trivial bundle $N = 0$ which also happens to be the canonical bundle of E. If $\Gamma_{v'}$ is of type 2-4, f is an unramified map (or unramified tuple of maps) whose source curve is also of genus 1. Without loss of generality we call potential component maps again f. We have $\frac{\mathcal{R}_f}{2} = 0$ since f is unramified and, hence, by theorem 2.19

$$L_f = f^*(N) \otimes \mathcal{O}(\frac{\mathcal{R}_f}{2}) \cong f^*(\omega_E) \otimes \mathcal{O}(\mathcal{R}_f) \cong \omega_C.$$

Thus, $p(f) = h^0(C, \omega_C) \mod 2 = 1$ since C has genus 1 as well.
If $\Gamma_{v'}$ is of type 1, f counts towards $H_{2\xrightarrow{3}1}((3))$. Note that in this case the Hurwitz and spin Hurwitz number differ only by a sign, i.e. $H_{2\xrightarrow{3}1}((3)) = 3 = -H^{(1,-)}_{(3)}$. We conclude that $p(f) = 1$ is the only possible parity. □

[5]This is true since E has genus 1.

By inspection of the proof of lemma 5.27 we have the following:

Lemma 5.30. The multiplicity of $(\pi, s) : \Gamma \to B$ is given by $mult(\pi, s) =$

$$\frac{1}{|Aut(\Gamma, s)|} \prod_{e \in E(\Gamma)\text{bounded}} \omega(e) \frac{1}{3}^{|\{v : g(v)=1\}|} 2^{I} \prod_{v \in \pi^{-1}(v')} n^{v}!\, H^{(1,-)}((\pi, s), v),$$

where I is the set of vertices as in figure 31 (2) and all possible local spin Hurwitz numbers $H^{(1,-)}((\pi, s), v)$ are listed below.

- $H^{(1,-)}((\pi, s), v) = 3$, if $v \in \pi^{-1}(v')$ and $\pi^{-1}(v')$ is of type 1 (figure 37).
- $H^{(1,-)}((\pi, s), v) = \frac{8}{3!}$, if $v \in \pi^{-1}(v')$ and $\pi^{-1}(v')$ is of type 2 (figure 37).
- $H^{(1,-)}((\pi, s), v) = \frac{9}{3!}$, if $v \in \pi^{-1}(v')$ and $\pi^{-1}(v')$ is of type 3 (figure 37).
- $H^{(1,-)}((\pi, s), v) = 1$, if $v \in \pi^{-1}(v')$ and $\pi^{-1}(v')$ is of type 4 (figure 37).

Proof. Since an admissible parity function on Γ is unique we have

- $H^{(0,+)}((\pi, s), v) = H_v$ for each vertex $v \in V(\Gamma) \setminus \pi^{-1}(v')$.
- $H^{(1,-)}((\pi, s), v) = H_{g(v) \xrightarrow{d(v)} 1}(n_v)$ for $v \in \pi^{-1}(v')$.

Hurwitz numbers corresponding to the first case were already computed in section 5.1.1. The second follows by inspection of the proof of lemma 5.27. □

We have a count analogous to the one in [LP13] in the tropical world.

Proposition 5.31. For $k \in \mathbb{N}$ we have $\mathbb{T}H^{(1,-)}_{(3)^k} = (-1)^k 2^k - 1$.

Example 5.32. We compute

$$\mathbb{T}H^{(1,-)}_{(3)^2} = 3 \cdot \frac{1}{3} \cdot 3 - 2 \cdot \frac{1}{3!} + 2 \cdot \frac{9}{3!} - 2 \cdot \frac{8}{3!} = 3.$$

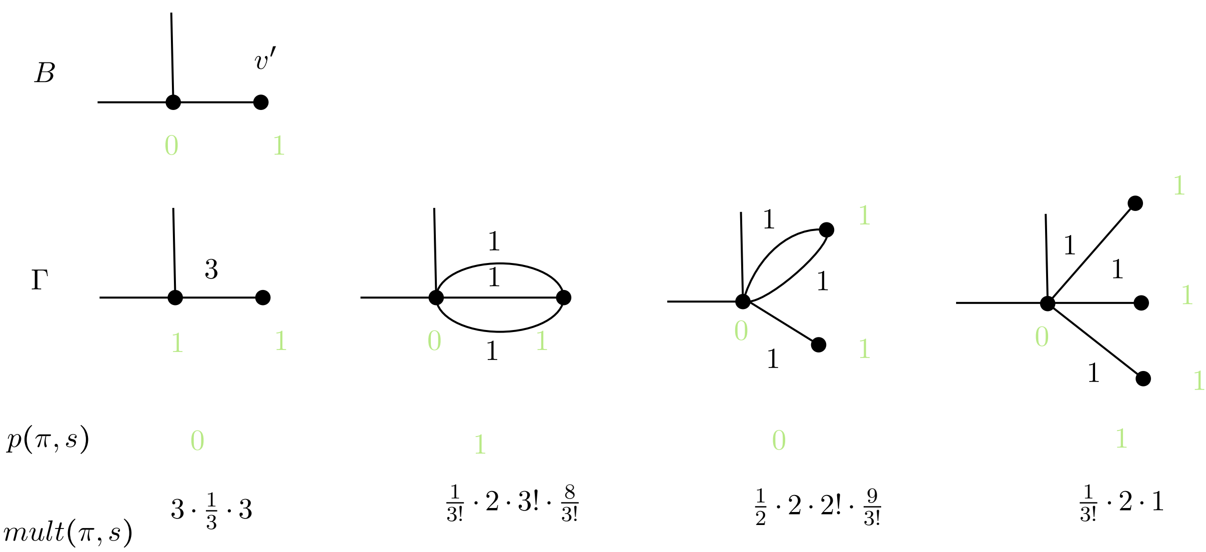


Figure 40: Combinatorial type of curves with parity and multiplicity that contribute to $\mathbb{T}H^{(1,-)}_{(3)^2}$. We omit weighting on the ends of Γ to avoid cluttering the picture.

The main ingredient for the proof of proposition 5.31 is the following observation:

Lemma 5.33. Let T_k be the set of tropical spin Hurwitz covers $(\pi, s) : (\Gamma, s) \to (B, s_B)$ such that (π, s) contributes to $\mathbb{T}H^{(1,-)}_{(3)^k}$ and $\pi^{-1}(v')$ is of type 2-4 (lemma 5.27) where v' is the genus 1 of B. Then

$$\sum_{(\pi,s)\in T_k} (-1)^{p(\pi,s)} \cdot mult(\pi, s) = 0.$$

For reason of practicality we rewrite the multiplicity of (π, s) in lemma 5.30 as

$$mult(\pi, s) = \frac{3!}{|Aut(\tilde{\Gamma}, \tilde{s})|} \prod_{e\in E(\Gamma)\text{bounded}} \omega(e) \frac{1}{3}^{|\{v:g(v)=1\}|} 2^I \cdot e_i,$$

where

- $\tilde{\Gamma} := \Gamma \smallsetminus \pi^{-1}(v')$ and $\tilde{s}$ is the restriction of s to $\tilde{\Gamma}$.
- $i = 1$ with $e_1 = 3$, if $\pi^{-1}(v')$ is as in figure 37 (type 1).
- $i = 2$ with $e_2 = \frac{8}{3!}$, if $\pi^{-1}(v')$ is as in figure 37 (type 2).
- $i = 3$ with $e_3 = \frac{9}{3!}$, if $\pi^{-1}(v')$ is as in figure 37 (type 3).
- $i = 4$ with $e_4 = \frac{1}{3!}$, if $\pi^{-1}(v')$ is as in figure 37 (type 4).

