Non-archimedean phenomena illustrated with Laurent series

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In the lecture "Introduction to Berkovich geometry" we were discussing non-archimedean fields and the strange phenomena that occur within them. While some of this might seem very mysterious in the abstract setting, many things are really "obvious" when working with concrete examples. This note illustrates this with a focus on the field of Laurent series.

1 The fields $k(T)$ and $k(\!(T)\!)$

Let k be any field (feel free to plug in $k = \mathbb{R}$ or $\mathbb{C}$). Take a variable T and consider the smallest field containing k and T: this is the field of rational functions $k(T)$. You can think of it as follows: the set of all expressions which can be build from elements in k and T using only addition, multiplication, and subtraction gives the polynomial ring $k[T]$. To turn this into a field, you also have to allow the inverses $1/f(T)$ for all polynomials $f(T)$. Hence

$$k(T) = \left\{ \frac{f}{g} \;\middle|\; f, g \in k[T], g \neq 0 \right\}$$

and this is in fact the *fraction field* of $k[T]$.

Now $k(T)$ is not complete: any rational function can be expressed as a *Laurent series*, i.e. a formal expression of the form

$$\sum_{n=v}^{\infty} a_n T^n, \qquad \text{for some } v \in \mathbb{Z}, a_n \in k.$$

This is something you typically do when discussing holomorphic functions in complex analysis (= "Funktionentheorie"). Just think of the geometric series

$$\frac{1}{1-T} = \sum_{n=0}^{\infty} T^n$$

as a template for turning rational functions into series. The field $k(\!(T)\!)$ is the field of all Laurent series where addition and multiplication are as expected:

$$\sum_{n=v}^{\infty} a_n T^n + \sum_{n=w}^{\infty} b_n T^n = \sum_{n=\min\{v,w\}}^{\infty} (a_n + b_n) T^n$$

$$\Big(\sum_{n=v}^{\infty} a_n T^n \Big) \Big(\sum_{n=w}^{\infty} b_n T^n \Big) = \sum_{n=v+w}^{\infty} \Big(\sum_{l=-\infty}^{\infty} a_l b_{n-l} \Big) T^n$$

where we take $a_n = 0$ for all $n < v$ for ease of notation and similar for b_n.

It is not hard to see that $k(T) \subseteq k(\!(T)\!)$ but it is much harder to see that this not an equality[1]. In fact, this is exactly the reason why $k(T)$ is not complete and $k(\!(T)\!)$ is its completion.

Once we write rational functions as series, we can easily see the *valuation* ord (algebra language) and non-archimedean absolute value:

$$\text{ord} : \sum_{n=v}^{\infty} a_n T^n \longmapsto v \qquad \text{and} \qquad |\cdot| = \exp(-\,\text{ord}(\cdot)).$$

[1] With enough effort one can show that a Laurent series is a rational function if and only if the sequence of coefficients is *linearly recurrent* after some n_0.

Here we assume that $a_v \neq 0$ and the Laurent series to be not 0[2] As a sanity check: the *value group* is

$$\left|k(\!(T)\!)^\times\right| = \exp(\mathbb{Z}) = \left\{ \exp(n) \;\middle|\; n \in \mathbb{Z}\right\}$$

and this is the same as $|k(T)^\times|$. With this we can now see much better why $k(\!(T)\!)$ is the completion of $k(T)$: Take a Laurent series $f = \sum_{n=v}^{\infty} a_n T^n$ and look at the partial sums $f_N = \sum_{n=v}^{N} a_n T^n$. One can show that all f_N are rational functions and that $(f_N)_N$ is a Cauchy sequence in $k(T)$. The limit of this sequence should be f since

$$|f - f_N| = \exp(-(N+1)) \longrightarrow 0 \qquad \text{for } N \longrightarrow \infty$$

but as we mentioned before, f might not be in $k(T)$.

2 Basic non-archimedean properties

2.1 The archimedean axiom does not hold on $k(\!(T)\!)$

Take two Laurent series $f = \sum_{n=v}^{\infty} a_n T^n$ and $g = \sum_{n=w}^{\infty} b_n T^n$. If $|f| < |g|$ then this means that $v > w$. But since the order of $f + f + \cdots + f$ is not going to be lower than that of f (no matter how many summands) we see that

$$|nf| < |g| \text{ for all } n \in \mathbb{N}.$$

Note that if $\operatorname{char}(k) = 0$ then in fact $\operatorname{ord}(f + \cdots + f) = \operatorname{ord}(f)$. But even for positive characteristic this is ok: we can only have the order $f + f + \cdots + f$ potentially higher than that of f, so still $|nf| \leq |f| < |g|$.

2.2 Integers have absolute value ≤ 1

In fact even more is true: k lives in $k(\!(T)\!)$ as the Laurent series consisting only of a constant term. Hence any $a \in k^\times$ has $\operatorname{ord}(a) = 0$ and hence $|a| = 1$. In particular this holds for the integers $\mathbb{Z} \subseteq k \subseteq k(\!(T)\!)$.

2.3 Ultrametric triangle inequality

Given two Laurent series $f = \sum_{n=v}^{\infty} a_n T^n$ and $g = \sum_{n=w}^{\infty} b_n T^n$ and try to add them. The lowest degree in which there will be a non-zero coefficient can surely not be lower than the order of f and the order of g. This is the ultrametric triangle inequality:

$$\operatorname{ord}(f+g) \geq \min\left\{ \operatorname{ord} f, \operatorname{ord} g\right\} \qquad \Longleftrightarrow \qquad |f+g| \leq \max\left\{|f|, |g|\right\}.$$

2.4 Every triangle is isosceles

This refines the previous point. Let's assume that $\operatorname{ord} f \neq \operatorname{ord} g$, so without loss of generality $v < w$. Then the sum $f + g$ will have coefficient $a_v + 0$ in front of T^v, i.e.

$$|f+g| = \exp(-v) = \max\left\{|f|, |g|\right\}.$$

If on the other hand $\operatorname{ord} f = \operatorname{ord} g$ then we have no control over what happens in $f + g$: the two Laurent series might have $a_n = -b_n$ for arbitrarily many $n = v, v+1, v+2, ...$, so $\operatorname{ord}(f+g)$ might get arbitrarily big and conversely $|f+g|$ arbitrarily small. But in this case $|f| = |g|$ are the two sides of same length.

[2] By definition $\operatorname{ord}(0) = \infty$ and $|0| = 0$.