The Curve Length Extremizing Flow in Riemannian and Lorentzian Manifolds

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Abstract

We consider the evolution of curves with (fixed end points) along the curve length extremizing flow. These curves are embedded in Riemannian manifolds or in Lorentzian manifolds. If the curvature is uniformly bounded, we can prove long-time existence and convergence to geodesics. Such a uniform bound can be established under natural assumptions on the manifold.

Keywords: Curve shortening flow, Lorentzian manifolds long-time existence, geodesics.

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Note to the reader.

Contents

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1 Introduction

Review: Recent related publications

Much work has been done in recent years on geometric evolution equations. In 1984 G. HUISKEN considered the evolution of convex surfaces along the *mean curvature flow* (cf. [Hui84]). A special outline of the principles which allow short-time existence for weakly elliptic operators was given in the work of R. HAMILTON which appeared in 1982 (cf. [Ham82]) and in which he proves that short-time existence holds as long as a particular integrability condition is satisfied. In 1989 M. GRAYSON published his work on the curve shortening flow for embedded curves (cf. [Gra89]), which a year later has been succeeded by M. GAGE's article on curve shortening on surfaces (cf. [Gag90]). S. ALTSCHULER has considered as part of his doctoral dissertation the evolution of spacelike closed immersed curves under the curve shortening flow ([AG92]). In 1996 M. HOLDER, a member of this research group, considered in his *Diplomarbeit* the evolution of hypersurfaces under the *inverse mean curvature flow* (cf. [Hol96]). Just to name a few more researchers who added valuable contributions to this problem: B. ANDREWS, S. ANGENENT, R. BARTNIK, M. GAGE and M. GRAYSON, which is by no means a comprehensive list.

Overview of this paper

This *Diplomarbeit* is a generalization of ALTSCHULER's paper and uses his ideas in large parts. We consider closed curves in Riemannian manifolds $\mathcal{M}^{n+1}$ and curves which are fixed at the end points $F(a)$ and $F(b)$ for Riemannian as well as Lorentzian manifolds, where in the latter case the curves have to be timelike. For these curves we apply the so-called curvature flow which yields a system of weakly parabolic partial differential equations. We consider several settings, all of which are also of importance in the physical sciences, under which a solution to this problem exists for all times t. We also prove under which circumstances this solution approaches a particular class of curves called geodesics.

The paper is organized as follows:

This section serves as an introduction to the general concept of evolution equations and shows applications to the curvature flow.

In the second section we give a mathematical definition of the curvature flow problem and we explain why this flow may be called "curve length extremizing flow".

In the third section, we introduce notation and a particular "triple" norm which we use for our estimates. In the Lorentzian case, we will replace the standard norm by this new norm.

In section 4, we derive the evolution equations of the curvature and the tangent vector. We also calculate the commutator of the tangent vector and the curvature vector and show that the speed of the curve is always well-defined and does not vanish.

Section 5 is devoted to the short-time existence: we use the method R. HAMILTON has established and show that some particular conditions hold such that we are allowed to apply his existence theorem.

In section 6, we consider a gradient function which measures the angle between the tangent vector and a given timelike unit vector field. This is in order to guarantee that the tangent vector remains bounded and does not tilt to the null cone. We show that parallel vector fields and warped product spaces are sufficient to establish such a bound on the angle.

Evolution equations of derivatives of the tangent vector of higher order in the arc length parameter are calculated in section 7.

In section 8, these evolution equations will be applied to the standard norm of the derivatives of the tangent vector. We prove that the evolution equations are such that the weak parabolic maximum principle in the exponential formulation holds, as long as the curvature remains bounded. This means that in between the fixed end points the maximum of the derivatives can at most grow exponentially.

In section 9, we show that the maximum principle in the exponential form also holds for the corresponding evolution equations in the Lorentzian case. Boundedness will then be defined by making use of the particular "triple" norm.

Section 10 deals with the boundedness of the higher derivatives of the tangent vector on the end points. Assuming that the curve is either without boundary or closed, or that the manifold is flat, and furthermore that the curvature is uniformly bounded then the derivatives can at most grow exponentially on the end points. Therefore it is — under these premises — possible to iterate the short-time existence theorem by applying the weak parabolic maximum principle in order to obtain long-time existence.

In section 11 we show that boundedness of the curvature even implies that a bound for the higher order derivatives of the tangent vector can be established uniformly for all positive times.

In section 12 we further investigate the evolution of the length and the conditions under which a geodesic between two fixed end points exists. We show that in the Lorentzian case the curvature will always be uniformly bounded.

We prove in section 13 that if there is a uniform bound for all positive times on the curvature, the curve must converge to a geodesic.

Therefore there remains only the restriction that the curvature has to be uniformly bounded on the whole time interval $[0, \infty)$. This can be established in the case of Lorentzian manifolds, as well as for manifolds which admit a parallel vector field such as the Euclidian space. Furthermore, if the manifold has a warped product structure, an assumption on the warping function which can be satisfied for a large class of applications in cosmology yields this bound as well.

Applications

In general relativity theory, particles travel on timelike paths. In particular, the paths of particles freely falling under the influence of gravity are geodesics. Therefore one often has to calculate the geodesics between two given points. Since usually the settings are very complex, there often are no analytic solutions and one needs powerful numerical methods and extensive computing resources to obtain a numerical solution. As for particle paths, one often employs numerical methods such as "shooting methods" and ray tracing. These methods start at one point, parallely translating constant sections of the curve along their tangent vector. Due to the error which is caused by the approximating time steps, there might be problems associated with the convergence of the solution, which is of particlular concern if one has a discreet metric.

One could think of a more sophisticated approach, such as computing the curve with a conventional ray tracing method, and then smoothening the curve between the fixed end points in order to obtain a well fitting approximation to a nearby geodesic. The approach of smoothening a given timelike curve with fixed end points seems to be natural, since it evolves a curve as a whole — instead of translating constant slices a certain number of times. Some study of such applications will probably be done as a part of the *Sonderforschungsbereich* (special research project) # 382 "*Verfahren und Algorithmen zur Simulation physikalischer Prozesse auf Höchstleistungsrechnern*" (methods and algorithms simulating physical processes on metacomputers).

In [Set90] J. Sethian gave an overview of the problems associated with a numerical treatment and suggested an approach which is adequate to this type of evolution equations. Unfortunately, some straightforward techniques such as numerical approximations to the Lagrangian formulation yield stability defects, and restricted space dimensions cause considerable trouble. He suggested using entropy-satisfying upwind algorithms such as the Engquist-Osher scheme instead. There are also numerically generated pictures which show curves that evolve under the curvature flow and other flows depending

on some function of the curvature.
We also have to mention the research group in applied mathematics of G. Dzuick at Universität Freiburg. Some of their research interests are the mean curvature flow, numerical calculations of viscosity solutions, and geometric variational problems.

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2 The Curve Length Extremizing Problem

In this paper, we want to investigate the following initial value problems:
Case 1. For a given initial curve $F_0(x) : [a,b] \to \mathcal{M}^{n+1}$ $(a, b \in \mathbb{R}, a < b)$, which has non-vanishing *speed*

$$c_0(x) = \sqrt{\left\langle \frac{\partial F_0(x)}{\partial x}, \frac{\partial F_0(x)}{\partial x} \right\rangle},$$

find a solution $F(t,x) : [0,T) \times [a,b] \to \mathcal{M}^{n+1}$ for some $T > 0$ such that

$$\begin{aligned} \frac{\partial}{\partial t} F(t,x) &= k(t,x) N(t,x) \\ F(0,x) &= F_0(x) \end{aligned} \tag{$\star$}$$

and either

$$F(t,a) = F_0(a) \qquad F(t,b) = F_0(b) \quad (F \text{ has fixed end points})$$

or

$$F(t,a) = F(t,b) \quad (F \text{ is a closed loop})$$

holds. For a closed loop, we assume that at the points a and b, F fits smoothly.
Case 2. In the case of a semi-Riemannian manifold $\mathcal{M}^{n,1}$, we consider a timelike initial curve $F_0(x) : [a,b] \to \mathcal{M}^{n,1}$. Let $|\cdot|$ denote the (positive) absolute value of a vector: here

$$\left| \frac{\partial}{\partial x} F(t,x) \right| = \sqrt{- \left\langle \frac{\partial}{\partial x} F(t,x), \frac{\partial}{\partial x} F(t,x) \right\rangle}.$$

The unit tangent vector, which we define by

$$T(t,x) = \frac{\frac{\partial}{\partial x} F(x,t)}{\left| \frac{\partial}{\partial x} F(x,t) \right|},$$

thus yields $\langle T, T \rangle = -1$. In this situation we want to find a solution $F(t,x) : [0, T_0) \times [a,b] \to \mathcal{M}^{n,1}$, for some $T_0 > 0$, to the *Dirichlet problem*

$$\begin{aligned} \frac{\partial}{\partial t} F(t,x) &= k(t,x) N(t,x) \\ F(0,x) &= F_0(x) \\ F(a,t) = F_0(a) \qquad & F(b,t) = F_0(b). \end{aligned} \tag{$\star\star$}$$

Note. In the fixed end point case, the variation vector field kN has to vanish at $F_t(a)$ and $F_t(b)$ for all positive times t; therefore we have for all $t > 0$ that $k(t,a) = 0 = k(t,b)$ (cf. [O'N83, p. 265]).

We say that the *chronology condition* holds for the manifold $\mathcal{M}^{n,1}$, if $\mathcal{M}^{n,1}$ contains no timelike closed curves (cf. [O'N83, p. 407]).

2.1. Lemma. *The direction of the curvature flows* ($\star$) *and* ($\star\star$) *yields a gradient flow in length of the curve if in the Lorentzian case only variation vector fields V are considered which are perpendicular to the tangent vector T: $\langle V, T\rangle = 0$ [which only requires that the variation vector field does not cause tangential "acceleration" of the curve]. For this reason, we call the flow the* curve length extremizing flow.

Proof. The notation will be introduced in the next section. The first variation formula [O'N83, p. 263–5] for a (timelike) variation vector field V of a fixed endpoint problem yields:

$$\begin{aligned}
\frac{\partial l}{\partial t} =& \varepsilon \int_a^b \left\langle T, \frac{\partial V}{\partial x}\right\rangle \, dx \overset{\text{integration by parts}}{=} \varepsilon \underbrace{\left.\langle T, V\rangle\right|_a^b}_{=0} - \varepsilon \int_a^b \left\langle \frac{\partial T}{\partial x}, V \right\rangle \, dx \\
=& - \varepsilon \int_a^b \left\langle \frac{\partial s}{\partial x}\frac{\partial T}{\partial s}, V\right\rangle \, dx = -\varepsilon \int_a^b \left\langle \frac{\partial T}{\partial s}, V\right\rangle c \, dx \\
=& - \varepsilon \int_0^l \left\langle \frac{\partial T}{\partial s}, V\right\rangle \, ds = -\varepsilon \int_0^l \langle kN, V\rangle \, ds
\end{aligned}$$

In the Riemannian case, the result follows by CAUCHY-SCHWARZ's inequality. In the Lorentzian case, the remaining part of the claim will be proven in the appendix [p. 85]. □

3 Notations and Construction of a Norm

Notations

Throughout this paper, we want to use the following notation:

We consider Riemannian and Lorentzian manifolds of dimension $n+1 \in \mathbb{N}$ with a torsion-free symmetric "metric" (indefinite bilinear form) $\langle\cdot,\cdot\rangle$ of signature $n+1$ or $n-1$ respectively. These manifolds will be denoted by $\mathcal{M}^{n+1}$, resp. $\mathcal{M}^{n,1}$. Our sign convention for the case $\mathcal{M} = \mathcal{M}^{n,1}$ will be

$$\langle\cdot,\cdot\rangle \cong \langle\langle -1, +1, \dots, +1\rangle\rangle \equiv \operatorname{diag}(-1, +1, \dots, +1).$$

To be able to treat $\mathcal{M}^{n+1}$ and $\mathcal{M}^{n,1}$ simultaneously, we let $\varepsilon := \operatorname{sgn}\langle T, T\rangle$ which is $+1/-1$, respectively. We write $\mathcal{M}$ and $T\mathcal{M}$ for its collection of tangent spaces if we have no need to specify the appropriate structure.

The Lorentzian manifold is supposed to be time orientable, i.e. we assume there exists a timelike vector field $e_0 \in T\mathcal{M}^{n,1}$ (cf. [O'N83, p.145]). Then a timelike vector field $V \in T\mathcal{M}^{n,1}$ is defined to be *future pointing* if $\langle V, e_0\rangle < 0$ (cf. [O'N83, p.144 f]). Furthermore, we assume that there is an extrinsic orthonormal frame field $\{e_i\}_{i=0}^n$, where e_0 is the future timelike vector field which establishes a time-orientability.