Proof. Let $i = 2, 3, 4$. We introduce the following notation: For an open cover of $B_{v'}$ of type i (lemma 5.29) we write $(\pi^i_{v'} : \Gamma^i_{v'} \to B_{v'}, s^i_{v'})$ such that $\Gamma^i_{v'}$ is of type i and $s^i_{v'}$ is given by $s^i_{v'}(v) = 1$ for all $v \in V(\Gamma^i_{v'})$. Observe that gluing a spin Hurwitz cover $(\tilde{\pi}, \tilde{s})$ of $\mathbb{TP}^1$ that counts towards $\mathbb{T}H^{(0,+)}_{(3)^k,(1,1,1)}$ to a cover $(\pi^i_{v'}, s^i_{v'})$ gives rise to a cover in T_k and that all covers in T_k arise in this way. Denote by (π_i, s_i) the resulting cover. We have

$$(-1)^{p(\pi_i,s_i)} mult(\pi_i, s_i) = (-1)^{p(\tilde{\pi},\tilde{s})} mult(\tilde{\pi}, \tilde{s}) \cdot (3!) \cdot \tilde{e}_i,$$

where $\tilde{e}_i := (-1)^{p((\pi^i_{v'},s^i_{v'}))} e_i$ is the signed weight of type i and 3! appears seeing that by gluing we loose a double balanced fork (the ends of $\tilde{\Gamma}$ over $+\infty$). With $\sum_{i=2}^4 \tilde{e}_i = 0$ we get

$$\begin{aligned}\sum_{(\pi,s)\in T_k} (-1)^{p(\pi,s)} \cdot mult(\pi, s) &= \sum_{i=2}^{4} \sum_{(\tilde{\pi},\tilde{s})} (-1)^{p(\pi_i,s_i)} mult(\pi_i, s_i)\\ &= \sum_{(\tilde{\pi},\tilde{s})} (-1)^{p(\tilde{\pi},\tilde{s})} mult(\tilde{\pi}, \tilde{s}) \cdot (3!) \cdot (\tilde{e}_1 + \tilde{e}_2 + \tilde{e}_3 + \tilde{e}_4) = 0.\end{aligned}$$

□

Proof of proposition 5.31. By lemma 5.33 we only have to count covers where $\pi^{-1}(v')$ is a vertex of type 1 (lemma 5.27). These arise from gluing a spin Hurwitz cover $(\tilde{\pi}, \tilde{s})$ of $\mathbb{TP}^1$ relevant to $\mathbb{T}H^{(0,+)}_{(3)^{k+1}}$ to a cover $(\pi_{v'}, s_{v'})$ of type 1 along a bounded edge of weight 3. The multiplicity and parity changes accordingly:

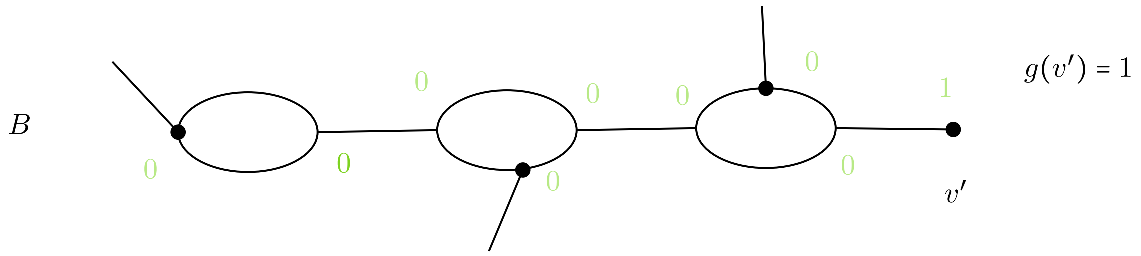
$$mult(\pi, s) = 3^2 mult(\tilde{\pi}, \tilde{s}) \text{ and } p(\pi, s) = (1 + p(\tilde{\pi}, \tilde{s})) \mod 2.$$

Together with proposition 5.17 we have:

$$\begin{aligned}\mathbb{T}H^{(1,-)}_{(3)^k} &= \sum_{(\pi,s)\in\mathbb{T}\mathcal{S}} (-1)^{p(\pi,s)} \cdot mult((\pi,s)) \\ &= \sum_{(\tilde{\pi},\tilde{s})\in M_{k+1}} -3^2(-1)^{p(\tilde{\pi},\tilde{s})} \cdot mult(\tilde{\pi},\tilde{s}) \\ &= -3^2 \cdot \frac{1}{9}((-1)^{k+1}2^k + 1) = (-1)^k 2^k - 1.\end{aligned}$$

□

Remark 5.34. By considering a target curve B of genus $h \geqslant 1$ whose combinatorial type is as shown below for $h = 4$, we see that the computation of $\mathbb{T}H^{(h,-)}_{(3)^k}$ is forced to run along the same patterns as the one of $\mathbb{T}H^{(h,+)}_{(3)^k}$ in subsection 5.1.2.



5.1.4 Degree d with genus 0 base curve and at most 4 branch points

Let us consider tropical spin Hurwitz cover of $\mathbb{TP}^1$ of arbitrary degree d with $k \geqslant, 4$ almost simple ramifications. Taking advantage of the well known structure of tropical 3-cycle Hurwitz covers ([Hah14]) we notice that only for 2-valent genus 1 vertices with edge weight $\omega(e) > 3$ the assignment of a parity is not clear from previous analysis. In particular, we do not know whether it is unique or not. However, we can circumvent this problem by restricting the number of branch points to at most 4. These numbers were not computed by Lee and Parker in [LP13] and, as far as the author knows, are also new to the classical world.

Example 5.35. Let $h = 0$ and $d = 5$ with $k = 4$ almost simple ramifications. Figure 41 shows all tropical spin covers that contribute to $\mathbb{T}H^{(0,+)}_{(311)^4}$. Note that these consist of vertices with parity and weight known from the degree 3 case. Hence, we can compute

$$\mathbb{T}H^{(0,+)}_{(311)^4} = \frac{1}{6} - 1 - 1 + 3 + 3 + 5 + \frac{1}{3} + 2 + 1 = \frac{25}{2}.$$

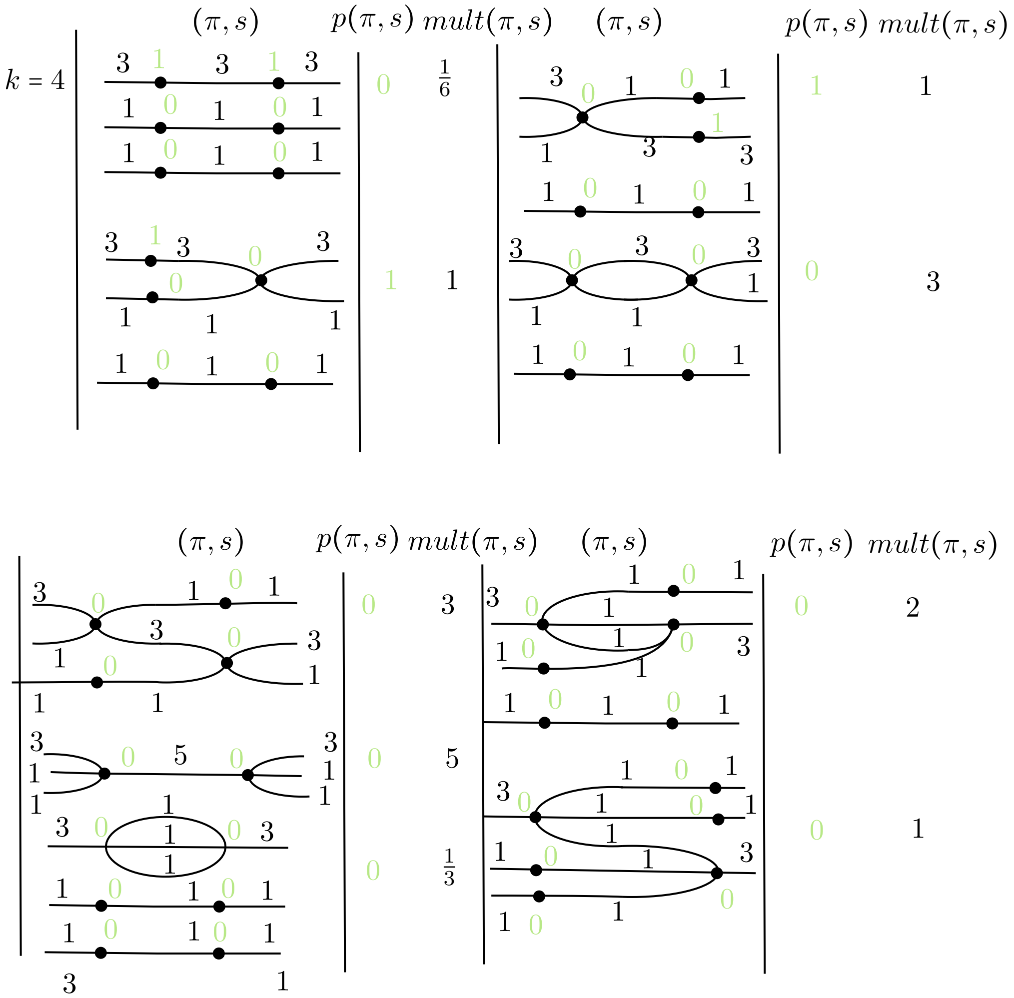
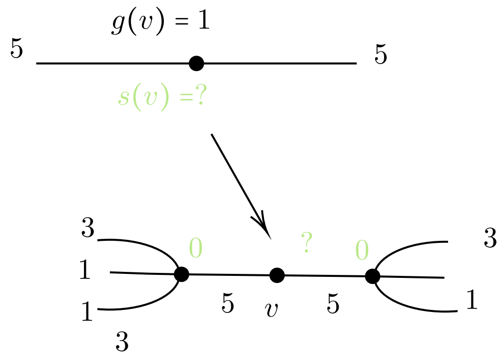


