Berkovich unit disc with trivial absolute value

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In the lecture "Introduction to Berkovich geometry" we have discussed the Berkovich unit disc $\mathcal{M}(T_1)$ for algebraically closed, complete, non-trivially valued base field k in great detail. Here we will see how the picture changes when k is algebraically closed with the trivial absolute value $|\cdot|_{\text{triv}}$. (Note that k is automatically complete in this setting, but this will not even matter.)

The Tate algebra is the polynomial ring

$$T_1 = \left\{ \sum_{n=0}^{\infty} a_n z^n \;\middle|\; |a_n| \to 0 \text{ for } n \to \infty \right\} = k[z]$$

and the Gauß-norm becomes

$$\begin{aligned} ||\cdot|| : k[z] &\longrightarrow \mathbb{R}_{\geq 0} \\ f &\longmapsto \begin{cases} 1 & \text{if } f \neq 0 \\ 0 & \text{if } f = 0. \end{cases} \end{aligned}$$

It is easy to see that $||\cdot||$ is a multiplicative non-archimedean norm and $(T_1, ||\cdot||)$ is a Banach algebra.

Claim 1. *The field k is spherically complete.*

Proof. Assume we have a descending chain of discs $(a_n, r_n)_n$ (as always $r_n \leq 1$ for all n). If $r_n = 1$ for all n, then

$$D_{r_n}(a_n) = D_1(0) = k \qquad \text{for all } n.$$

If $r_n < 1$ for some n, then $D_{r_m}(a_m) = \{a_m\}$ for all $m \geq n$. By the containment condition $D_{r_{n+1}}(a_{n+1}) \subseteq D_{r_n}(a_n)$ we even get $D_{r_m}(a_m) = \{a_n\}$ for all $m > n$.

In particular, in both cases the intersection over all $D_{r_n}(a_n)$ is not empty. ☐

For the following note that k algebraically closed implies that every $f \in k[z]$ can be written uniquely as a product of a unit in k and linear factors of the form $z - a$.

Claim 2. *The Berkovich unit disc as a set is precisely*

$$\mathcal{M}(T_1) = \left\{ \zeta_{a,r} \;\middle|\; a \in k, r \in [0,1] \right\} \cup \left\{ ||\cdot|| \right\},$$

where

$$\begin{aligned} \zeta_{a,r} : T_1 &\longrightarrow \mathbb{R}_{\geq 0} \\ (z-b) &\longmapsto \begin{cases} 1 & \text{if } b \neq a \\ r & \text{if } b = a \end{cases} \end{aligned}$$

extended by multiplicativity to all of T_1. The points of type I are the $\zeta_{a,0}$, the Gauß-norm is the only point of type II, and all other points are of type III. There are no points of type IV.

Proof. It is quite easy to see that the $\zeta_{a,r}$ are all in $\mathcal{M}(T_1)$. To show that every point $\gamma \in \mathcal{M}(T_1)$ is of this form we verify that $\gamma(z-a) < 1$ implies $\gamma(z-b) = 1$ for all $b \neq a$. Indeed:

$$\gamma(z-b) = \gamma(z-a+a-b) \leq \max\Big\{ \underbrace{\gamma(z-a)}_{<1}, \underbrace{\gamma(b-a)}_{=1} \Big\} = 1$$

and the inequality is in fact an equality. ☐

Now let's talk about the topology.

Claim 3. *A basis for the topology of $\mathcal{M}(T_1)$ is given by the sets*

- $\Delta^\circ(a,r) = \{\zeta_{a,r} \mid 0 \leq r < r_1\}$ *for* $a \in k$ *and* $0 < r_1 \leq 1$.
- $\Delta^\circ(a,r_1) \setminus \Delta(a,r_0) = \{\zeta_{a,r} \mid r_0 < r < r_1\}$ *for* $a \in k$ *and* $0 \leq r_0 < r_1 \leq 1$.
- $\mathcal{M}(T_1) \setminus \bigcup_{i=1}^N \Delta(a_i,r_i) = \mathcal{M}(T_1) \setminus \bigcup_{i=1}^N \{\zeta_{a_i,r} \mid 0 \leq r \leq r_i\}$ *for* $a_i \in k$ *and* $0 \leq r_i < 1$.

In particular, note that $\mathcal{M}(T_1)$ is **not** homeomorphic to $|k|$ many copies of $[0,1] \subseteq \mathbb{R}$ glued in one point (the neighborhood of $||\cdot||$ is different from this picture).

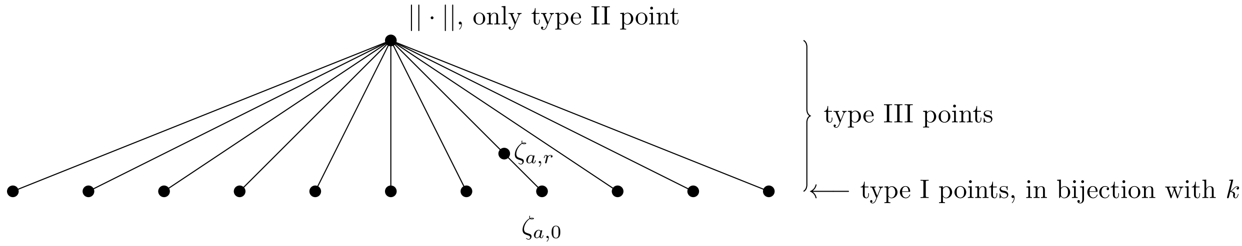

Figure 1: $\mathcal{M}(T_1)$ for trivially valued base field.

It is still true that

1. $D_1(0) = k$ embeds into $\mathcal{M}(T_1)$, and
2. $\mathcal{M}(T_1)$ is (uniquely) path connected, Hausdorff, and compact.

But $D_1(0)$ (i.e. the set of type I points) is not dense. The set $\{||\cdot||\}$ of type II points is closed and hence not dense either. Moreover, we still see that the Berkovich tree only branches at points of type II and there the set of directions of branches is in bijection with the residue field of k (which is just k itself).