By t we denote the evolution parameter, and $Y := \partial/\partial t$ the corresponding tangent vector. The "spatial" parameter x will most often be replaced by the arc length parameter s, where

$$s(x) = \int_a^x \sqrt{\varepsilon \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle} d\tilde{x}, \tag{3.1}$$

$$\text{and therefore } c := \frac{\partial s_t}{\partial x} = \sqrt{\varepsilon \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle}. \tag{3.2}$$

Then $T = \partial/\partial s$, and for completeness, we let $X := \partial/\partial x$. We have $T = X/c$.

In order to facilitate computations, we would like to use a unit normal vector field $N \perp T$, which we would like to define by $N = Y/k$. Obviously, if $k(x,t) = 0$, then there is not a unique choice of such a normal, and at these points the direction of N may vary discontinuously. Additionally, even for points $k(x,t) \neq 0$ there still remains a freedom of choice of orientation of N. We could require that initially $\int_a^b k_0(x)dx \geq 0$, k_0 continuous, and demand that for all $(x,t) \in [a,b] \times [0,T)$ the curvature k and the normal N are chosen

such that they are continuous in x and t (which indeed is possible).
We mention that very often a sign convention for k and N is chosen such that for the evolution of F_t rather $(\partial/\partial t)F = -\varepsilon kN$ holds. Actually, the definition of N as, say, "outer" or "inner" normal does not matter, since it is the flow Y we are interested in. We will avoid having to deal with singular points $k = 0$ by considering $k^2 = \langle Y, Y\rangle$, which is always well defined.
In theorem (5.1) of [Alt91, p. 503], ALTSCHULER mentions a method suggested by R. HAMILTON of how to avoid this difficulty by using some $\kappa_\delta := \sqrt{k^2+\delta} > k$, where $\delta > 0$.
Due to the smoothening properties of the curvature flow, which yields a weakly parabolic system of evolution equations (cf. section 5), $k^2(s,t)$ is smooth for all times $t > 0$ for which F_t exists.

Let $T^{(n)} := (\partial^n/\partial s^n)T := (\nabla_T)^n T$, and, in particular, $T' := T^{(1)}$, $T'' := T^{(2)}$ and $T''' := T^{(3)}$.

We let $\|v\|^2 := \langle v, v\rangle$ — note: $\|v\|^2$ may be negative since the scalar product $\langle\cdot,\cdot\rangle$ is not a positive definite form.

We say that a function $g : [a,b] \times [0,\infty) \to \mathbb{R}$ is *uniformly bounded* if there exists a $G > O$ such that $|g(x,t)| \leq G$ for all $(x,t) \in [a,b] \times [0,\infty)$.

By writing $x \in \{a,b\}$ we mean that x is an element of this set of two elements, i.e. $x = a$ or $x = b$.

Construction of a Norm and a Scalar Product

The projections of a vector $v \in T\mathcal{M}$ onto the tangential and normal space (with respect to the tangent vector T) will be computed as follows:

$$\Pi^\top(v) := \langle v, \varepsilon T\rangle\, T \tag{3.3}$$
$$\Pi^\perp(v) := v - \Pi^\top(v) = v - \langle v, \varepsilon T\rangle\, T \tag{3.4}$$

We therefore have

$$\left\langle v, \Pi^\top(v)\right\rangle = \left\langle \Pi^\top(v), \Pi^\top(v)\right\rangle = \varepsilon\,\langle v, T\rangle^2 \tag{3.5}$$

and

$$\begin{aligned}\left\langle v, \Pi^\perp(v)\right\rangle &= \left\langle \Pi^\perp(v), \Pi^\perp(v)\right\rangle \\ &= \langle v, v\rangle - 2\varepsilon\,\langle v, T\rangle^2 + \varepsilon\langle v, T\rangle^2 = \langle v, v\rangle - \varepsilon\,\langle v, T\rangle^2\end{aligned} \tag{3.6}$$

In the remaining part of this section, we will pay special attention to the case of Lorentzian manifolds.

Since for $v \in T\mathcal{M}^{n,1}$, $\|\Pi^\top(v)\|^2 \leq 0$, we introduce a norm $|||.|||$ which we define as

$$|||v|||^2 := -\left\langle v, \Pi^\top(v)\right\rangle + \left\langle v, \Pi^\perp(v)\right\rangle \tag{3.7}$$

$$= \left\langle v, \Pi^\perp(v) - \Pi^\top(v)\right\rangle = \langle v, v\rangle + 2\langle v, T\rangle^2 \tag{3.8}$$

With respect to the basis $\{e_i\}_{0\leq i\leq n}$, where $e_0 = T$, $e_1 \parallel N$ and $\langle e_i, e_j\rangle = \pm\delta_{ij}$ for all $0 \leq i, j \leq n$, this norm corresponds to a positive-definite bilinear form with a representation $\langle\langle 1, \ldots, 1\rangle\rangle$. It will be used to bound T'' and its higher derivatives $T^{(n)}$.

Unlike for $\|.\|$, we obtain a Cauchy-Schwarz type inequality:

$$\langle v, w\rangle^2 \leq |||v|||^2 \cdot |||w|||^2 \quad \text{for all } v, w \in T\mathcal{M} \tag{3.9}$$

(For verification, consider the orthonormal frame $\{e_i\}_{0\leq i\leq n}$ as defined above).

Since the norm $|||.|||$ is defined using the tangent vector T, which may change in time, we have to take care that T does not tilt asymptotically to the null cone. Therefore we need an additional future timelike unit vector field U, which we use to bound the angle $\langle U, \varepsilon T\rangle = \cosh \angle(U, T)$ between U and T.

For the reason of completeness, we proceed with this construction: Assume there is a given orthonormal system $\{e_i\}_{0\leq i\leq n}$, where e_0 is the (only) timelike basis vector; e_0 will be used for the projections $\Pi^\perp(\cdot)$ and $\Pi^\top(\cdot)$. The standard definition of a norm in terms of the bilinear form $\langle\cdot,\cdot\rangle$ yields the one-form $\|\cdot\|^2$, which is *not* nonnegative. An appropriate type of projection $[\Pi^\perp(\cdot) - \Pi^\top(\cdot)]$ yields a norm $|||\cdot|||$, which in turn *is* positive definite:

$$|||v||| := \langle v, v\rangle + 2\langle v, e_0\rangle^2 \quad \text{for all } v \in T\mathcal{M}\,. \tag{3.10}$$

The usual polarization procedure (note that $\operatorname{char}\mathbb{R} \neq 2$) yields a positive definite bilinear form $\mathrm{b}(\cdot,\cdot)$:

$$\mathrm{b}(v, w) = \langle v, w\rangle + 2\langle v, e_0\rangle\langle w, e_0\rangle \quad \text{for all } v, w \in T\mathcal{M}\,. \tag{3.11}$$

Now [HE73, p. 38–40] suggest a procedure which yields a semi-Riemannian metric out of a Riemannian one, using a line element field $(e_0, -e_0)$:

$$\begin{aligned}(v, w) :=& \mathrm{b}(v, w) - 2\frac{\mathrm{b}(v, e_0)\mathrm{b}(w, e_0)}{\underbrace{\mathrm{b}(e_0, e_0)}_{=+1}} \\ =& \mathrm{b}(v, w) - 2\mathrm{b}(v, e_0)\mathrm{b}(w, e_0) \quad \text{for all } v, w \in T\mathcal{M}\,.\end{aligned} \tag{3.12}$$

Defining a reflection σ_ξ by means of the vector field $\xi \in T\mathcal{M}$, on the hyperplane $\xi^\perp$, by

$$\sigma_\xi(v) := v - 2\frac{\mathrm{b}(v,\xi)}{\mathrm{b}(\xi,\xi)}\xi, \tag{3.13}$$

we get

$$(v,w) = \mathrm{b}(\sigma_{e_0}(v), w) = \mathrm{b}(v, \sigma_{e_0}(w)) \tag{3.14}$$

We observe that $(\cdot,\cdot)$ is just the same as $\langle\cdot,\cdot\rangle$; therefore the entire procedure is bijective:

$$\begin{aligned}(v,w) &= \langle v,w\rangle + 2\langle v,e_0\rangle\langle w,e_0\rangle \\ &\qquad - 2(\langle v,e_0\rangle + 2\langle v,e_0\rangle\langle e_0,e_0\rangle)\cdot(\langle w,e_0\rangle + 2\langle w,e_0\rangle\langle e_0,e_0\rangle) \\ &= \langle v,w\rangle + 2\langle v,e_0\rangle\langle w,e_0\rangle - 2(-)\langle v,e_0\rangle(-)\langle w,e_0\rangle = \langle v,w\rangle\end{aligned}$$

The procedure can be represented by the following commutative diagram:

$$\begin{tikzcd}
\langle\cdot,\cdot\rangle \arrow[r, "\text{contraction}", "\mathcal{M}^{n,1}"'] & \|.\|^2 \arrow[d, "\Pi^\perp(\cdot)-\Pi^\top(\cdot)"] \\
\mathrm{b}(\cdot,\cdot) \arrow[u, "\text{reflection } \mathrm{b}(\sigma_{e_0}(\cdot),\cdot)", "\text{line element } (e_0,-e_0)"'] & |||.|||^2 \arrow[l, "\text{polarization}"', "\mathcal{M}^{n+1}"]
\end{tikzcd}$$

4 Basic Evolution Equations

In this section we calculate the commutator of $T = \partial/\partial s$ and $Y = \partial/\partial t$, and the evolution equations of the tangent vector T and the curvature k. We also show that if $F_0(x)$ is timelike, then it will remain so for all $t > 0$.

We start with some basic calculations:

4.1. Remark. We get these explicit formulae for T, Y, and their derivatives: Since $\varepsilon = \langle T, T\rangle$, we obtain

$$0 = \langle T, \nabla_T T\rangle = \langle T, Y\rangle \; (= k\,\langle T, N\rangle).$$

Using $k^2 = \langle Y, Y\rangle$, we can calculate

$$\begin{aligned} -k^2 &= \langle T, \nabla_T Y\rangle = \langle T, T''\rangle\,, \\ \frac{\partial k^2}{\partial s} &= \langle Y, \nabla_T Y\rangle = \langle Y, T''\rangle\,, \\ \frac{\partial k^2}{\partial t} &= \langle Y, \partial Y/\partial t\rangle\,, \end{aligned}$$

and then

$$\left\langle \frac{\partial Y}{\partial t}, T\right\rangle = -\left\langle Y, \frac{\partial T}{\partial t}\right\rangle.$$

If the curvature $k \neq 0$, we may define N. Due to the orthonormality conditions $1 = \langle N, N\rangle$ and $0 = \langle T, N\rangle$ there is:

$$0 = \langle N, \nabla_T N\rangle$$

and

$$0 = \langle \nabla_T T, N\rangle + \langle T, \nabla_T N\rangle = \langle kN, N\rangle + \langle T, \nabla_T N\rangle\,.$$

Let $\nabla_T N = \sum_{i=0}^n \alpha_i E_i$, where $\{E_i\}_{i=0}^n$ is an adapted orthonormal system such that $E_0 = T$ and $E_1 = N$ (there *is* such a system!). Since

$$-k = \langle T, \nabla_T N\rangle = \varepsilon\alpha_0$$

it follows that

$$\alpha_0 = -\varepsilon k$$

and therefore

$$\nabla_T N = -\varepsilon k T + \sum_{i=2}^{n} \alpha_i E_i. \tag{4.1}$$

Note: If we have a moving (FRENET) frame, we then may obtain $\alpha_2 = \tau$, $\alpha_3 = \ldots = \alpha_n = 0$, where τ is the torsion.

4.2. Remark.
(i) Since x and t are independent parameters of the two-parameter variation which yields the curve $F(x,t)$, we have

$$\nabla_X Y = \frac{\partial^2 F}{\partial x \partial t} = \frac{\partial^2 F}{\partial t \partial x} = \nabla_Y X, \tag{4.2i}$$

which is equivalent to

$$[X, Y] = 0. \tag{4.2ii}$$

For further references, consult [Kli82],[O'N83, p. 123, (4.44)] and [D'I92, p. 137].
(ii) Since the lenght l changes with the time parameter t [cf. (12.2)], the "spatial" arc length parameter s, which is the reparametrization of F to give the unit length tangent vector T, and t are *not* independent. In fact,

$$[Y, T] = \nabla_Y \left(\frac{X}{c}\right) - \nabla_{X/c} Y = X \frac{\partial}{\partial t}\left(\frac{1}{c}\right) \overset{(4.3)}{=} \varepsilon k^2 T, \tag{4.2iii}$$

where the calculation of the "speed" c is done in the proof of lemma 4.4.