Figure 41: Tropical computation of $\mathbb{T}H^{(0,+)}_{(311)^4}$.

For $k > 4$ a cover might have an inner vertex whose parity (possibly parities) and weight is not known yet.



Proposition 5.36. We have

- $\mathbb{T}H^{(0,+)}_{(31...1)^0} = \frac{1}{d!}$ and $\mathbb{T}H^{(0,+)}_{(31...1)^1} = 0$ for all d.
- $\mathbb{T}H^{(0,+)}_{(31...1)^2} = \frac{1}{3(d-3)!}$ and $\mathbb{T}H^{(0,+)}_{(31...1)^3} = \frac{3d-10}{3(d-3)!}$ for $d \geqslant 4$.

- $\mathbb{T}H^{(0,+)}_{(31\ldots1)^4} = \frac{2d^3+57d^2-446d+780}{9(d-3)!}$ for $d \geqslant 6$.

Proof. If $k = 1$, the set of relevant spin Hurwitz covers $\mathbb{T}S$ is empty. Figure 42 shows the computation for $k = 0$ and $k = 2$.

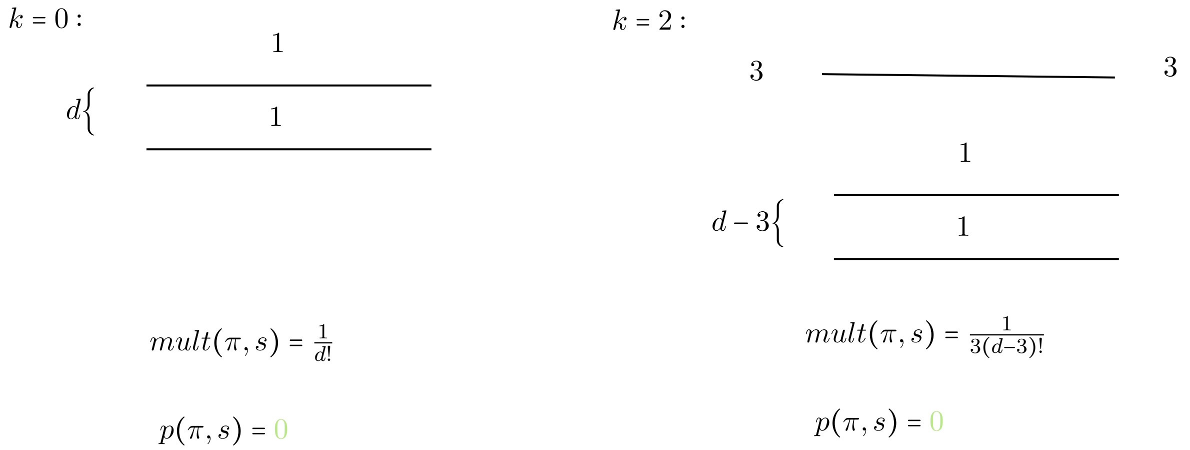
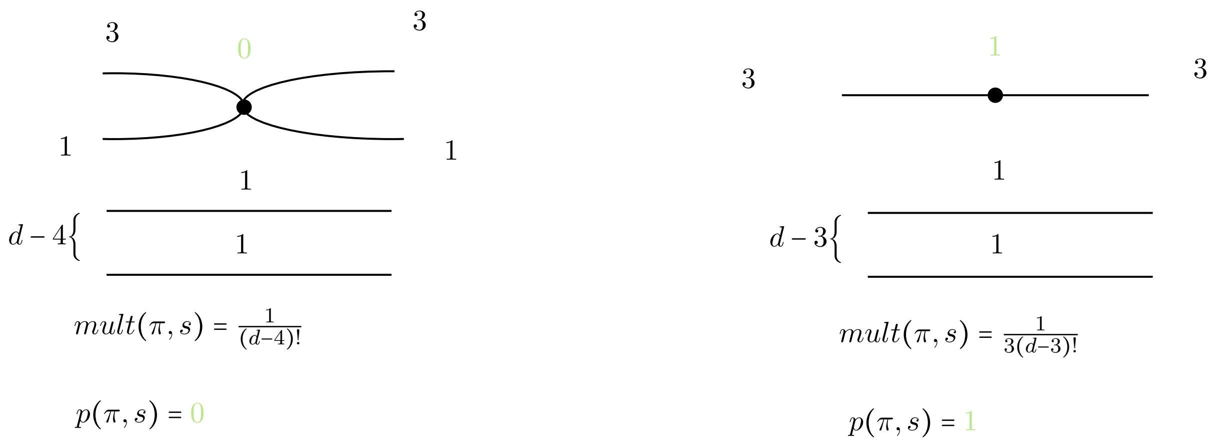


Figure 42: Computation of $\mathbb{T}H^{(0,+)}_{(31\ldots1)^0}$ and $\mathbb{T}H^{(0,+)}_{(31\ldots1)^2}$.

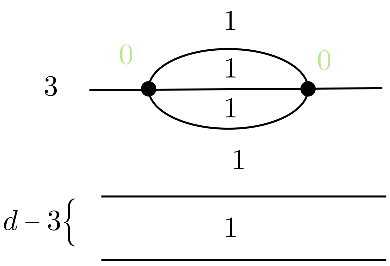
If $d \geqslant 4$, the two tropical spin curves (together with their multiplicity) that contribute to $\mathbb{T}H^{(0,+)}_{(31\ldots1)^3}$ are shown below. We conclude $\mathbb{T}H^{(0,+)}_{(31\ldots1)^3} = \frac{1}{(d-4)!} + \frac{-1}{3(d-3)!} = \frac{3d-10}{3(d-3)!}$.



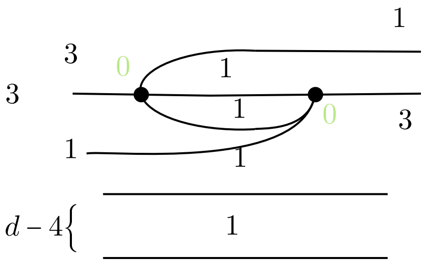
For $\mathbb{T}H^{(0,+)}_{(31\ldots1)^4}$ we notice that example 5.35 contains essentially all relevant tropical spin Hurwitz covers. Let $B = \mathbb{T}\mathbb{P}^1$ with two inner vertices and s_B the parity function from subsection 5.1.1.
Construction of $(\pi, s) : (\Gamma, s) \to (B, s_B)$. We start with a strand of weight 3 and $(d-3)$ strands of weight 1 over $-\infty$. Above the first vertex the following options are available to us, i.e. we can

1. split the strand of weight 3.
2. form a butterfly vertex.
3. form a genus 1 vertex.
4. join 3 strands.

