UNIVERSITÄT TÜBINGEN
FACHBEREICH MATHEMATIK

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Convex Geometry

Winter term 2024/25

Exercise sheet 3

Due on: Thursday, 14.11.2024, 10:00

Exercise 1 (4 Points)

Let V be a finite dimensional vector space over $\mathbb{K}$ with basis B and $b : V \times V \to \mathbb{K}$ a non-degenerate bilinear form. Consider the linear map

$$b'' : V \to V^\vee, w \mapsto b_w$$

and show: ${}_B M_{B^\vee}(b'') = M_B(b)$, where ${}_B M_{B^\vee}(b'')$ denotes the representation matrix of b'' with respect to the basis B and $B^\vee$ and $M_B(b)$ is the Gram matrix of B.

Exercise 2 (4 Points)

Let $\mathbb{R}[x]_{\leq 3} := \{p := \sum_{i=0}^{3} a_i x^i \mid a_i \in \mathbb{R}, deg(p) \leqslant 3\}$ be the real vector space of polynomials of degree ≤ 3 with standard basis $B_3 := (x^0, x^1, x^2, x^3)$ and dual basis $B_3^\vee$. The map

$$ev_{a,3} : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}, p \mapsto p(a)$$

is called *evaluation at* $a \in \mathbb{R}$. Write $ev_{0,3}$ and $ev_{4,3}$ as linear combination of vectors in $B_3^\vee$.

Exercise 3 (4 Points)

A subset K' of $\mathbb{R}^n$ is called a *cone*, if for all $x_1, ..., x_k \in K'$ and $\lambda_1, ..., \lambda_k \in \mathbb{R}_+$, we have $\sum_{i=1}^{k} \lambda_i x_i \in K'$. Let $K \subset \mathbb{R}^n$. Define in analogy to the convex hull of K, the *conical hull* $c(K)$ and show: If K is finite, then $c(K) = cone(K)$.

Exercise 4 (6 Points)

Let $c \subset \mathbb{R}^n$ be a strictly convex cone of dimension d, i.e. c does not contain a line through the origin. Then c is called *simplicial*, if c can be generated by d points. Show: Every strictly convex cone c of dimension 2 is simplicial. Give an example for a strictly convex cone that is not simplicial.

Hint: Assume w.l.o.g. that $c = cone(S)$, where $u_n > 0$ for all $u \in S$, and use the correspondence between cones and polytopes in this case (see lecture notes p.66) to express the claim in terms of polytopes.

Hand in via URM. Exercise classes take place on Wednesdays 12-14, in S11.