4.3. Lemma. *If F_t is a solution of the curvature flow equation $(\star)$ or $(\star\star)$ for some time t, then the evolution of the "speed" $c > 0$, where $c^2 := \varepsilon \left\|\partial F/\partial x\right\|^2$, is given by:*

$$\text{(i) } \frac{\partial}{\partial t} c^2 = -2\varepsilon k^2 c^2 \tag{4.3i}$$

$$\text{(ii) } \frac{\partial c}{\partial t} = -\varepsilon k^2 c \tag{4.3ii}$$

$$\text{(iii) } 0 = \left\langle T, \varepsilon \nabla_T Y + k^2 T \right\rangle \tag{4.3iii}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}c^2 =&\frac{\partial}{\partial t}\langle \varepsilon X, X\rangle = 2\varepsilon \langle X, \nabla_Y X\rangle = 2\varepsilon \langle X, \nabla_X Y\rangle \\ =&2\varepsilon c^2 \langle T, \nabla_T Y\rangle = 2\varepsilon c^2(\nabla_T \langle T, Y\rangle - \langle \nabla_T T, Y\rangle) \\ =&-2\varepsilon c^2 k^2 = -2\varepsilon k^2 \varepsilon \langle X, X\rangle = -2k^2 \langle X, X\rangle . \end{aligned}$$ □

4.4. Lemma. *We have for any vector field $V \in T\mathcal{M}$:*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}V = \frac{\partial}{\partial s}\frac{\partial}{\partial t}V + \varepsilon k^2 \frac{\partial}{\partial s}V + R(Y,T)V, \tag{4.4}$$

where $R : T\mathcal{M} \times T\mathcal{M} \times T\mathcal{M} \to T\mathcal{M}$ is the Riemann tensor in $\mathcal{M}$. *

Proof.

$$\begin{aligned}\nabla_Y\nabla_T V =&\nabla_Y(\frac{1}{c}\nabla_X V) = \frac{-1}{c^2}\frac{\partial c}{\partial t}\nabla_X V + \frac{1}{c}\nabla_Y\nabla_X V \\ =&\frac{-1}{c^2}(-\varepsilon k^2 c)\nabla_X V + \frac{1}{c}\{\nabla_X\nabla_Y V + R(Y,X)V\} \\ =&\frac{\varepsilon k^2}{c}\nabla_X V + \frac{1}{c}\nabla_X\nabla_Y V + R(Y,T)V \\ =&\varepsilon k^2 \nabla_T V + \nabla_T\nabla_Y V + R(Y,T)V. \end{aligned}$$ □

4.5. Lemma. *The tangent vector T satisfies the evolution equation*

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + \varepsilon k^2 T. \tag{4.5}$$

If $k^2 \neq 0$, and where $\sum_{i\geq 2}\alpha_i E_i$ is defined as in (4.1), we get the following expression in terms of an orthonormal frame:

$$\frac{\partial T}{\partial t} = \frac{\partial k}{\partial s}N + k\sum_{i\geq 2}\alpha_i E_i$$

Proof.

$$\nabla_Y T = \nabla_Y(\frac{1}{c}X) = \frac{-1}{c^2}(-\varepsilon k^2 c)X + \frac{1}{c}\nabla_X Y = \frac{\varepsilon k^2}{c}X + \nabla_T Y$$

and if $k^2 \neq 0$

$$= \varepsilon k^2 T + \nabla_T(kN) = \frac{\partial k}{\partial s}N + k\sum_{i\geq 2}\alpha_i E_i.$$ □

*We use the "Landau-Lifschitz Spacelike Convention"; cf. [MTW73] for a comparison chart on the signs used by various authors. Note that [O'N83] uses the negative sign for the Riemann tensor.

4.6. Lemma. *If the curvature $k^2 \neq 0$, its evolution is determined by*

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + \varepsilon k^3 - k\mu + k \langle N, R(N,T)T \rangle , \tag{4.6}$$

where $\mu = \mu(t,s) = \sum_{i=2}^n \alpha_i^2$ (cf. (4.1)) is some nonnegative parameter.

Proof.
We will explicitly calculate each term of the following commutator relation:

$$\langle N, \nabla_Y Y \rangle = \langle N, \nabla_Y \nabla_T T \rangle \overset{(4.4)}{=} \left\langle N, \varepsilon k^2 \nabla_T T + \nabla_T \nabla_Y T + R(Y,T)T \right\rangle .$$

- $\langle N, \nabla_Y Y \rangle = \langle N, \nabla_Y (kN) \rangle = k \langle N, \nabla_Y N \rangle + \left\langle N, \dfrac{\partial k}{\partial t} N \right\rangle = \dfrac{\partial k}{\partial t},$

  since $2 \langle N, \nabla_Y N \rangle = \nabla_Y \langle N, N \rangle = \nabla_Y(1) = 0.$

- $\left\langle N, \varepsilon k^2 \nabla_T T \right\rangle = \varepsilon k^2 \langle N, kN \rangle = \varepsilon k^3$

- $$\begin{aligned} \langle N, \nabla_T \nabla_Y T \rangle &= \left\langle N, \nabla_T (\varepsilon k^2 T + \nabla_T Y) \right\rangle = \left\langle N, \nabla_T (\varepsilon k^2 T + k \nabla_T N + \frac{\partial k}{\partial s} N) \right\rangle \\ &= \left\langle N, 2\varepsilon k \frac{\partial k}{\partial s} T + \varepsilon k^2 (kN) + \frac{\partial k}{\partial s} \nabla_T N + k \nabla_T^2 N + \frac{\partial k}{\partial s} \nabla_T N + N \frac{\partial^2 k}{\partial s^2} \right\rangle \\ &= \varepsilon k^3 + k \left\langle N, \nabla_T^2 N \right\rangle + \frac{\partial^2 k}{\partial s^2} \end{aligned}$$

- $\nabla_T N = -\varepsilon k T + \displaystyle\sum_{i=2}^n \alpha_i E_i$

Therefore

$$\nabla_T^2 N = -\varepsilon \frac{\partial k}{\partial s} T - \varepsilon k (kN) + \sum_{i \geq 2} \frac{\partial \alpha_i}{\partial s} E_i + \alpha_i \nabla_T E_i$$

such that

$$k \left\langle N, \nabla_T^2 N \right\rangle = -\varepsilon k^3 + k \sum_{i=2}^n \alpha_i \langle N, \nabla_T E_i \rangle$$

- $$\frac{\partial k}{\partial t} = \langle N, \nabla_Y Y\rangle = \varepsilon k^3 + k\,\langle N, R(N,T)T\rangle + \varepsilon k^3 + \frac{\partial^2 k}{\partial s^2} - \varepsilon k^3 + k \sum_{i=2}^{n} \alpha_i \,\langle N, \nabla_T E_i\rangle\,.$$

Since for all $i \geq 2$, $\langle N, E_i\rangle = 0$, we therefore get

$$0 = \nabla_T \langle N, E_i\rangle = \langle \nabla_T N, E_i\rangle + \langle N, \nabla_T E_i\rangle = \alpha_i + \langle N, \nabla_T E_i\rangle$$

and

$$\alpha_i \langle N, \nabla_T E_i\rangle = \alpha_i(-\alpha_i) = -\alpha_i^2.$$

Let $\mu := \sum_{i=2}^n \alpha_i^2 \geq 0$, then we finally have

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + \varepsilon k^3 - k\mu + k\,\langle N, R(N,T)T\rangle\,.$$ □

Note. In [Gra89, p. 75], Grayson set

$$kR := k\,\langle R(T,N)T, N\rangle$$

and considered R as the ambient Gaussian curvature of the manifold $\mathcal{M}^{n,1}$ restricted to $F_t(\cdot)$

4.7. Theorem. *Case $\mathcal{M} = \mathcal{M}^{n+1}$:*
If the initial curve $F_0(x) : [a,b] \to \mathcal{M}^{n+1}$ has non-vanishing speed $c_0(x) = c(0,x) = |\partial F_0(x)/\partial x|$ and the curvature k is uniformly bounded, then the speed $c(t,x) = |\partial F(t,x)/\partial x|$ is decreasing at most exponentially and thus $c(t,x) > 0$ for all $t \geq 0$.

Proof.

$$\frac{\partial}{\partial t} c^2 = \frac{\partial}{\partial t} \left\| \frac{\partial F}{\partial x} \right\|^2 = -2k^2c^2 \geq -2k_{\max}^2 c^2$$

Standard comparison methods (cf. [McO96, p. 322]), for instance Gronwall's inequality ([McO96, p. 327]), yield:

$$c^2(t,x) \geq \exp\left(-2k_{\max}^2 t\right) c^2(0,x)$$ □

4.8. Theorem. *Case $\mathcal{M} = \mathcal{M}^{n,1}$:*
If the initial curve $F_0(x) : [a,b] \to \mathcal{M}^{n,1}$ is timelike, then the evolving curve $F_t(x)$ stays timelike for all $t \geq 0$.

Proof. In the case of $\varepsilon = -1$, we have that

$$\frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right\rangle = -\frac{\partial c^2}{\partial t} = -2k^2c^2 \leq 0.$$

Therefore $\langle X, X\rangle$ shrinks for all $t \geq 0$, which implies that if $\langle X, X\rangle|_{t=0} \leq \alpha < 0$, then $\langle X, X\rangle \leq \alpha$ for all $t \geq 0$. $\square$

4.9. Corollary. *For all $t \geq 0$ such that the evolving curve F_t exists, c is well-defined and positive for both $\mathcal{M}^{n+1}$ and $\mathcal{M}^{n,1}$, and c^{-1} is defined for all such times t.*

5 Short-Time Existence

In this section we show that the curvature flow has a solution at least on a short time interval $[0, T_0[$. This result allows us to conclude that if the solution F_t exists on some closed time interval $[0, T_0]$, it can be extended at least for some small time τ to exist for times t in $[0, T_0 + \tau[$. Once we have shown that the derivatives of the tangent vector to the curve, including the curvature k^2, do not blow up for any finite time t, for any time interval $[0, T_0[$, the curve may be extended by some well-defined limiting process to the closed interval $[0, T_0]$ and therefore — by the short-time existence result — we obtain long-time existence, i.e. existence of F_t for all $t \geq 0$.

5.1. Theorem. *The evolution equations* $(\star)$ *and* $(\star\star)$ *for the curve* F_t*,*

$$\frac{\partial}{\partial t} F = kN = Y = \nabla_T T = \frac{\partial^2}{\partial s^2} F,$$

yield a weakly parabolic system.

Proof. In the previous section we have shown that for all time intervals $[0, T_0[$ such that F_t exists as a solution of the curvature flow, $1/c$ is well defined.

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{c} =& \frac{\partial}{\partial x}\left(\varepsilon \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x}\right\rangle\right)^{-\frac{1}{2}} = \frac{-1}{2|\partial F/\partial x|^3} 2\varepsilon \left\langle \frac{\partial^2 F}{\partial x^2}, \frac{\partial F}{\partial x}\right\rangle \\ =& \frac{-\varepsilon}{|\partial F/\partial x|^3}\left\langle \frac{\partial^2 F}{\partial x^2}, \frac{\partial F}{\partial x}\right\rangle = \frac{-\varepsilon}{c^3}\left\langle \frac{\partial^2 F}{\partial x^2}, \frac{\partial F}{\partial x}\right\rangle\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t} F =& \frac{\partial}{\partial s} T = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} T = \frac{1}{c}\left(\frac{\partial}{\partial x}\left(\frac{X}{c}\right)\right) = \frac{1}{c}\left\{\frac{-\varepsilon}{c^3}\left\langle \frac{\partial^2 F}{\partial x^2}, \frac{\partial F}{\partial x}\right\rangle X + \frac{1}{c}\frac{\partial^2 F}{\partial x^2}\right\} \\ =& \frac{1}{c^2}\left\{\frac{\partial^2 F}{\partial x^2} - \left\langle \varepsilon T, \frac{\partial^2 F}{\partial x^2}\right\rangle T\right\} = \frac{1}{c^2}\Pi^\perp\left(\frac{\partial^2 F}{\partial x^2}(t,x)\right)\end{aligned}$$

Since $\Pi^\perp \circ (\partial^2/\partial s^2)$ is weakly elliptic with eigenvalues in $\{0, 1\}$, the result follows. $\square$

5.2. Corollary. *The results of [HG86] and [Ham82] guarantee short-time existence of the evolution equations* $(\star)$ *and* $(\star\star)$*.*

Proof. In terms of the notation of the papers cited above, we have a system

$$\frac{\partial F}{\partial t} = \frac{1}{c^2}\Pi^\perp\left(\frac{\partial^2 F}{\partial x^2}\right) = E(F),$$

where $E(F) = 1/c^2\Pi^\perp\left(\partial^2/\partial x^2\right)(F)$ is weakly elliptic of second order.