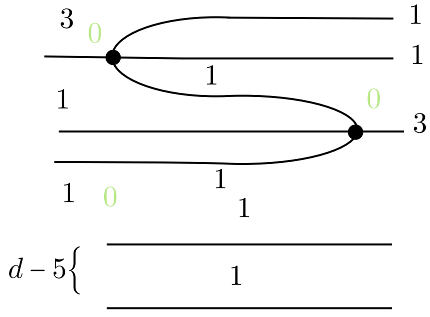
Option 1. After splitting the strand of weight 3 into three strands of weight 1, we are forced to join three strands of weight 1 above the second vertex to create the ramification profile $(3, 1, ..., 1)$ over $+\infty$. All covers with multiplicity and parity are shown below



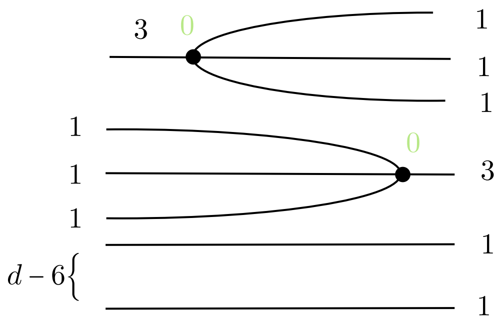
$$mult(\pi, s) = \frac{2^2}{3!(d-3)!}$$



$$mult(\pi, s) = \frac{2^2}{2(d-4)!}$$

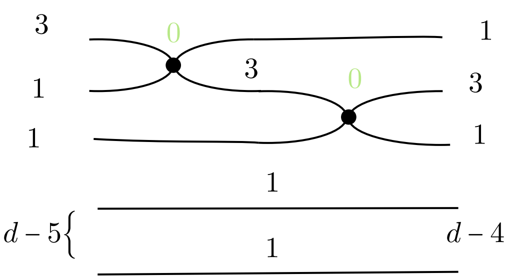


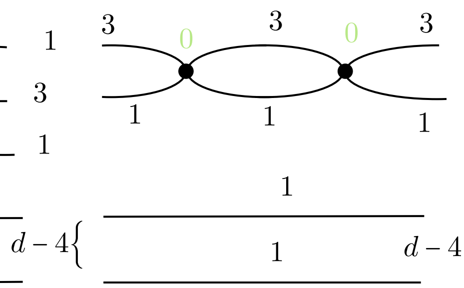
$$mult(\pi, s) = \frac{2^2}{2^2(d-5)!}$$

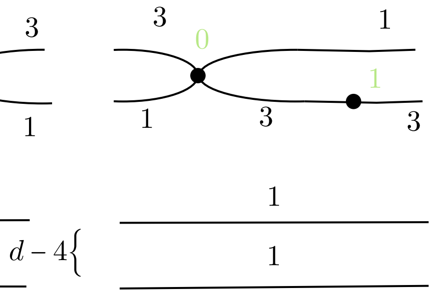


$$mult(\pi, s) = \frac{2^2}{(3!)^2(d-6)!}$$

Option 2. The only possibility for a butterfly vertex of non-zero multiplicity has in-going and out-going edges of weight 3 and 1. Above the second vertex we are left with the choice of either creating a genus one vertex on the strand of weight 3 or forming another butterfly vertex.

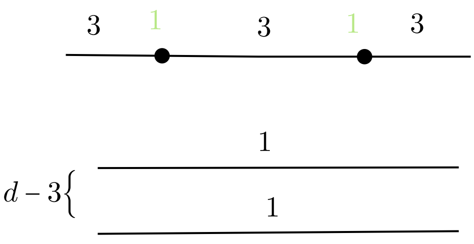


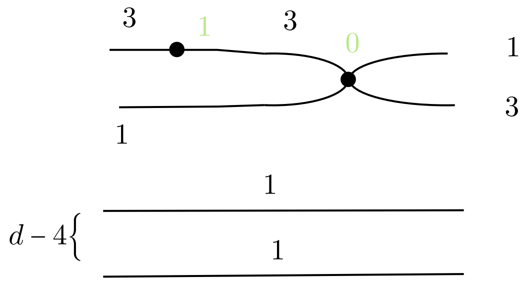




$$mult(\pi, s) = \frac{3}{(d-5)!} \qquad mult(\pi, s) = \frac{3}{(d-4)!} \qquad mult(\pi, s) = \frac{3}{3(d-4)!}$$

Option 3. After creating a vertex of genus 1 on the strand of weight 3, we can either create another genus 1 vertex or form a butterfly vertex above the second branch point.

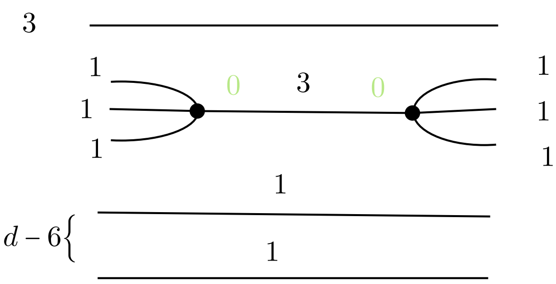


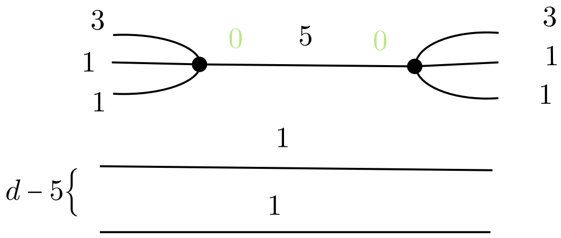


$mult(\pi, s) = \frac{3}{3^2(d-3)!}$ $\qquad$ $mult(\pi, s) = \frac{3}{3(d-4)!}$

Option 4.1. We join 3 strands of weight 1. Above the second branch point we can either split this strand into 3 strands of weight 1 or choose to split the other strand of weight 3 and obtain a cover as in option 1 with two disconnected balanced double forks.

Option 4.2. We join 3 strands, two of weight 1 and one of weight 3. The only possibility above the second vertex is to split again.





$mult(\pi, s) = \frac{2^2 \cdot 3}{3 \cdot (3!)^2 (d-6)!}$ $\qquad$ $mult(\pi, s) = \frac{2^2 \cdot 5}{2^2 (d-5)!}$

Adding all contributions together and taking into account the sign provided by the parity functions (in green) yields the claim.

□

5.2 Correspondence theorems

In this section, we state and prove two correspondence theorems, theorem 5.37 and 5.41, declaring the equality of spin Hurwitz numbers as defined in 4.2 to their tropical counterparts defined in 5. We can use theorem 5.37 and the computations in proposition 5.36 to conclude

- $H^{(0,+)}_{(31...1)^0} = \frac{1}{d!}$ and $H^{(0,+)}_{(31...1)^1} = 0$ for all d.
- $H^{(0,+)}_{(31...1)^2} = \frac{1}{3(d-3)!}$ and $H^{(0,+)}_{(31...1)^3} = \frac{3d-10}{3(d-3)!}$ for $d \geqslant 4$.
- $H^{(0,+)}_{(31...1)^4} = \frac{2d^3+57d^2-446d+780}{9(d-3)!}$ for $d \geqslant 6$.

5.2.1 Even case

Theorem 5.37 (Correspondence Theorem 1). For the numbers $\mathbb{T}H^{(h,+)}_{m^1,\dots,m^k}$ defined in 5.9 and the numbers $H^{(h,+)}_{m^1,\dots,m^k}$ defined in 4.9 we have $\mathbb{T}H^{(h,+)}_{m^1,\dots,m^k} = H^{(h,+)}_{m^1,\dots,m^k}$.