We take as integrability condition $B(F)$:

$$B(F)V := \langle V, \varepsilon T\rangle\, T = \Pi^\top(V) \quad \forall\, V \in T\mathcal{M}\,.$$

$B(F)$ is linear in V of degree 0 and we get:

$$B(F)\circ E(F) = \Pi^\top \circ \frac{1}{c^2}\Pi^\perp\left(\frac{\partial^2 F}{\partial x^2}\right) = 0 \tag{A}$$

(that is, $B\circ E$ is exact), and

$$\ker B(F) = \{V \in T\mathcal{M} \mid \langle V, T\rangle = 0\} = \{V \in T\mathcal{M} \mid \Pi^\top(V) = 0\} \tag{B}$$

On $\ker B(F)$, the symbol $\sigma DE(F)$ acts by multiplication by $1/c^2$, which is positive for all $c \neq 0$.

In the previous section we have established that $c \neq 0$ for all times t. Therefore the results of [Ham82, theorem (5.1)] immediately apply to the case of closed curves, where no boundary values are to be satisfied. For the fixed end point case, we consider solutions in the space $\tilde{\mathcal{C}}(a,b) := \mathcal{J}^+(a)\cap\mathcal{J}^-(b)$ (if $b \in \mathcal{J}^+(a)$) ([Wal84, p. 206 f]), the space of continuous (timelike) curves joining a and b. Then the short time existence theorem applies to operators $E(\cdot)$ on the space $\tilde{\mathcal{C}}(a,b)$ and a slight modification gives the desired result. ☐

Note. A corresponding short-time existence theorem for hypersurfaces can be found in [Pol96, section 2.5].

5.3. Corollary. *An important feature of this flow is the smoothness of the short-time solutions $F_t(x)$.*

Therefore we did not have to impose a condition assuring the compatibility of the initial curve F_0 and the boundary values of F_t at $x \in \{a,b\}$: the compatibility is satisfied for all times $t > 0$, while it might fail at $t = 0$ (e.g. it actually will if $k(a,0) \neq 0$).

This yields the following important result:
If our system admits a solution $F(t,x)$ on the interval $[0,\delta)\times[a,b]$, we can consider the modified *initial curve $\tilde{F} = F_{\delta/2} = F(\delta/2,\cdot)$, which is smooth and satisfies $k(\delta/2,x) = 0$ for $x \in \{a,b\}$, i.e. which is compatible to the fixed end point condition.*
In the following, we can proceed using this modified initial curve $\tilde{F}$ and therefore assume without loss of generality that $k(0,x) = 0$ for $x \in [a,b]$.

6 Evolution of the Angle

In this section we are considering the "angle" between the tangent vector and a given unit vector field. There are several reasons why this is of importance to our problem.

- Case $\varepsilon = -1$:
  In section 3, we have already mentioned that the evolution of T needs special consideration. The unit timelike tangent vector T is used to enable us to extract (by projection) the timelike part of an arbitrary vector. Unfortunately, if T tilts to the null cone, experience tells us that, while the value of $\langle T, T\rangle$ remains $= -1$, T will become longer and longer if measured by a positive definite metric. Therefore we need some appropriate timelike unit vector field U which enables us to calculate the evolution of the angle $\langle U, T\rangle$ in order to ensure that T will not asymptotically become null.

- Case $\varepsilon = -1$:
  Another phenomenon occurs if we want to estimate the value of the Riemannian tensor Riem by its norm. Its norm is *not* invariant under a coordinate transformation, since the transformation from the basis $\{e_i\}_{0\le i\le n}$ to another basis $\{f_i\}_{0\le i\le n}$ yields factors $\langle e_i, f_j\rangle_{0\le i,j\le n}$. The value of these factors can be estimated by $\langle e_i, f_j\rangle \le |e_i| \cdot |f_j| = 1$ for a positive definite bilinear form, i.e. if e_i and f_j are spacelike. If both e_i and f_j are timelike, we get $\langle e_i, f_j\rangle = -\cosh\angle(e_i, f_j) \le -1$ [O'N83, p. 144]. Therefore, if the norm of Riem is given with respect to some extrinsic orthonormal system, with U being its unit timelike vector, the value of $\langle Y, R(Y,T)T\rangle$ (which we are interested in) can be estimated by:

  $$\langle Y, R(Y,T)T\rangle \le k^2 \langle U, T\rangle^4 \,|||\mathrm{Riem}|||_{\mathrm{op}} \tag{6.1}$$

  (for a more precise account on this problem, cf. [EH91, p. 599] and lemma 7.11).

- Case $\varepsilon = +1$:
  A third reason for calculating the evolution of the angle is that this enables us to conclude that k^2 has to remain bounded (this is relevant to the Riemannian case, since, under particular assumptions, there will be a theorem which guarantees boundedness of k^2 in the Lorentzian case). Of course, this result is not straightforward, and it is due to an idea which has already been successfully used in [AG92].

There will be several possibilities in order to obtain such a result.

6.1. Remark. Let $U \in T\mathcal{M}$ be a timelike unit vector field. The full evolution equation of $\langle U, \varepsilon T\rangle$ is as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\langle U, \varepsilon T\rangle =& \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle U, \varepsilon T\rangle \\ &+ \left\langle \frac{\partial}{\partial t}U - \frac{\partial^2}{\partial s^2}U, \varepsilon T\right\rangle - 2\left\langle \frac{\partial}{\partial s}U, \varepsilon Y\right\rangle\end{aligned}$$

which is an immediate consequence of the evolution equation (4.5) of T.

In the following subsections we will impose particular assumptions on U and we will show that under theses assumptions the angle will be uniformly bounded for all times $t \geq 0$.

6.1 Parallel Vector Fields

6.2. Definition. Parallel vector fields are vector fields $Z \in T\mathcal{M}$ such that

$$\nabla Z = 0.$$

6.3. Theorem. *If our manifold $\mathcal{M}$ admits a timelike parallel vector field $\tilde{U}$, $\langle U, \varepsilon T\rangle$ obeys the following equation:*

$$\frac{\partial}{\partial t}\langle U, \varepsilon T\rangle = \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle U, \varepsilon T\rangle, \tag{6.2}$$

where $U = \tilde{U}/\sqrt{\varepsilon\left\langle \tilde{U}, \tilde{U}\right\rangle}$.

Proof. If $\tilde{U}$ is parallel, then so is U.

$$\begin{aligned}\frac{\partial}{\partial t}\langle \varepsilon U, T\rangle =& \left\langle \varepsilon U, \frac{\partial}{\partial t}T\right\rangle \overset{(4.5)}{=} \left\langle \varepsilon U, T'' + \varepsilon k^2 T\right\rangle \\ =& \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle \varepsilon U, T\rangle\end{aligned}$$ □

Note. It would have been "nearly sufficient" to assume that U is a unit Killing vector field†. Then the evolution equation would read

$$\frac{\partial}{\partial t}\langle U, \varepsilon T\rangle = \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle U, \varepsilon T\rangle + 2\left\langle \frac{\partial}{\partial t}U, \varepsilon T\right\rangle,$$

with the last term not being known well enough to be handled properly.

†Killing vector fields will be defined in 6.9.

6.4. Theorem. *Assume that the short-time existence theorem guarantees the existence of the curve F_t on $U_T := [a,b] \times (0,T)$ for some time T. If U is a parallel vector field on $\mathcal{M}$, then:*

(i) For Riemannian manifolds $\mathcal{M}^{n+1}$, we have that the infimum $\inf_{U_T} \langle U,T \rangle$ is nondecreasing with t. If $\langle U,T \rangle$ is not constant, then the infimum is strictly increasing with t: $\inf_{U_T} \langle U,T \rangle = \inf_{[a,b]\times\{0\}} \langle U,T \rangle$.

(ii) For semi-Riemannian manifolds $\mathcal{M}^{n,1}$, the supremum $\sup_{U_T} \langle U,-T \rangle$ is nonincreasing with t. If $\langle U,-T \rangle$ is not constant on U_T, then the supremum is strictly decreasing with t: $\sup_{U_T} \langle U,-T \rangle = \sup_{[a,b]\times\{0\}} \langle U,-T \rangle$.

Proof. We use the standard strong parabolic maximum principle, cf. [McO96, p. 308–9] and [Sta96, p. 7–8 (weak case only)]. In terms of the notation of [McO96], for $\varepsilon = -1$ we have the case $c = -k^2 < 0$.

For $\varepsilon = +1$, we apply the maximum principle to $-\langle U,T \rangle$ and immediately get

$$\frac{\partial}{\partial t}(-\langle U,T \rangle) = \frac{\partial^2}{\partial s^2}(-\langle U,T \rangle) - k^2 \langle U,T \rangle \leq \frac{\partial^2}{\partial s^2}(-\langle U,T \rangle).$$

Considering the end points, we observe that for all $t > 0$

$$\left.\frac{\partial}{\partial s} \langle U,T \rangle\right|_{\{a,b\}} = \langle U,T' \rangle|_{\{a,b\}} = 0,$$

since $k = 0$ on the boundary of $[a,b]$. Assume the extremum lies on the boundary $\{a,b\} \times (0,T)$, but *not* in U_T. Then the exterior derivative $\pm\partial/\partial s$ must be positive, which yields a contradiction. This leaves the two cases as mentioned above. □

6.5. Corollary. *(i) Since for Riemannian manifolds $\mathcal{M}^{n+1}$, $\langle U,T \rangle \leq 1$, $\langle U,T \rangle$ is uniformly bounded from above and below for all times t.*
(ii) Case $\varepsilon = -1$: Since both U and T are future timelike, $\langle U,-T \rangle \geq 1$, and therefore $\langle U,-T \rangle$ is uniformly bounded from above and below for all times t.

6.6. Definition. The *sectional curvature* of the plane generated by the tangent vectors A and B of a tangent space $T_p\mathcal{M}$, $p \in \mathcal{M}$, is a symmetric form $K : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ which is defined by:

$$K(A,B) := \frac{\langle R(A,B)B, A \rangle}{\|A\|^2 \|B\|^2 - \langle A,B \rangle^2}.$$

We assume that the tangent plane generated by A and B is not degenerated, which guarantees that the denominator of the defining term of the sectional curvature is not vanishing. This definition is independent of the particular choice of basis for the tangent plane (cf. [O'N83, p. 77 and the remark on p. 89]).

6.7. Theorem. *If there is a parallel unit vector field U on $\mathcal{M}$, and if the sectional curvature $K(A,B)$ of the manifold $\mathcal{M}$ is nonnegative for all pairs (A,B) of tangent vector fields in $T\mathcal{M}$ where A is timelike and B spacelike, then the curvature k^2 of the evolving curve is uniformly bounded (in terms of the initial data): $k^2 \leq k^2_{\max}$, where $k^2_{\max}$ depends on the value of k^2 and the angle $\langle U, \varepsilon T\rangle^2$ at time $t=0$.*

Proof. We use an argument which can be found in [AG92].
Let $w := \langle U, \varepsilon T\rangle$ and let $u^2 := k^2/w^2$. By ∇ we want to denote the covariant derivative $\nabla_T = \partial/\partial s$ and we write Δ for $\Delta = \partial^2/\partial s^2$.

Note: the previous corollary guarantees that $w \geq \min_{(t=0,x)} w > 0$ for all times t, which ensures that k^2/w^2 is well-defined.

According to (7.4) the evolution equation for k^2 is as follows:

$$\frac{\partial}{\partial t}k^2 = \Delta k^2 - 2\left|\left|\left|\Pi^{\perp}(T'')\right|\right|\right|^2 + 2\varepsilon k^4 + 2\langle R(Y,T)T, Y\rangle$$

Let $P := \left|\left|\left|\Pi^{\perp}(T'')\right|\right|\right|^2$ and $k^2 R := \langle R(Y,T)T, Y\rangle$.

We compute the derivatives of $u^2 = (k/w)^2$:

$$\text{(i)}\quad \frac{\partial}{\partial t}\frac{k^2}{w^2} = \frac{1}{w^2}\left(\Delta k^2 - 2P + 2\varepsilon k^4 + 2k^2 R\right) - 2\frac{k^2}{w^3}\left(\Delta w + \varepsilon k^2 w\right)$$

$$\begin{aligned}\text{(ii)}\quad \frac{\partial^2}{\partial s^2}\frac{k^2}{w^2} &= \frac{\partial}{\partial s}\left(\frac{1}{w^2}\frac{\partial k^2}{\partial s} - \frac{2k^2}{w^3}\frac{\partial w}{\partial s}\right)\\ &= -\frac{2}{w^3}\frac{\partial w}{\partial s}\frac{\partial k^2}{\partial s} + \frac{1}{w^2}\Delta k^2 - \frac{2k^2}{w^3}\Delta w - \frac{2}{w^3}\frac{\partial k^2}{\partial s}\frac{\partial w}{\partial s} + \frac{6k^2}{w^4}\left(\frac{\partial w}{\partial s}\right)^2\end{aligned}$$

and

$$\text{(iii)}\quad \left(\frac{\partial}{\partial t} - \Delta\right)u^2 = -\frac{2P}{w^2} + 2Ru^2 + \frac{4}{w^3}\frac{\partial w}{\partial s}\frac{\partial k^2}{\partial s} - \frac{6k^2}{w^4}\left(\frac{\partial w}{\partial s}\right)^2$$

Since

$$\frac{3}{w}\frac{\partial w}{\partial s}\frac{\partial}{\partial s}\left(\frac{k^2}{w^2}\right) = \frac{3}{w^3}\frac{\partial w}{\partial s}\frac{\partial k^2}{\partial s} - \frac{6k^2}{w^4}\left(\frac{\partial w}{\partial s}\right)^2$$

we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)u^2 = -\frac{2P}{w^2} + 2Ru^2 + \frac{3}{w}\nabla w \nabla u^2 + \frac{1}{w^3}\nabla w \nabla k^2$$

We distinguish two cases:

Case 1: $u^2 = 0$, i.e. $k^2 = \langle Y, Y\rangle = 0$, which implies, if $\varepsilon = +1$, that $Y = 0$.
In the case $\varepsilon = -1$, we know that T is timelike and $Y \perp T$. Then $Y = 0$ or Y is spacelike. In the latter case, since $\langle Y, Y\rangle = 0$, we also have $Y = 0$. Therefore $\nabla k^2 = 2\langle Y, T''\rangle = 0$, and we get the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u^2 = -\frac{2P}{w^2} + 2Ru^2 + \frac{3}{w}\nabla w \nabla u^2.$$

Case 2: $u^2 \neq 0$. In this situation, k and the normal $N = Y/k$ are well defined. In particular, we can calculate the basis representation of P with respect to the adapted orthonormal frame $\{E_i\}_{i=0}^n$, $E_0 = T$, $E_1 = N$:

$$P = \left\|\Pi^\perp(T'')\right\|^2 = \left\|\sum_{i=1}^n \beta_i E_i\right\|^2 = \underbrace{\sum_{i=2}^n \beta_i^2}_{=:\tilde{\mu}} + \beta_1^2$$

Since

$$\beta_1 = \langle N, T''\rangle = \frac{1}{k}\langle Y, T''\rangle = \frac{1}{2k}\nabla\langle Y, Y\rangle = \frac{1}{2k}\nabla k^2$$

this yields

$$-\frac{2P}{w^2} = -\frac{2}{w^2}\tilde{\mu} - \frac{1}{2w^2k^2}\left(\nabla k^2\right)^2.$$

Furthermore

$$\begin{aligned} -\frac{2P}{w^2} &+ \frac{1}{w}\nabla w \nabla u^2 + \frac{1}{w^3}\nabla w \nabla k^2 \\ &= -\frac{2}{w^2}\tilde{\mu} - \frac{1}{2w^2k^2}\left(\nabla k^2\right)^2 + \frac{k^2}{w}\nabla w \nabla\frac{1}{w^2} + \frac{2}{w^3}\nabla w \nabla k^2 \end{aligned}$$

Since $2\nabla w = -w^3\nabla(1/w^2)$, this equals to

$$\begin{aligned} &= -\frac{2}{w^2}\tilde{\mu} - \frac{1}{2w^2k^2}\left(\nabla k^2\right)^2 - \frac{k^2w^2}{2}\left(\nabla\frac{1}{w^2}\right)^2 - \nabla k^2 \nabla\frac{1}{w^2} \\ &= -\frac{2}{w^2}\tilde{\mu} - \frac{1}{2u^2}\frac{1}{w^4}\left(\nabla k^2\right)^2 - \frac{1}{2u^2}k^4\left(\nabla\frac{1}{w^2}\right)^2 - \nabla k^2\nabla\frac{1}{w^2} \\ &= -\frac{2}{w^2}\tilde{\mu} - \frac{1}{2u^2}\left(\nabla u^2\right)^2, \end{aligned}$$

due to

$$\begin{aligned}(\nabla u^2)^2 =& \left(\frac{1}{w^2}\nabla k^2 + k^2\nabla\frac{1}{w^2}\right)^2\\ =&\frac{1}{w^4}\left(\nabla k^2\right)^2 + k^4\left(\nabla\frac{1}{w^2}\right)^2 + 2u^2\nabla k^2\nabla\frac{1}{w^2}.\end{aligned}$$

This finally yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)u^2 = 2Ru^2 + \frac{2}{w}\nabla w\nabla u^2 - \frac{2}{w^2}\tilde{\mu} - \frac{1}{2u^2}\left(\nabla u^2\right)^2.$$

In consequence, we can summarize both cases by noting that the evolution of $u^2 = k^2/w^2$ is determined by the operator

$$\frac{\partial}{\partial t} \leq \Delta + \max\left(\frac{2}{w}\nabla w, \frac{3}{w}\nabla w\right)\nabla + 2R \tag{6.3}$$

$$\leq \Delta + 3\left|\frac{\nabla w}{w}\right|\nabla + 2R. \tag{6.4}$$

The most natural assumption which yields $k^2R \equiv \langle R(Y,T)T,Y\rangle \leq 0$ without knowledge of the particular vector fields Y and T seems to be a general assumption on the sectional curvature K. If we assume that this quadratic form is nonnegative, we know that for arbitrary vector fields A and B, one of which is assumed to be timelike and the other one spacelike, $\|A\|^2\|B\|^2 < 0$ and therefore $\langle R(A,B)B,A\rangle \leq 0$.
Therefore the maximum principle holds for the evolution of k^2/w^2.
Fortunately, this time we do not have to bother about the behavior of k^2/w^2 at the end points, since at $x \in \{a,b\}$, $k^2 = 0$, $w^2 > 0$ and therefore $k^2/w^2 = 0$. As a consequence, the maximum principle implies that the maximum of k^2/w^2 is not increasing.

Finally, we may conclude that the maximum of k^2 is bounded for all times and remains less than some positive constant:

$$\text{Case } \varepsilon = +1: \quad w^2 \leq 1 \quad \Rightarrow \quad k^2 \leq \frac{k^2}{w^2} \leq \max_{(t,x)}\frac{k^2}{w^2} = \max_{(t=0,x)}\frac{k^2}{w^2}$$

$$\begin{aligned}\text{Case } \varepsilon = -1: \quad k^2 = \frac{k^2}{w^2}\cdot w^2 &\leq \max_{(t,x)}\frac{k^2}{w^2}\cdot\max_{(t,x)} w^2\\ &= \max_{(t=0,x)}\frac{k^2}{w^2}\cdot\max_{(t=0,x)} w^2\end{aligned}$$

□

6.8. Corollary. *In Euclidian and Minkowskian manifolds $\mathcal{M}$ there are two pairs of fixed positive constants, $0 < \delta_1^a < \delta_2^a < \infty$, and $0 < \delta_1^c < \delta_2^c < \infty$ independent of time such that both the angle $\langle e_0, \varepsilon T\rangle$ and the curvature k^2 are bounded uniformly in time by these constants (here e_0 denotes the unit vector field which is parallel to the canonical time axis or, in the Euclidian case, any unit vector field which is parallel).*

$$\delta_1^a \leq \langle e_0, \varepsilon T\rangle \leq \delta_2^a \quad \textit{for all } (x,t) \in [a,b] \times [0,T)$$

and

$$\delta_1^c \leq k^2(x,t) \leq \delta_2^c \quad \textit{for all } (x,t) \in [a,b] \times [0,T).$$

Proof. In Euclidian and Minkowskian manifolds there always is a (timelike) parallel vector field. The sectional curvatures vanish since Riem $\equiv 0$: these manifolds are flat. □

6.2 Killing Vector Fields

6.9. Definition. [O'N83, pp. 250–256]. A *Killing vector field* is a vector field X for which the Lie derivative $\mathfrak{L}$ of the metric $\mathrm{g}(\cdot,\cdot) = \langle\cdot,\cdot\rangle$ vanishes:

$$\mathfrak{L}_X \mathrm{g} = 0.$$

For practical purposes we state an equivalent definition: X is a Killing vector field iff the covariant derivative ∇X is skew-adjoint relative to the bilinear form $\langle\cdot,\cdot\rangle$:

$$\langle \nabla_V X, W\rangle + \langle \nabla_W X, V\rangle = 0 \quad \text{for all } V, W \in T\mathcal{M}\,.$$

As mentioned in the previous subsection, assuming U is Killing but not parallel would yield an extra term $2\,\langle \partial U/\partial t, \varepsilon T\rangle$ for the evolution equation:

$$\begin{aligned}\frac{\partial}{\partial t}\langle U, \varepsilon T\rangle =& \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle U, \varepsilon T\rangle + 2\,\langle \partial U/\partial t, \varepsilon T\rangle \\ =& \frac{\partial^2}{\partial s^2}\langle U, \varepsilon T\rangle + \varepsilon k^2 \langle U, \varepsilon T\rangle - 2\,\langle \partial U/\partial s, \varepsilon Y\rangle\end{aligned}$$

Since we could not find a way of how to deal with it, we did not further investigate this case.

We would have liked to deduce conditions under which a spacelike unit Killing vector field U would have enabled us to conclude that

- $\langle U, \varepsilon T\rangle$ remains bounded, or, even better, $\langle U, \varepsilon T\rangle \leq \langle U, \varepsilon T\rangle|_{t=0}$

- $\max(k/\langle U, \varepsilon T\rangle)$ does not increase, which yields that k is bounded.

Since U is Killing, $\langle \partial U/\partial s, T\rangle = 0$ and therefore we obtain this particular behavior of U that

$$\left\langle \frac{\partial^2}{\partial s^2}U, T\right\rangle = -\left\langle \frac{\partial}{\partial s}U, \frac{\partial}{\partial s}T\right\rangle = -\left\langle \frac{\partial}{\partial s}U, Y\right\rangle = \left\langle \frac{\partial}{\partial t}U, T\right\rangle,$$

which can be stated as:

$$\Pi^\top\left(\frac{\partial^2}{\partial s^2}U\right) = \Pi^\top\left(\frac{\partial}{\partial t}U\right).$$

6.3 Warped Product Spaces

Warped product spaces are a generalization of direct product spaces and often occur as examples in cosmological models. Consult [O'N83, p. 204 ff., and in particular pp. 343–345] for an introduction.

We consider the special case where the base $\mathcal{B}$ is one-dimensional and spanned by the future timelike unit vector field U.

In this case, we have a product manifold $\mathcal{M} = \mathcal{I} \times_f \mathcal{F}$, $\mathcal{I} = (uU)_{u\in I\subset \mathbb{R}}$, which has been established with a bilinear form $\langle .,. \rangle$ and a positive warping function $f = f(u)$. This bilinear form is equivalent to a representation

$$\langle .,. \rangle \cong \begin{pmatrix} \varepsilon & 0 \\ 0 & f^2(u) g_{ij} \end{pmatrix}, \tag{6.5}$$

where g_{ij} is a positive definite bilinear form on $\mathcal{F}$.

The formulas which can be found in [O'N83] have been adapted to this situation in [Hol96] :

6.10. Theorem. *If $\mathcal{M} = \mathcal{I} \times_f \mathcal{F}$, $U \in \mathcal{I}$, V and W in the lift $\mathcal{L}(\mathcal{F})$ of $\mathcal{F}$ to $\mathcal{M}$, then*

$$\text{(i) } \nabla_U U = 0 \tag{6.6i}$$

$$\text{(ii) } \nabla_U V = \nabla_V U = \frac{f'}{f} V \tag{6.6ii}$$

$$\text{(iii) } \langle \nabla_V W, U\rangle = -\frac{f'}{f}\langle V, W\rangle, \tag{6.6iii}$$

where $f' = U(f)$ is the derivative of f (note: $V(f) = 0$ for all $V \in \mathcal{L}(\mathcal{F})$).

Proof. [Hol96, p. 41]; note that for Riemannian manifolds, there are no sign changes, which can be checked if one considers the corresponding proof in [O'N83]. □

Since the tangent vector field T changes its position in space-time continuously (cf. section 5), it has a constant time orientation with respect to the timelike unit vector field U. Without loss of generality we may assume that the vector fields U and T are pointing in the same direction of time (otherwise use $-U$), which we will call the "future" time cone.

For the tangent vector T and the curvature vector $Y = kN$, we obtain the decomposition

$$T = t_1 U + \tilde{T}, \quad t_1 > 0, \text{ since } T \textit{ future} \text{ timelike,} \tag{6.7}$$

and

$$Y = y_1 U + \tilde{Y} \tag{6.8}$$

where $\tilde{T}$ and $\tilde{Y}$ are spacelike vectors such that $\left\langle U, \tilde{T} \right\rangle = 0 = \left\langle U, \tilde{Y} \right\rangle$. In order to derive the evolution equation of $\langle \varepsilon T, U \rangle$, we first have to calculate various relations which are summed up in the following remark. They are all easily verified.