The equalities of the numbers $\mathbb{T}H^{(h,+)}_{m^k}$ and $H^{(h,+)}_{m^k}$ for the special cases considered in 5.1.1, 5.1.2 and 5.1.3 can be used to give an indirect proof of the correspondence theorem (in these cases). The advantage is, of course, that a tropical proof is much simpler. Instead of a complicated algebraic degeneration, the proof reduces to a mere count of graphs. The disadvantage, however, is that it keeps the relation between the tropical and the algebraic count hidden. Therefore, we give an alternative proof that makes use of a degeneration argument.
The idea of the proof is simple: Iterate the degeneration procedure in 4.4 until the base curve D is a maximal quasistable degeneration. We obtain 1-parameter families of stable spin curves $(\hat{\mathcal{Y}},\hat{\mathcal{N}})$ and $(\hat{\mathcal{X}},\hat{\mathcal{L}})$ whose special fibres are the maximal quasistable degeneration of D and its quasistable cover. By taking dual graphs we translate the count into the tropical world and record the number of maps (weighted by their automorphism group) that tropicalize to the same object. This gives us the weight with which we should count tropical spin Hurwitz covers. The procedure of tropicalizing algebraic families of covers can be made concrete in terms of Berkovich skeleta by using the tropicalization map in Definition 5.1.1 ([CMP20]). First, we consider the case $h = 0$, i.e. spin Hurwitz numbers of $\mathbb{P}^1$, in detail before deducing the statement for the general situation.
Preparing the proof of theorem 5.37. Let us investigate the iterated degeneration from the perspective of numbers only. Start with a spin Hurwitz number $H^{(0,+)}_{m^1,\dots,m^k}$ and apply $k-3$ times degeneration formula (1) in theorem 4.19:

$$\begin{aligned} H^{(0,+)}_{m^1,\dots,m^k} &= \sum_{m_1} |m_1| m_1! H^{(0,+)}_{m^1,\dots,m^{k-2},m_1} H^{(0,+)}_{m_1,m^{k-1},m^k} \\ &= \sum_{m_1} |m_1| m_1! H^{(0,+)}_{m_1,m^{k-1},m^k} \Big(\sum_{m_2} |m_2| m_2! H^{(0,+)}_{m^1,\dots,m^{k-3},m_2} H^{(0,+)}_{m_2,m^{k-2},m_1}\Big) \\ &= \sum_{m_1}\sum_{m_2} |m_1||m_2| m_1! m_2!\, H^{(0,+)}_{m_1,m^{k-1},m^k} H^{(0,+)}_{m_2,m^{k-2},m_1} H^{(0,+)}_{m^1,\dots,m^{k-3},m_2} \\ &= \cdots \\ &= \sum_{m_1}\cdots\sum_{m_{k-3}} \underbrace{|m_1|\cdot\ldots\cdot|m_{k-3}| m_1!\cdot\ldots\cdot m_{k-3}!}_{F_{m_1,\dots,m_{k-3}}} H^{(0,+)}_{m_1,m^{k-1},m^k}\cdot\ldots\cdot H^{(0,+)}_{m^1,m^2,m_{k-3}}. \end{aligned}$$

Consider a partition of $\mathcal{M}$, the set of isomorphism classes of maps contributing to $H^{(0,+)}_{m_1,m_2,m_3}$, into the spaces of maps with even and odd parity, $\mathcal{M} = \mathcal{M}^0 \cup \mathcal{M}^1$, and write

$$\begin{aligned} &H^{(0,+)}_{m_1,m_2,m_3} = (H^{(0,+)}_{m_1,m_2,m_3})_0 - (H^{(0,+)}_{m_1,m_2,m_3})_1, \text{ where} \\ &(H^{(0,+)}_{m_1,m_2,m_3})_1 := \sum_{f\in\mathcal{M}^1} \frac{1}{|Aut(f)|} \text{ and } (H^{(0,+)}_{m_1,m_2,m_3})_0 := \sum_{f\in\mathcal{M}^0} \frac{1}{|Aut(f)|}. \end{aligned}$$

$$
\begin{aligned}
H^{(0,+)}_{m^1,\dots,m^k} &= \sum_{m_1}\cdots\sum_{m_{k-3}} F_{m_1,\dots,m_{k-3}} \underbrace{H^{(0,+)}_{m_1,m^{k-1},m^k}}_{=:H_1^{(0,+)}} \cdot \dots \cdot \underbrace{H^{(0,+)}_{m^1,m^2,m_{k-3}}}_{=:H_{k-2}^{(0,+)}} \\
&= \sum_{m_1}\cdots\sum_{m_{k-3}} F_{m_1,\dots,m_{k-3}} ((H_1^{(0,+)})_0 - (H_1^{(0,+)})_1) \cdot \dots \cdot ((H_{k-2}^{(0,+)})_0 - (H_{k-2}^{(0,+)})_1) \\
&= \sum_{m_1}\cdots\sum_{m_{k-3}} F_{m_1,\dots,m_{k-3}} \sum_{\nu\in(\mathbb{Z}/2\mathbb{Z})^{k-3}} \prod_{j=1}^{k-2} (-1)^{\nu_j} (H_j^{(0,+)})_{\nu_j} \\
&= \sum_{m_1}\cdots\sum_{m_{k-3}} F_{m_1,\dots,m_{k-3}} \sum_{\nu\in(\mathbb{Z}/2\mathbb{Z})}^{k-2} (-1)^{\sum_{j=1}^{k-2}\nu_j} \prod_{j=1}^{k-2} (H_j^{(0,+)})_{\nu_j}.
\end{aligned}
$$

We will see that a tropical spin Hurwitz cover (π, s) corresponds to the choice of $k-3$ odd partitions $(m_1, \dots, m_{k-3})$ and an element $\nu \in (\mathbb{Z}/2\mathbb{Z})^{k-2}$ such that

$$
mult(\pi, s) = F_{m_1,\dots,m_{k-3}} \prod_{j=1}^{k-2} (H_j^{(0,+)})_{\nu_j} \text{ and } p(\pi, s) = \sum_{j=1}^{k-2} \nu_j \mod 2.
$$

We record the discussion in the following lemma.

Lemma 5.38. All (disconnected) degree d spin Hurwitz numbers $H^{(0,+)}_{m^1,\dots,m^k}$ are determined in terms of (disconnected) spin Hurwitz numbers of the form $H^{(0,+)}_{n_1,n_2,n_3}$. Using the notation from the previous discussion we have

$$
H^{(0,+)}_{m^1,\dots,m^k} = \sum_{m_1}\cdots\sum_{m_{k-3}} F_{m_1,\dots,m_{k-3}} \sum_{\nu\in(\mathbb{Z}/2\mathbb{Z})^{k-2}} (-1)^{\sum_{j=1}^{k-2}\nu_j} \prod_{j=1}^{k-2} (H_j^{(0,+)})_{\nu_j}.
$$

Remark 5.39. Note that Hurwitz and spin Hurwitz numbers are related by

$$
H_{g\xrightarrow{d}0}(m_1, m_2, m_3) = (H^{(0,+,c)}_{m_1,m_2,m_3})_0 + (H^{(0,+,c)}_{m_1,m_2,m_3})_1,
$$

where c indicates that we consider connected spin Hurwitz numbers (see section 5). In the case $d = 3$ or $d = 4$, we have either $(H^{(0,+,c)}_{m_1,m_2,m_3})_1 = 0$ or $(H^{(0,+,c)}_{m_1,m_2,m_3})_0 = 0$. We can rephrase the uniqueness question from the beginning of section 5 in the following way: A tropical Hurwitz cover

π has a unique parity function if and only if $(H^{(0,+,c)}_{m_1,m_2,m_3})_1 = 0$ or $(H^{(0,+,c)}_{m_1,m_2,m_3})_0 = 0$.

Proof of theorem 5.37 for $h = 0$. The proof consists of three main steps.

Main steps 5.40.

1. Iterate degeneration procedure in section 4.4.
2. Tropicalize the obtained stable spin curves.
3. Construct a surjection $\phi : \mathcal{S} \to \mathbb{T}\mathcal{S}$ that takes a degenerate spin Hurwitz cover to a tropical spin Hurwitz cover.

Preliminaries. For the discrete data in theorem 5.37, that is d, h, k and odd partitions $m^1, \dots, m^k$, fix

- a spin curve (D, N) of genus h with parity $p := h^0(N) \mod 2 = 0 \mod 2$.
- a collection of distinct points $V := \{q^1, ..., q^k\} \subset D$, the prescribed branch points.