6.11. Remark. We obtain the following list of equations:

(ia) $\nabla_Y U = \nabla_{y_1 U + \tilde{Y}} U = \nabla_{\tilde{Y}} U = \dfrac{f'}{f} \tilde{Y}$

(ib) $\langle \nabla_Y U, T \rangle = \dfrac{f'}{f} \left\langle \tilde{Y}, \tilde{T} \right\rangle$

(iia) $\nabla_T U = \nabla_{\tilde{T}} U = \dfrac{f'}{f} \tilde{T}$

(iib) $\langle \nabla_T U, T \rangle = \dfrac{f'}{f} \left\langle \tilde{T}, \tilde{T} \right\rangle$

(iiia)
$$\begin{aligned} \nabla_T T =& t_1 \nabla_U (t_1 U + \tilde{T}) + \nabla_{\tilde{T}} (t_1 U + \tilde{T}) = (t_1 U(t_1) + \tilde{T}(t_1)) U \\ &+ 2 t_1 \frac{f'}{f} \tilde{T} + \nabla_{\tilde{T}} \tilde{T} = Y = y_1 U + \tilde{Y} \end{aligned}$$

(iiib) $\left\langle U, \nabla_{\tilde{T}}\tilde{T}\right\rangle = -\frac{f'}{f}\left\langle \tilde{T}, \tilde{T}\right\rangle$

(iiic) $\langle U, \nabla_T T\rangle = \varepsilon t_1 U(t_1) + \varepsilon \tilde{T}(t_1) - \frac{f'}{f}\left\langle \tilde{T}, \tilde{T}\right\rangle = \langle U, Y\rangle = \varepsilon y_1$

(iiid) $y_1 = t_1 U(t_1) + \tilde{T}(t_1) - \varepsilon\frac{f'}{f}\left\langle \tilde{T}, \tilde{T}\right\rangle$

(iiie) $\tilde{Y} = Y - y_1 U = \nabla_T T - y_1 U = 2t_1\frac{f'}{f}\tilde{T} + \nabla_{\tilde{T}}\tilde{T} + \frac{f'}{f}\varepsilon\left\langle \tilde{T}, \tilde{T}\right\rangle U$

(iiif) $\left\langle \tilde{Y}, \tilde{T}\right\rangle = 2t_1\frac{f'}{f}\left\langle \tilde{T}, \tilde{T}\right\rangle + \left\langle \nabla_{\tilde{T}}\tilde{T}, \tilde{T}\right\rangle = -\varepsilon t_1 y_1$

(iiig)
$$\begin{aligned}\left\langle \tilde{Y}, \tilde{Y}\right\rangle =& 2t_1\frac{f'}{f}\left\langle \tilde{T}, \tilde{Y}\right\rangle + \left\langle \nabla_{\tilde{T}}\tilde{T}, \tilde{Y}\right\rangle \\ =& \left\| 2t_1\frac{f'}{f}\tilde{T} + \nabla_{\tilde{T}}\tilde{T} + \varepsilon\frac{f'}{f}\left\langle \tilde{T}, \tilde{T}\right\rangle U\right\|^2 = 4t_1^2(\frac{f'}{f})^2\left\langle \tilde{T}, \tilde{T}\right\rangle \\ &+ 4t_1\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T}, \tilde{T}\right\rangle + \left\langle \nabla_{\tilde{T}}\tilde{T}, \nabla_{\tilde{T}}\tilde{T}\right\rangle + 2\varepsilon\frac{f'}{f}\left\|\tilde{T}\right\|^2\left\langle U, \nabla_{\tilde{T}}\tilde{T}\right\rangle \\ &+ \varepsilon(\frac{f'}{f})^2\left\langle \tilde{T}, \tilde{T}\right\rangle^2 = k^2 - \varepsilon y_1^2 = \langle Y, Y\rangle - \langle y_1 U, y_1 U\rangle\end{aligned}$$

(iva)
$$\begin{aligned}\nabla_Y T =& y_1\nabla_U(t_1 U + \tilde{T}) + \nabla_{\tilde{Y}}(t_1 U + \tilde{T}) \\ =& (y_1 U(t_1) + \tilde{Y}(t_1))U + y_1\frac{f'}{f}\tilde{T} + t_1\frac{f'}{f}\tilde{Y} + \nabla_{\tilde{Y}}\tilde{T}\end{aligned}$$

(ivb) $\langle \nabla_Y T, U\rangle = \varepsilon y_1 U(t_1) + \varepsilon\tilde{Y}(t_1) - \frac{f'}{f}\left\langle \tilde{Y}, \tilde{T}\right\rangle$

6.12. Lemma. *Using these relations, we get the evolution of $\langle \varepsilon T, U\rangle$:*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}\right)\langle \varepsilon T, U\rangle = \varepsilon k^2\langle \varepsilon T, U\rangle + \varepsilon t_1\left\|\tilde{T}\right\|^2\left(\left(\frac{f'}{f}\right)^2 - \frac{f''}{f}\right) + 2t_1\frac{f'}{f}\nabla_T\langle \varepsilon T, U\rangle \tag{6.9}$$

Proof. Due to the algebra of covariant derivatives we have:

$$\frac{\partial}{\partial t}\langle T,U\rangle = \left\langle \frac{\partial^2}{\partial s^2}T,U\right\rangle + \varepsilon k^2\langle T,U\rangle + \left\langle T,\frac{\partial}{\partial t}U\right\rangle$$
$$\frac{\partial^2}{\partial s^2}\langle T,U\rangle = \langle T'',U\rangle + 2\left\langle Y,\frac{\partial U}{\partial s}\right\rangle + \left\langle T,\frac{\partial^2 U}{\partial s^2}\right\rangle$$

and therefore

$$\begin{aligned}(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial s^2})\langle T,U\rangle =&\varepsilon k^2\langle T,U\rangle + \left\langle T,\frac{\partial}{\partial t}U\right\rangle - 2\left\langle Y,\frac{\partial U}{\partial s}\right\rangle - \left\langle T,\frac{\partial^2 U}{\partial s^2}\right\rangle\\ =&\varepsilon k^2\langle T,U\rangle - \frac{f'}{f}\left\langle \tilde{Y},\tilde{T}\right\rangle - \left\langle T,\frac{\partial^2 U}{\partial s^2}\right\rangle.\end{aligned}$$

We will consider each term separately:

$$\text{(ia)}\quad \left\langle T,\frac{\partial^2 U}{\partial s^2}\right\rangle = \frac{\partial}{\partial s}\left\langle T,\frac{\partial U}{\partial s}\right\rangle - \left\langle T',\frac{\partial U}{\partial s}\right\rangle = \frac{\partial}{\partial s}\left(\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle\right) - \frac{f'}{f}\left\langle \tilde{T},\tilde{Y}\right\rangle$$

Since the warping function $f = f(u)$ is constant on the leaves $\{u = \text{constant}\}$, $\nabla_{\tilde{T}} f = 0$:

$$\begin{aligned}\text{(ib)}\quad \frac{\partial}{\partial s}\left\langle T,\frac{\partial U}{\partial s}\right\rangle =&(t_1\nabla_U + \nabla_{\tilde{T}})\left(\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle\right)\\ =&t_1\left(\frac{f'}{f}\right)'\left\langle \tilde{T},\tilde{T}\right\rangle + 2t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle + 2\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle\end{aligned}$$

such that we obtain

$$\begin{aligned}\text{(ic)}\quad \left\langle T,\frac{\partial^2 U}{\partial s^2}\right\rangle =&t_1\left(\frac{f'}{f}\right)'\left\langle \tilde{T},\tilde{T}\right\rangle + 2t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle + 2\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle\\ &-\frac{f'}{f}\left\langle \tilde{Y},\tilde{T}\right\rangle = t_1\left(\frac{f'}{f}\right)'\left\langle \tilde{T},\tilde{T}\right\rangle + 2t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle\\ &+2\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle - 2t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle - \frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle\\ =&t_1\left(\frac{f'}{f}\right)'\left\langle \tilde{T},\tilde{T}\right\rangle + \frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle.\end{aligned}$$

By the basis representation (6.7) of T:

$$\begin{aligned}\text{(iia)}\quad \nabla_T\langle U,T\rangle =&\nabla_T(\varepsilon t_1) = \varepsilon t_1 U(t_1) + \varepsilon\tilde{T}(t_1)\\ =&\left\langle \frac{\partial U}{\partial s},T\right\rangle + \langle U,Y\rangle = \frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle + \varepsilon y_1\\ =&\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle + \varepsilon t_1 U(t_1) + \varepsilon\tilde{T}(t_1) - \frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle\end{aligned}$$

which yields for the second last terms of the evolution equation:

$$\text{(iib)}\quad -\frac{f'}{f}\left\langle \tilde{Y},\tilde{T}\right\rangle - \langle U'',T\rangle = -\frac{f'}{f}\left(2t_1\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle + \left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle\right)$$

$$- t_1\left(\frac{f'}{f}\right)'\left\langle \tilde{T},\tilde{T}\right\rangle - \frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle$$

$$= -t_1\left\langle \tilde{T},\tilde{T}\right\rangle\left(\left(\frac{f'}{f}\right)^2 + \frac{f''}{f}\right) - 2\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle.$$

This gives the following equation:

$$\text{(iiia)}\quad (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2})\left\langle \varepsilon T,U\right\rangle$$

$$= \varepsilon k^2\left\langle \varepsilon T,U\right\rangle - \varepsilon t_1\left\langle \tilde{T},\tilde{T}\right\rangle\left(\left(\frac{f'}{f}\right)^2 + \frac{f''}{f}\right) - 2\varepsilon\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle$$

where the last term is

$$-2\varepsilon\frac{f'}{f}\left\langle \nabla_{\tilde{T}}\tilde{T},\tilde{T}\right\rangle = -2\varepsilon\frac{f'}{f}\left(-2t_1\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle - \varepsilon y_1 t_1\right)$$

which yields

$$\text{(iiib)}\quad (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2})\left\langle \varepsilon T,U\right\rangle = \varepsilon k^2\left\langle \varepsilon T,U\right\rangle + 4\varepsilon t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle$$

$$+ 2\frac{f'}{f}y_1 t_1 - \varepsilon t_1\left\langle \tilde{T},\tilde{T}\right\rangle\left(\left(\frac{f'}{f}\right)^2 + \frac{f''}{f}\right)$$

$$= \varepsilon k^2\left\langle \varepsilon T,U\right\rangle + \varepsilon t_1\left\langle \tilde{T},\tilde{T}\right\rangle\left(3\left(\frac{f'}{f}\right)^2 - \frac{f''}{f}\right) + 2\frac{f'}{f}y_1 t_1.$$

Furthermore

$$\nabla_T\left\langle \varepsilon T,U\right\rangle = \varepsilon\frac{f'}{f}\left\langle \tilde{T},\tilde{T}\right\rangle + y_1$$

such that

$$2\frac{f'}{f}y_1 t_1 = -2\varepsilon t_1\left(\frac{f'}{f}\right)^2\left\langle \tilde{T},\tilde{T}\right\rangle + 2t_1\frac{f'}{f}\nabla_T\left\langle \varepsilon T,U\right\rangle.$$

Therefore we may conclude that

$$
\begin{aligned}
\text{(iiie)}\quad (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial s^2}) \langle \varepsilon T, U \rangle &= \varepsilon k^2 \langle \varepsilon T, U \rangle + \varepsilon t_1 \left\langle \tilde{T}, \tilde{T} \right\rangle \left(3 \left(\frac{f'}{f} \right)^2 - \frac{f''}{f} \right) \\
&- 2\varepsilon t_1 \left(\frac{f'}{f} \right)^2 \left\langle \tilde{T}, \tilde{T} \right\rangle + 2t_1 \frac{f'}{f} \nabla_T \langle \varepsilon T, U \rangle = \varepsilon k^2 \langle \varepsilon T, U \rangle \\
&+ \varepsilon t_1 \left\| \tilde{T} \right\|^2 \left(\left(\frac{f'}{f} \right)^2 - \frac{f''}{f} \right) + 2t_1 \frac{f'}{f} \nabla_T \langle \varepsilon T, U \rangle . \qquad \square
\end{aligned}
$$

6.13. Theorem. *Assume that the short-time existence theorem guarantees the existence of the curve F_t on $U_T := [a,b] \times (0,T)$ for some time T. Also assume that $\left((f'/f)^2 - f''/f\right) = -(f'/f)' \geq 0$. Then*

(i) for Riemannian manifolds $\mathcal{M}^{n+1}$, there exists a $\delta > 0$ such that $\inf_{U_T} \langle U, T \rangle \geq \delta$.

(ii) for Lorentzian manifolds $\mathcal{M}^{n,1}$, $\langle U, -T \rangle$ is constant on U_T or the maximum $\sup_{U_T} \langle U, -T \rangle$ is decreasing: $\sup_{U_T} \langle U, -T \rangle = \sup_{[a,b]\times\{0\}} \langle U, -T \rangle$

Proof. This proof works the same way as the one in theorem 6.4. Recall that $t_1 > 0$ since T is *future* timelike. Also $\tilde{T} \in \mathcal{L}(\mathcal{F})$, which implies that $\tilde{T}$ is spacelike: $\left\|\tilde{T}\right\|^2 \geq 0$

If $\varepsilon = +1$, there exists a $\delta < 0$ such that $\sup -\langle U, T \rangle \leq \delta < 0$. Otherwise $\sup -\langle U, T \rangle = 0$ and the strong maximum principle would apply. Since the normal derivative of $\langle U, T \rangle$ on the boundary is zero, and initially, $\langle U, T \rangle > 0$, this would yield a contradiction.

If $\varepsilon = -1$, $(\partial/\partial t - \partial^2/\partial s^2) \langle -T, U \rangle \leq 0$ and the maximum principle applies immediately to $\langle -T, U \rangle$. $\square$

6.14. Remark. Warped product spaces are intimately linked to the modelling of cosmology. In particular, the condition $-(f'/f)' \geq 0$ is satisfied in most of modern evolutionary theories:

(i) The condition $-(f'/f)' \geq 0$ holds for all $f = u^\alpha$ if $\alpha \geq 0$ and u positive $[(f'/f)' = (\alpha/u)' = -\alpha/u^2]$.
If we consider a cosmological Friedmann model [a Robertson-Walker space-time], cf. [Hol96, sec. 9.4.] and [HE73], with constant vanishing curvature $K = 0$, the warping function is $f(u) = cu^{2/3}$, $c > 0$, which satisfies our assumption. Furthermore, for all big bang cosmological models, the rate of expansion initially increases like $u^{2/3}$.

(ii) [O'N83, p. 347] In the setting of warped product spaces, the Hubble parameter $H(u)$ is defined by $H(u) := f'/f$ [current estimates are

$H(u_0) \approx 75 \pm 25$ km/s Mpc$^{-1} \approx [13 \cdot 10^9$ a$]^{-1}$ for the rate of expansion at this epoch [Ger97]]. Furthermore, the deceleration parameter $q(u)$ is $q(u) := -ff''/(f')^2$. Notice that $f'' < 0$ iff $q > 0$. Since $q_0 \approx 1 \pm 0.5$ [D'I92, section 22.12], which is positive, the condition $-(f'/f)' \geq 0$ is satisfied for actual cosmological models.

(iii) [O'N83, pp. 343–345] If U is the flow vector field on a Robertson-Walker spacetime $\mathcal{M}(K, f) = \mathcal{I} \times_f \mathcal{F}$, where $K \in \{-1, 0, +1\}$ denotes the (constant) curvature, then we have the following relation [O'N83, p. 345]:

$$\mathrm{Ric}(U, U) = -3\frac{f''}{f} = 4\pi(\rho + 4p),$$

where ρ is the energy density and p the (radiation) pressure of a perfect fluid. If we assume the timelike convergence condition $\mathrm{Ric}(A, A) \geq 0$ holds for all timelike vector fields $A \in T\mathcal{M}$, then once again the condition for the boundedness of the angle $\langle U, \varepsilon T \rangle$ is satisfied.

6.15. Remark. [‡] Suppose $\mathcal{M}$ is a cosmological globally hyperbolic space time. A well-known conjecture states that if the timelike convergence condition holds, then there occurs exactly one of the two cases:

(i) There exists a (timelike) line. Then $\mathcal{M}$ splits as a metric product (this has been proven by Eschenburg), *or*

(ii) The manifold is timelike geodesic *in*complete (e.g. the 'counterexample to almost everything' the Taub-NUT space-time).

If this conjecture holds true, we already have a warped product space in the first case. Therefore only the case of a timelike geodesic incomplete manifold does not yet yield an *a priori* bound on the angle between the tangent vector T of the evolving curve and an exterior timelike unit vector field. For a further discussion of this conjecture we refer the reader to [Bar88].

[‡] I would like to thank R. Bartnik for a discussion on this topic.

7 Evolution Equations of the Derivatives $T^{(n)}$

In the succeeding three sections, we prove, under the condition that k^2 is bounded for all times t, that the evolution equations of the norms of the derivatives $T^{(n)}$ satisfy the parabolic maximum principle.

This requires to have T'' and its higher derivatives be bounded (since the length l of F_t (cf. (12.3)) resp. the curvature k^2 are bounded, T resp. $T' = Y(= kN)$ are well defined). In the case $\mathcal{M} = \mathcal{M}^{n+1}$, this will be proven by showing that $\|T^{(n)}\|^2 = \left\langle T^{(n)}, T^{(n)} \right\rangle$ is bounded.

If $\mathcal{M} = \mathcal{M}^{n,1}$, we have to use $\left|\left|\left|T^{(n)}\right|\right|\right|^2 := -\|\Pi^\top\left(T^{(n)}\right)\|^2 + \|\Pi^\perp\left(T^{(n)}\right)\|^2$ in order to be able to guarantee boundedness.

Since $\left|\left|\left|T^{(n)}\right|\right|\right|^2 = \left\langle T^{(n)}, T^{(n)} \right\rangle + 2\left\langle T^{(n)}, T \right\rangle^2$, we calculate the evolution of $\left\|T^{(n)}\right\|^2$ for both $\mathcal{M}^{n+1}$ and $\mathcal{M}^{n,1}$, and then consider each case separately.

7.1. Lemma. *The Laplacian acts on* $\left\|T^{(n)}\right\|^2$ *as follows:*

$$\frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 = 2\left\|T^{(n+1)}\right\|^2 + 2\left\langle T^{(n)}, T^{(n+2)} \right\rangle \tag{7.1}$$

Proof. This is a direct computation:

$$\frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 = \frac{\partial}{\partial s}(2\left\langle T^{(n)}, T^{(n+1)} \right\rangle) = 2\left\|T^{(n+1)}\right\|^2 + 2\left\langle T^{(n)}, T^{(n+2)} \right\rangle .$$

□

7.2. Lemma. *The evolution of the first derivative of the tangent vector is:*

$$\frac{\partial}{\partial t}\left\|T'\right\|^2 = \frac{\partial}{\partial t}k^2 = \frac{\partial^2}{\partial s^2}\left\|T'\right\|^2 - 2\left\|T''\right\|^2 + 4\varepsilon k^4 + 2\left\langle R(Y,T)T, Y \right\rangle \tag{7.2}$$

Proof. Use the commutator as derived in section 4.

$$\begin{aligned}
\frac{\partial}{\partial t}T' &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}T \overset{(4.4)}{=} \frac{\partial}{\partial s}\frac{\partial T}{\partial t} + \varepsilon k^2 \frac{\partial}{\partial s}T + R(Y,T)T \\
&\overset{(4.5)}{=} \frac{\partial}{\partial s}\Big(\frac{\partial^2 T}{\partial s^2} + \varepsilon \underbrace{k^2}^{\equiv\|T'\|^2} T\Big) + \varepsilon k^2 \frac{\partial}{\partial s}T + R(Y,T)T \\
&= \frac{\partial^2}{\partial s^2}T' + 2\varepsilon\left\langle T', T'' \right\rangle T + \varepsilon k^2 T' + \varepsilon k^2 T' + R(Y,T)T \\
&= \frac{\partial^2}{\partial s^2}T' + 2\varepsilon\left\langle T', T'' \right\rangle T + 2\varepsilon k^2 T' + R(Y,T)T
\end{aligned} \tag{7.3}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}k^2 =&\frac{\partial}{\partial t}\left\|T'\right\|^2 = 2\left\langle T', \frac{\partial}{\partial t}T'\right\rangle = 2\left\langle \frac{\partial^2}{\partial s^2}T', T'\right\rangle \\ &+ 4\varepsilon \left\langle T', T''\right\rangle \underbrace{\left\langle T, T'\right\rangle}_{=0,\text{ since } T\perp T'} + 4\varepsilon k^2 \underbrace{\left\|T'\right\|^2}_{\equiv k^2} + 2\left\langle R(Y,T)T, T'\right\rangle \\ =&\frac{\partial^2}{\partial s^2}\left\|T'\right\|^2 - 2\left\|T''\right\|^2 + 4\varepsilon k^4 + 2\left\langle R(Y,T)T, T'\right\rangle. \end{aligned}$$

□

7.3. Corollary. *In particular, we get the following evolution equation for k^2:*

$$\frac{\partial}{\partial t}k^2 = \frac{\partial^2}{\partial s^2}k^2 - 2\left|\left\|\Pi^{\perp}\left(T''\right)\right\|\right|^2 + 2\varepsilon k^4 + 2\left\langle R(Y,T)T, Y\right\rangle, \tag{7.4}$$

where in the case $\varepsilon = +1$ the triple norm shall be replaced by the double norm.

Proof.
Case $\varepsilon = +1$:

$$\begin{aligned}\left|\left\|\Pi^{\perp}\left(T''\right)\right\|\right|^2 \equiv \left\|\Pi^{\perp}\left(T''\right)\right\|^2 &= \left\|T''\right\|^2 - \left\|\Pi^{\top}\left(T''\right)\right\|^2 \\ &= \left\|T''\right\|^2 - \left\langle T, T''\right\rangle^2 = \left\|T''\right\|^2 - k^4.\end{aligned}$$

Case $\varepsilon = -1$:

$$\left|\left\|\Pi^{\perp}\left(T''\right)\right\|\right|^2 = \left\|T''\right\|^2 + \left\langle T, T''\right\rangle^2 = \left\|T''\right\|^2 + k^4.$$

This, in conjecture with the preceding lemma, immediately yields the result.

□

7.4. Corollary. *In terms of the* FRENET *frame notation, the evolution equation of k^2 can be stated as*

$$\frac{\partial}{\partial t}k^2 = \frac{\partial^2}{\partial s^2}k^2 - 2\left|\frac{\partial k}{\partial s}\right|^2 + 2\varepsilon k^4 - 2k^2\mu + 2k^2\left\langle R(N,T)T, N\right\rangle \tag{7.5}$$

if the curvature $k \neq 0$ and where $\mu = \sum_{i=2}^n \alpha_i^2 \geq 0$ is defined as in (4.1).

Proof. We have

$$\begin{aligned}\left\|T''\right\|^2 = \left\|\frac{\partial}{\partial s}(kN)\right\|^2 &\overset{(4.1)}{=} \left\|\frac{\partial k}{\partial s}N\right\|^2 + k^2\left\langle -\varepsilon kT, -\varepsilon kT\right\rangle \\ &+ k^2\sum_{i=2}^n \left\|\alpha_i E_i\right\|^2 = \left|\frac{\partial k}{\partial s}\right|^2 + \varepsilon k^4 + k^2\sum_{i=2}^n \alpha_i^2.\end{aligned}$$

Therefore the previous corollary yields

$$\Rightarrow \quad \begin{aligned}\frac{\partial}{\partial t}k^2 =& \frac{\partial^2}{\partial s^2}k^2 - 2\left|\frac{\partial k}{\partial s}\right|^2 - 2\varepsilon k^4 \\ &- 2k^2\sum_{i=2}^{n}\alpha_i^2 + 4\varepsilon k^4 + 2k^2\left\langle R(N,T)T, N\right\rangle . \end{aligned} \qquad \square$$

7.5. Definition.
We introduce a useful abbreviation:

$$R^{(i)} := \partial^i/\partial s^i\{R(Y,T)\},$$

for the i-th spatial derivative of the Riemannian tensor applied to T and Y.