Recall that the spin curve degeneration in 4.4 relates the count of spin Hurwitz covers of (D, N) to the count of degenerate spin Hurwitz covers of a nodal curve $D_0 = D_2 \cup E \cup D_1$ that was fixed in advance. In fact, Theorem 5.1. [LP13] produces (provided we choose a ramification behaviour, i.e. an odd partition m of d, over the nodes) for any degenerate map $f_0 = (f_2, f_e, f_1) \in \mathcal{M}_{m,0}$, a family of stable spin curve $(\mathcal{D}, \mathcal{N})$ and "the right number" of families of covering spin curves $(\mathcal{C}_\zeta, \mathcal{L}_\zeta)$ (with parameter s) together with holomorphic maps $\mathcal{F}_\zeta : \mathcal{C}_\zeta \to \mathcal{D}$, such that the map between special fibres is given by f_0 and $p(f_{\zeta,s}) = p(f_2) + p(f_1)$, where $f_{\zeta,s} := \mathcal{F}_{\zeta|\mathcal{C}_{\zeta,s}}$. Twisting the construction around we can think of taking $f : C \to D \in \mathcal{M}^V_{\chi,m^1,...,m^k}$ ($\mathcal{M}^V_{\chi,m^1,...,m^k}$ is relative Gromov-Witten space of $V-$regular maps defined in 4.4 (step 0)) together with the theta characteristic L_f and degenerating both cover and base curve simultaneously to D_0 to obtain

- a degenerate spin Hurwitz cover $f_0 := (f_2, f_e, f_1) : C_0 = C_2 \cup \bigcup_{i=1}^{l} E_i \cup C_1 \to D_0$ and
- a spin structure on C_0 that restricted to the smooth curve C_i is given by L_{f_i} for $i = 1, 2$ and $p(f) = p(f_2) + p(f_1)$.

1. Iterated degeneration. We assume $h = 0$ and describe the iterated degeneration procedure in this case. The case $h = 1$ is similar, but will not be carried out in detail. We start by setting D_0 to be a chain of three $\mathbb{P}^1$ meeting in two nodes, p^1 and p^2, together with $k + 3$ distinct points (fixing 3 additional points where f is unramified ensures that all automorphism groups of V-regular maps are trivial) such that

$$q^{k+1}, q^1, ..., q^{k-2}, p^2 \in D_2,\ p^2, q^{k+2}, p^1 \in E \text{ and } q^{k+3}, q^{k-1}, q^k, p^1 \in D_1.$$

Given $f \in \mathcal{M}^V_{\chi,m^1,...,m^k,(1^d)^3}$ together with theta characteristic L_f deform domain and target (meaning construct stable spin curves $(\mathcal{D}, \mathcal{N})$ and $(\mathcal{C}, \mathcal{L})$ where we omit the index ζ to make notation lighter) to obtain a limit map f_0 with odd ramification profile m_1 over the nodes p^1 and p^2. Memorize the component maps f_e and f_1 and the pair $(\bigcup_{i=1}^{l} E_i \cup C_1, \mathcal{L}_{\bigcup_{i=1}^{l} E_i \cup C_1})$. Set $D := D_2$ with theta characteristic $N := \mathcal{N}_{|D_2}$, $f := f_2$, $C := C_2$ and $q^{k-1} := p^2$. Fix $D_0 := D_2 \cup E \cup D_1$ with nodes p^1, p^2 and two additional point $q^k, q^{k+1} \in D_0$, such that $q^k, q^1, ..., q^{k-3}, p^2 \in D_2$ and $p^1, q^{k+2}, q^{k-2}, q^{k-1} \in D_1$.
Repeat the previous step with the map f ramified over $k - 1$ branch points $(q^1, ..., q^{k-1})$ with ramification profile $m^1, ..., m^{k-2}, m_1$. Continue until D contains only three branch points. The process terminates after $k - 3$ steps.

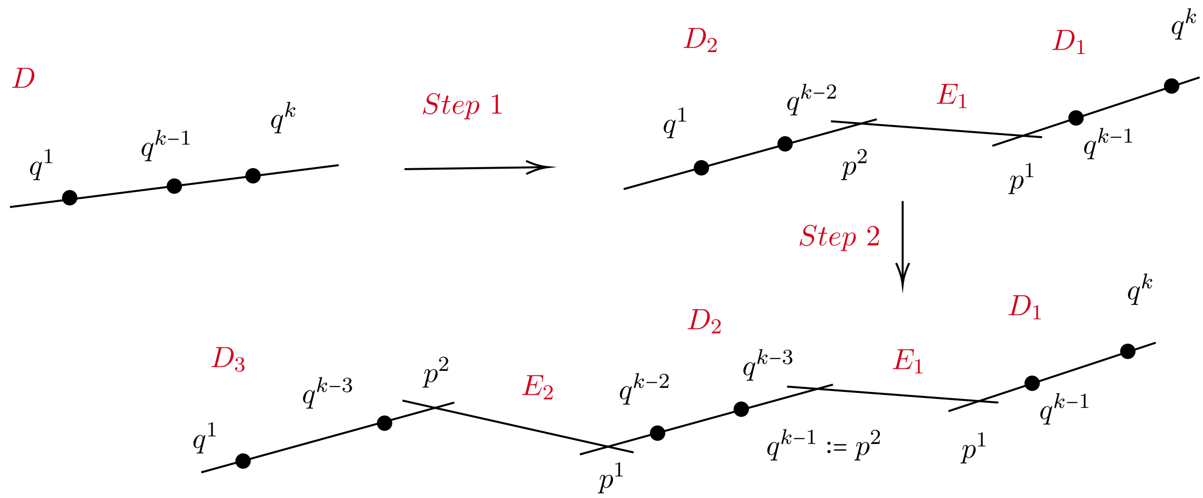


Figure 43: The first two degeneration steps on the base curve.

Collecting all memorized data together we end up with a holomorphic map $\mathcal{F}$, stable spin structures $(\hat{\mathcal{D}}, \hat{\mathcal{N}})$ and $(\hat{\mathcal{C}}, \hat{\mathcal{L}})$ such that the special fibres encode the following data:

1. A maximal degenerate target curve: $D_0 = D_{k-2} \cup E \cup D_{k-3} \cup \ldots \cup D_2 \cup E \cup D_1$, that is a chain of $2k-5$ copies of $\mathbb{P}^1$.
2. A degenerate domain curve: $C_0 = C_{k-2} \cup \bigcup_{i=1}^{l_{k-3}} E_i \cup C_{k-3} \cup \ldots \cup C_2 \cup \bigcup_{i=1}^{l_1} E_i \cup C_1$.
3. A degenerate map $f_0 = (f_{k-2}, f_e, f_{k-3}, \ldots, f_1)$.
4. Theta characteristics $\mathcal{L}_{|C_i} = L_{f_i}$ for $i = 1, \ldots, k-2$.

The family of quasistable curves $\hat{\mathcal{C}}$ is made by gluing the smooth family $\mathcal{C}$ from degeneration step $k-3$ to the nodal rest, $C_{k-3} \cup \ldots \cup C_2 \cup \bigcup_{i=1}^{l_1} E_i \cup C_1$, from the previous one: For $s \neq 0$ the generic fibre of $\hat{\mathcal{C}}$ is the quasistable curve $C_s \cup C_{k-3} \cup \ldots \cup C_2 \cup \bigcup_{i=1}^{l_1} E_i \cup C_1$ where C_s is the generic fibre of $\mathcal{C}$. The family $\hat{\mathcal{D}}$ is constructed analogously.
Let us call the set of these glued maps $\mathcal{M}_{m_1,\ldots,m_{k-3},0}$ (in the style of the space $\mathcal{M}_{m,0}$). As we have seen in the proof of theorem 4.19 (here just in iterated form) these contribute to the count in the following way:

$$H^{(0,+)}_{m^1,\ldots,m^k} = \frac{1}{(d!)^{2k-5} \prod_{i=1}^{k} m^i!} \sum_{(m_1,\ldots,m_{k-3})} \prod_{i=1}^{k-3} |m_i|^2 \sum_{f_0 \in \mathcal{M}_{m_1,\ldots,m_{k-3},0}} (-1)^{p(f_0)}. \tag{6}$$

Recall, the first factor accounts for counting spin Hurwitz covers with a marking of all ramification points. Keeping track of the number of maps degenerating to a nodal cover yields a factor $|m_i|^2$ in each degeneration step (each node in the domain curve can be smoothed in $|m_i|$ ways).

2. Tropicalization. Now, we are ready to tropicalize the families of stable spin structures $(\hat{\mathcal{D}}, \hat{\mathcal{N}})$ and $(\hat{\mathcal{C}}, \hat{\mathcal{L}})$ by taking dual graphs of both special fibre, $(C_0, \hat{\mathcal{L}}_{|C_0})$ and $(D_0, \hat{\mathcal{N}}_{|D_0})$, like in definition 3.23. The tropical spin curve $(\Gamma, \varnothing, s)$ $((B, \varnothing, \tilde{s}))$ is just the dual spin graph (definition 4.18) of $(C_0, \hat{\mathcal{L}}_{|C_0})$ $((D_0, \hat{\mathcal{N}}_{|D_0}))$ with metric l_1 (l_2) given in [CMP20] (Definition/Lemma 5.1.1):

Indeed, denote $\tilde{C}$ ($\tilde{D}$) the stable model of C_0 (D_0) with set of nodes F_1 (F_2). We see that C_0 (D_0) is the blow up of $\tilde{C}$ ($\tilde{D}$) at $R_1 = F_1$ ($R_2 = F_2$). For $i = 1, 2$ this yields $P_i = F_i \setminus R_i = \varnothing$.
The corresponding map $\pi : \Gamma \to B$ is the obvious map provided by the geometric covers. Note that (π, s) is a tropical spin Hurwitz cover relevant to the tropical count. Indeed, the family of target curves $(\hat{\mathcal{D}}, \hat{\mathcal{N}})$ is identical for all maps f that undergo step 1. Its tropicalization is just the line $\mathbb{TP}^1$ with $k-2$ inner vertices and carries the right parity.

3. Construction of ϕ. To alleviate notation set $m^1 = ... = m^k = m$ for an odd partitition of d and denote by $\mathcal{S}$ the set of all (equivalence classes of) degenerate spin coverings $(f_0 : C_0 \to D_0, \mathcal{L}_0)$ of $(D_0, \hat{\mathcal{N}}_{|D_0})$, where

- $(C_0, \mathcal{L}_0)$ is a stable spin curve.
- $(f_i, \mathcal{L}_{0|C_i}) = (f_i, L_{f_i})$ for $i = 1, ..., k-2$.
- for $i = 2, ..., k-3$ the map f_i counts towards $H_i^{(0,+)} := H_{m_1,m,m_2}^{(0,+)}$, where m_1 and m_2 are arbitrary odd partitions of d.
- f_1 and f_{k-2} count towards $H_1^{(0,+)} := H_{k-2}^{(0,+)} := H_{m_1,m,m}^{(0,+)}$.

Equivalently we have

$$\mathcal{S} := \{(f_0 = (f_1, f_e^1, ..., f_e^{k-3}, f_{k-2}), \mathcal{L}_0) \text{ is obtained via step 1. }\}$$

Let $\mathbb{T}\mathcal{S}$ be the set of tropical spin covers (definition 5.9). We construct a map

$$\phi : \mathcal{S} \to \mathbb{T}\mathcal{S}, (f_0, \mathcal{L}_0) \mapsto (\pi, s)$$

and show that

1. ϕ is well-defined.
2. $\phi(\mathcal{S}) = \mathbb{T}\mathcal{S}$, i.e. ϕ is surjective.
3. $$(-1)^{p((\pi,s))} mult((\pi, s)) = \frac{1}{(d!)^{2k-5}(m!)^k} \sum_{(f_0,\mathcal{L}_0)\in\phi^{-1}((\pi,s))} \prod_{i=1}^{k-3} |m_i|^2 (-1)^{p(f_0)} \qquad (7)$$
   where $m_1, ..., m_{k-3}$ are determined by the edge weights of Γ.

Together with equation (6) these statements prove theorem 5.37 for $h = 0$.
Let $(f_0, \mathcal{L}_0) \in \mathcal{S}$. Following step 2 yields a tropical spin curve (Γ, s) together with a cover π. Define $\phi((f_0, \mathcal{L}_0) := (\pi, s)$. Clearly π is a tropical Hurwitz cover with the right ramification profile over $\pm\infty$ and s is an admissible parity function. Hence, ϕ is well-defined, i.e. (π, s) is relevant to the tropical count. Statement two and three follow essentially from going step 2 backwards, that is reconstructing all elements in $\phi^{-1}((\pi, s))$ from a given tropical spin Hurwitz cover. To simplify this reconstruction we start by working on the right-hand side of (7).
The following computation is just an iterated version of the one in step 6 in section 4.4. We will use the same notation to make the connection clear: Denote by $\mathcal{P}_{m_1,...,m_{k-3}}$ the product space

$$\mathcal{M}_{m_1} \times \mathcal{M}_{m_1}^e \times ... \times \mathcal{M}_{m_{k-3}}^e \times \mathcal{M}_{m_{k-3}}$$

where $\mathcal{M}_{m_1}$ and $\mathcal{M}^e_{m_1}$ are the spaces in remark 4.24. We relate the count of glued maps $f_0 \in \mathcal{M}_{m_1,\dots,m_{k-3},0}$ to the count of disjoint maps in $\mathcal{P}_{m_1,\dots,m_{k-3}}$ obtained via pull back under the covering map $\mathcal{P}_{m_1,\dots,m_{k-3}} \to \mathcal{M}_{m_1,\dots,m_{k-3},0}$ (this means we count disjoint tuple instead) and get

$$\sum_{(f_0,\mathcal{L}_0)\in\pi^{-1}((\pi,s))} (-1)^{p(f_0)} = \frac{1}{\prod_{i=1}^{k-3}(m_i!)^2} \sum_{(f_1,f_e^1,\dots,f_e^{k-3},f_{k-2})} (-1)^{p(f_1)+\dots+p(f_{k-2})}$$
$$= \frac{1}{\prod_{i=1}^{k-3}(m_i!)^2} \sum_{f_e^1\in\mathcal{M}^e_{m_1}} \cdots \sum_{f_e^{k-3}\in\mathcal{M}^e_{m_{k-3}}} (-1)^{p(f_1)+\dots+p(f_{k-2})}$$

where the sum goes over $(f_1,f_e^1,\dots,f_e^{k-3},f_{k-2}) \in \mathcal{P}_{m_1,\dots,m_{k-3}}$ such that glueing yields a map $f_0 \in \pi^{-1}((\pi,s))$. The second equality follows from the fact that for $i = 1,\dots,k-3$ forgetting contact marked points of maps in $\mathcal{M}^e_{m_i}$ gives exactly one map (proof lemma 2.2. in [Lee13]). Hence, even if the parities of the component maps f_i are fixed by s, they can be combined with *all* labelled maps in the spaces $\mathcal{M}^e_{m_i}$. This is good news, since going over to the tropical world we shrink all exceptional components and thus forget about the intermediate maps f_e^i. With $|\mathcal{M}^e_{m_i}| = \frac{d! m_i!}{|m_i|}$ (lemma 2.2, [Lee13]) we get

$$\frac{1}{\prod_{i=1}^{k-3}(m_i!)^2} \sum_{f_e^1\in\mathcal{M}^e_{m_1}} \cdots \sum_{f_e^{k-3}\in\mathcal{M}^e_{m_{k-3}}} (-1)^{p(f_1)+\dots+p(f_{k-2})} = (d!)^{k-3} \prod_{i=1}^{k-3} \frac{m_i!}{|m_i|} \sum_{(f_1,f_2,\dots,f_{k-2})} (-1)^{p(f_1)+\dots+p(f_{k-2})}.$$

Switching back to counting tuple whose components are usual spin Hurwitz covers $\mathring{f} := (f_1, f_2, \dots, f_{k-2})$ (without marking of ramification points) yields

$$\sum_{(f_0,\mathcal{L}_0)\in\pi^{-1}((\pi,s))} (-1)^{p(f_0)} = (m!)^k (d!)^{2k-5} \prod_{i=1}^{k-3} \frac{m_i!}{|m_i|} \sum_{\mathring{f}} \frac{(-1)^{p(\mathring{f})}}{|Aut(\mathring{f})|}.$$

where the sum goes over all tuple of maps such that f_i counts towards $H_i^{(0,+)}$ (note that the numbers $H_i^{(0,+)}$ are fixed by the edge weights of Γ) and thus

$$\frac{1}{(d!)^{2k-5}(m!)^k} \sum_{(f_0,\mathcal{L}_0)\in\pi^{-1}((\pi,s))} \prod_{i=1}^{k-3} |m_i|^2 (-1)^{p(f_0)} = \prod_{j=1}^{k-3} m_j! |m_j| \sum_{\mathring{f}} \frac{(-1)^{p(\mathring{f})}}{|Aut(\mathring{f})|}. \quad (8)$$

To see that the right-hand side matches the tropical multiplicity we reconstruct $\mathring{f}$ from $(\pi,s) \in \mathbb{T}\mathcal{S}$. In passing, this allows us to see that ϕ is surjective as well.

Let $(\pi,s) \in \mathbb{T}\mathcal{S}$. For each vertex $v \in V(\Gamma)$ let $f_v : C_v \to D_v$ be a spin Hurwitz cover counting towards $H^{(0,+)}((\pi,s),v)$ together with its theta characteristic L_{f_v} such that $p(f_v) = s(v)$. Let $\mathring{C}$ be the disjoint union of all curves C_v and $\mathring{f}$ be the map defined by $\mathring{f}_{|C_v} := f_v$.
We see immediately

$$\prod_{j=1}^{k-3} m_j! \sum_{\mathring{f}} \frac{(-1)^{p(\mathring{f})}}{|Aut(\mathring{f})|} = \frac{1}{|Aut(\Gamma)|} (-1)^{\sum_{v\in V(\Gamma)} s(v)} \prod_{v\in V(\Gamma)} n_1^v! n_2^v! n_3^v!\, H^{(0,+)}((\pi,s),v).$$

Thus the right-hand side of (8) is almost the tropical multiplicity. Using $\prod_{j=1}^{k-3} |m_j| = \prod_{e\in E(\Gamma)} \omega(e)$ completes the proof

$$\prod_{j=1}^{k-3} |m_j| m_j! \sum_{\mathring{f}} \frac{(-1)^{p(\mathring{f})}}{|Aut(\mathring{f})|} = (-1)^{p(\pi,s)} mult(\pi,s).$$

□

Proof of theorem 5.37 for $h>0$. If $h>0$ the proof consists of the same three steps 5.40. In order to obtain a maximally quasistable degeneration of a base curve D of genus h, we apply degeneration procedure (2) (theorem 4.19) h times first, i.e. reduce the genus by 1 in each step. We end up with a nodal curve of genus 0. Now, we can continue just as in step 1 of the proof of theorem 5.37. We tropicalize the special fibres according to step 2 and construct the map ϕ in the same way.
The only thing left open is to notice that the tropical spin curves from step 2 are counted with the right multiplicity. To see this let us look at the degeneration from the perspective of numbers only:
Let $(h,p)=(h,+)$, where $h>0$. Start with a spin Hurwitz number $H^{(h,+)}_{n^1,\dots,n^k}$ and apply h times degeneration formula (2) in theorem 4.19:

$$\begin{aligned}H^{(h,+)}_{n^1,\dots,n^k} &= \sum_{m_1}|m_1|m_1!H^{(h-1,+)}_{m_1,m_1,n^1,\dots,n^k}\\ &= \cdots\\ &= \sum_{m_1}\dots\sum_{m_h}\underbrace{|m_1|\cdot\ldots\cdot|m_h|m_1!\cdot\ldots\cdot m_h!}_{F_{m_1,\dots,m_h}}H^{(0,+)}_{m_1,m_1,\dots,m_h,m_h,n^1,\dots,n^k}\end{aligned}$$

Now, set $\tilde{k}:=2h+k$ and apply $\tilde{k}-3$ times degeneration formula (1) in theorem 4.19:

$$\begin{aligned}H^{(h,+)}_{n^1,\dots,n^k} &= \sum_{m_1}\dots\sum_{m_h}F_{m_1,\dots,m_h}\Big(\sum_{\tilde{m}_1}\dots\sum_{\tilde{m}_{\tilde{k}-3}}F_{\tilde{m}_1,\dots,\tilde{m}_{\tilde{k}-3}}\sum_{\nu\in(\mathbb{Z}/2\mathbb{Z})^{\tilde{k}-2}}(-1)^{\sum_{j=1}^{\tilde{k}-2}\nu_j}\prod_{j=1}^{\tilde{k}-2}(H_j^{(0,+)})_{\nu_j}\Big)\\ &= \sum_{m_1,\dots,m_h}\sum_{\tilde{m}_1,\dots,\tilde{m}_{\tilde{k}-3}}F_{m_1,\dots,m_h}F_{\tilde{m}_1,\dots,\tilde{m}_{\tilde{k}-3}}\sum_{\nu\in(\mathbb{Z}/2\mathbb{Z})^{\tilde{k}-2}}(-1)^{\sum_{j=1}^{\tilde{k}-2}\nu_j}\prod_{j=1}^{\tilde{k}-2}(H_j^{(0,+)})_{\nu_j},\end{aligned}$$

where $H_j^{(0,+)}$ is the spin Hurwitz number with 3 special points (branch points or nodes) we split off in the the j–th step. We see that a tropical spin Hurwitz cover (π,s) corresponds to an ordered choice of $\tilde{k}-3$ odd partitions and an element $\nu\in(\mathbb{Z}/2\mathbb{Z})^{\tilde{k}-2}$ such that

$$\begin{aligned}mult(\pi,s) &= F_{m_1,\dots,m_h}F_{\tilde{m}_1,\dots,\tilde{m}_{\tilde{k}-3}}\prod_{j=1}^{\tilde{k}-2}(H_j^{(0,+)})_{\nu_j} \text{ and}\\ p(\pi,s) &= \sum_{j=1}^{\tilde{k}-2}\nu_j \mod 2.\end{aligned}$$

□

5.2.2 Odd case

Theorem 5.41 (Correspondence Theorem 2). For the numbers $\mathbb{T}H^{(h,-)}_{m^1,\dots,m^k}$ defined in 5.13 and the numbers $H^{(h,-)}_{m^1,\dots,m^k}$ defined in 4.9, where $h>1$, we have $\mathbb{T}H^{(h,-)}_{m^1,\dots,m^k}=H^{(h,-)}_{m^1,\dots,m^k}$.

Proof. The proof is analogous to the one of theorem 5.37 and consists of the same three steps 5.40. Start with a spin Hurwitz number $H^{(h,-)}_{m^1,\dots,m^k}$ In order to obtain an almost maximally quasistable degeneration (section 5.1) of a base curve D of genus h, we apply degeneration procedure of theorem 4.19 (1) with $h=(h-1)+1$ and $k_0=k$ first:

$$H^{(h,-)}_{m^1,\dots,m^k}=\sum_m|m|m!H^{(h-1,+)}_{m,m^1,\dots,m^k}H^{(1,-)}_m.$$

Now, we can continue just as in the proof of theorem 5.37 for the numbers $H^{(h-1,+)}_{m,m^1,\dots,m^k}$. We tropicalize the special fibres according to step 2. Notice that after degeneration the base curve has a dual spin graph with a 1-valent vertex of genus 1 (dual to the only genus 1 component with odd theta characteristic) and 3-valent genus 0 vertices else. We construct the map ϕ in the same way. The only difference to the even case is that the degeneration procedure yields numbers of the form $H^{(1,-)}_m$. These correspond to the preimages of the 1-valent genus 1 vertex v' of B and appear, as expected, in the weighting of a tropical spin Hurwitz cover (π, s) as local spin Hurwitz numbers $H^{(1,-)}((\pi, s), v)$ for $v \in \pi^{-1}(v')$. □

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Versicherung der selbstständigen Arbeit

Sehr geehrte Damen und Herren,

Hiermit versichere ich, dass ich die beiliegende Masterarbeit mit dem Titel *Tropical spin Hurwitz numbers* selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe, sowie die Arbeit weder vollständig noch in wesentlichen Teilen bereits veröffentlicht oder für ein anderes Prüfungsverfahren verwendet habe und das in Dateiformat eingereichte Exemplar mit dem eingereichten gebundenen Exemplar übereinstimmt.

Mit freundlichen Grüßen

Lou-Jean Leila Cobigo