7.6. Lemma. *We proceed to the calculation of the evolution equation of the second derivative of the tangent vector:*

$$\begin{aligned}\frac{\partial}{\partial t}\|T''\|^2 =& \frac{\partial^2}{\partial s^2}\|T''\|^2 - 2\|T'''\|^2 + 4\varepsilon\|T''\|^2\langle T'', T\rangle \\ &+ 4\varepsilon\langle T', T'''\rangle\langle T'', T\rangle + 12\varepsilon\langle T', T''\rangle^2 + 6\varepsilon k^2\|T''\|^2 \\ &+ 2\left\langle T'', R^{(1)}T + 2R^{(0)}T'\right\rangle .\end{aligned} \tag{7.6}$$

Proof. The commutator relation (4.4) gives:

$$\begin{aligned}\frac{\partial}{\partial t}T'' =& \frac{\partial}{\partial t}\frac{\partial}{\partial s}T' = \frac{\partial}{\partial s}\frac{\partial T'}{\partial t} + \varepsilon k^2\frac{\partial}{\partial s}T' + R(Y,T)T' \\ \overset{(7.3)}{=}& \frac{\partial}{\partial s}\{\frac{\partial^2}{\partial s^2}T' + 2\varepsilon\langle T', T''\rangle T + 2\varepsilon k^2 T' + R(Y,T)T\} + \varepsilon k^2 T'' + R(Y,T)T' \\ =& \frac{\partial^2}{\partial s^2}T'' + 2\varepsilon\|T''\|^2 T + 2\varepsilon\langle T', T'''\rangle T + 2\varepsilon\langle T', T''\rangle T' \\ &+ 4\varepsilon\langle T', T''\rangle T' + 2\varepsilon k^2 T'' + \frac{\partial(R^{(0)}T)}{\partial s} + \varepsilon k^2 T'' + R^{(0)}T' \\ =& \frac{\partial^2}{\partial s^2}T'' + 2\varepsilon\|T''\|^2 T + 2\varepsilon\langle T', T'''\rangle T + 6\varepsilon\langle T', T''\rangle T' \\ &+ 3\varepsilon k^2 T'' + R^{(1)}T + 2R^{(0)}T'.\end{aligned} \tag{7.7}$$

This yields

$$\begin{aligned}\frac{\partial}{\partial t}\|T''\|^2 =& 2\left\langle\frac{\partial}{\partial t}T'', T''\right\rangle = 2\left\langle T'', \frac{\partial^2}{\partial s^2}T'' + 2\varepsilon\|T''\|^2 T\right. \\ &\left. + 2\varepsilon\langle T', T'''\rangle T + 6\varepsilon\langle T', T''\rangle T' + 3\varepsilon k^2 T'' + R^{(1)}T + 2R^{(0)}T'\right\rangle \\ =& \frac{\partial^2}{\partial s^2}\|T''\|^2 - 2\|T'''\|^2 + 4\varepsilon\|T''\|^2\langle T'', T\rangle + 4\varepsilon\langle T', T'''\rangle\langle T'', T\rangle \\ &+ 12\varepsilon\langle T', T''\rangle^2 + 6\varepsilon k^2\|T''\|^2 + 2\left\langle T'', R^{(1)}T + 2R^{(0)}T'\right\rangle .\end{aligned}$$

$\square$

7.7. Lemma. *Furthermore, the third derivative $T^{(3)}$ has the following evolution equation:*

$$\begin{aligned}\frac{\partial}{\partial t}\left\|T^{(3)}\right\|^2 =& \frac{\partial^2}{\partial s^2}\left\|T^{(3)}\right\|^2 - 2\left\|T^{(4)}\right\|^2 + 12\varepsilon\left\langle T'', T^{(3)}\right\rangle\left\langle T^{(3)}, T\right\rangle \\ &+ 16\varepsilon\left\|T''\right\|^2\left\langle T^{(3)}, T'\right\rangle + 4\varepsilon\left\langle T', T^{(4)}\right\rangle\left\langle T^{(3)}, T\right\rangle \\ &+ 16\varepsilon\left\langle T', T^{(3)}\right\rangle^2 + 24\varepsilon\left\langle T', T''\right\rangle\left\langle T'', T^{(3)}\right\rangle \\ &+ 8\varepsilon k^2\left\|T^{(3)}\right\|^2 + 2\sum_{i=0}^{2}\left\langle\frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)}), T^{(3)}\right\rangle.\end{aligned} \tag{7.8}$$

Proof. Again using the commutator for vector fields, we get

$$\begin{aligned}\frac{\partial}{\partial t}(T^{(3)}) =& \frac{\partial}{\partial s}\frac{\partial}{\partial t}(T'') + \varepsilon k^2 T^{(3)} + R^{(0)}T^{(2)} \\ =& \frac{\partial}{\partial s}\Big(\frac{\partial^2}{\partial s^2}T'' + 2\varepsilon\left\|T''\right\|^2 T + 2\varepsilon\left\langle T', T'''\right\rangle T + 6\varepsilon\left\langle T', T''\right\rangle T' \\ &+ 3\varepsilon k^2 T'' + R^{(1)}T + 2R^{(0)}T'\Big) + \varepsilon k^2 T^{(3)} + R^{(0)}T^{(2)} \\ =& \frac{\partial^2}{\partial s^2}(T^{(3)}) + 4\varepsilon\left\langle T'', T^{(3)}\right\rangle T + 2\varepsilon\left\|T''\right\|^2 T' + 2\varepsilon\left\langle T'', T'''\right\rangle T + 2\varepsilon\left\langle T', T^{(4)}\right\rangle T \\ &+ 2\varepsilon\left\langle T', T'''\right\rangle T' + 6\varepsilon\left\|T^{(2)}\right\|^2 T' + 6\varepsilon\left\langle T', T'''\right\rangle T' + 6\varepsilon\left\langle T', T''\right\rangle T'' \\ &+ 6\varepsilon\left\langle T', T''\right\rangle T'' + 3\varepsilon k^2 T''' + \varepsilon k^2 T^{(3)} + \sum_{i=0}^{2}\frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)}) \\ =& \frac{\partial^2}{\partial s^2}(T^{(3)}) + 6\varepsilon\left\langle T'', T^{(3)}\right\rangle T + 8\varepsilon\left\|T''\right\|^2 T' + 2\varepsilon\left\langle T', T^{(4)}\right\rangle T \\ &+ 8\varepsilon\left\langle T', T'''\right\rangle T' + 12\varepsilon\left\langle T', T''\right\rangle T'' + 4\varepsilon k^2 T''' + \sum_{i=0}^{2}\frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)}).\end{aligned} \tag{7.9}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial t}\left\|T^{(3)}\right\|^2 =& 2\left\langle\frac{\partial}{\partial t}T^{(3)}, T^{(3)}\right\rangle = 2\left\langle T^{(3)}, \frac{\partial^2}{\partial s^2}(T^{(3)})\right\rangle \\ &+ 12\varepsilon\left\langle T'', T^{(3)}\right\rangle\left\langle T^{(3)}, T\right\rangle + 16\varepsilon\left\|T''\right\|^2\left\langle T^{(3)}, T'\right\rangle + 4\varepsilon\left\langle T', T^{(4)}\right\rangle\left\langle T^{(3)}, T\right\rangle \\ &+ 16\varepsilon\left\langle T', T^{(3)}\right\rangle^2 + 24\varepsilon\left\langle T', T''\right\rangle\left\langle T'', T^{(3)}\right\rangle + 8\varepsilon k^2\left\|T^{(3)}\right\|^2\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{i=0}^{2}\left\langle \frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)}), T^{(3)}\right\rangle = \frac{\partial^2}{\partial s^2}\left\|T^{(3)}\right\|^2 - 2\left\|T^{(4)}\right\|^2 \\
&+ 12\varepsilon\left\langle T'', T^{(3)}\right\rangle\left\langle T^{(3)}, T\right\rangle 16\varepsilon\left\|T''\right\|^2\left\langle T^{(3)}, T'\right\rangle \\
&+ 4\varepsilon\left\langle T', T^{(4)}\right\rangle\left\langle T^{(3)}, T\right\rangle + 16\varepsilon\left\langle T', T^{(3)}\right\rangle^2 + 24\varepsilon\left\langle T', T''\right\rangle\left\langle T'', T^{(3)}\right\rangle \\
&+ 8\varepsilon k^2\left\|T^{(3)}\right\|^2 + 2\sum_{i=0}^{2}\left\langle \frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)}), T^{(3)}\right\rangle. \qquad \square
\end{aligned}
$$

7.8. Lemma. *For general $n \in \mathbb{N}$, $n \geq 3$, we obtain the evolution equation for $T^{(n)}$ as follows:*

$$
\begin{aligned}
\frac{\partial}{\partial t}T^{(n)} =& \frac{\partial^2}{\partial s^2}T^{(n)} + 2\varepsilon\left\langle T^{(n+1)}, T'\right\rangle T + (n+1)\varepsilon k^2 T^{(n)} \\
&+ 2n\varepsilon\left\langle T^{(n)}, T''\right\rangle T + 2(n+1)\varepsilon\left\langle T^{(n)}, T'\right\rangle T' \\
&+ \varepsilon\sum_{i=0}^{n-1} N_i T^{(i)} + \sum_{i=0}^{n-1}\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)}),
\end{aligned}
\tag{7.10}
$$

where $N_i = N_i(T', \ldots, T^{(n-1)}; n; s, t) \in \mathbb{R}$.

Proof. We use induction by n:

$$
\begin{aligned}
n = 3: \frac{\partial}{\partial t}T^{(3)} =& \frac{\partial^2}{\partial s^2}(T^{(3)}) + 2\varepsilon\left\langle T', T^{(4)}\right\rangle T + 4\varepsilon k^2 T''' + 6\varepsilon\left\langle T'', T^{(3)}\right\rangle T \\
&+ 8\varepsilon\left\langle T', T'''\right\rangle T' + \{8\varepsilon\left\|T''\right\|^2 T' + 12\varepsilon\left\langle T', T''\right\rangle T''\} + \sum_{i=0}^{2}\frac{\partial^{2-i}}{\partial s^{2-i}}(R^{(0)}T^{(i)})
\end{aligned}
$$

Now assume that $(n-1)$ has been proven. We then have

$$
\begin{aligned}
\frac{\partial}{\partial t}T^{(n)} =& \frac{\partial}{\partial s}\frac{\partial}{\partial t}T^{(n-1)} + \varepsilon k^2 T^{(n)} + R^{(0)}T^{(n-1)} = \varepsilon k^2 T^{(n)} + R^{(0)}T^{(n-1)} \\
&+ \frac{\partial^2}{\partial s^2}T^{(n)} + 2\varepsilon\left\langle T^{(n+1)}, T'\right\rangle T + 2\varepsilon\left\langle T^{(n)}, T''\right\rangle T + 2\varepsilon\left\langle T^{(n)}, T'\right\rangle T' \\
&+ 2n\varepsilon\left\langle T', T''\right\rangle T^{(n-1)} + n\varepsilon k^2 T^{(n)} + 2(n-1)\varepsilon\left\langle T^{(n)}, T''\right\rangle T \\
&+ 2(n-1)\varepsilon\left\langle T^{(n-1)}, T^{(3)}\right\rangle T + 2(n-1)\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' \\
&+ 2n\varepsilon\left\langle T^{(n)}, T'\right\rangle T' + 2n\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' + 2n\varepsilon\left\langle T^{(n-1)}, T'\right\rangle T'' \\
&+ \varepsilon\sum_{i=0}^{n-2} N_i T^{(i+1)} + \varepsilon\sum_{i=0}^{n-2}\frac{\partial N_i}{\partial s}T^{(i)} + \sum_{i=0}^{n-2}\frac{\partial^{i+1}}{\partial s^{i+1}}(R^{(0)}T^{(n-2-i)})
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\partial^2}{\partial s^2}T^{(n)} + (n+1)\varepsilon k^2 T^{(n)} + 2\varepsilon\left\langle T^{(n+1)}, T'\right\rangle T + 2n\varepsilon\left\langle T^{(n)}, T''\right\rangle T\\
&+ 2(n+1)\varepsilon\left\langle T^{(n)}, T'\right\rangle T' + \{2(n-1)\varepsilon\left\langle T^{(n-1)}, T^{(3)}\right\rangle T\\
&+ 2n\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' + 2n\varepsilon\left\langle T^{(n-1)}, T'\right\rangle T''\}\\
&+ \{\varepsilon\sum_{i=1}^{n-1} N_{i-1}T^{(i)} + \varepsilon\sum_{i=0}^{n-2}\frac{\partial N_i}{\partial s}T^{(i)}\}\\
&+ \{R^{(0)}T^{(n-1)} + \sum_{i=1}^{n-1}\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})\}. \qquad \square
\end{aligned}
$$

7.9. Corollary. *For all $n \in \mathbb{N}$, $n \geq 3$, we have:*

$$
\begin{aligned}
\frac{\partial}{\partial t}\left\|T^{(n)}\right\|^2 =&2\left\langle \frac{\partial}{\partial t}T^{(n)}, T^{(n)}\right\rangle = \frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 - 2\left\|T^{(n+1)}\right\|^2\\
&+ 4\varepsilon\left\langle T^{(n+1)}, T'\right\rangle\left\langle T, T^{(n)}\right\rangle + 2(n+1)\varepsilon k^2\left\|T^{(n)}\right\|^2\\
&+ 4n\varepsilon\left\langle T^{(n)}, T''\right\rangle\left\langle T, T^{(n)}\right\rangle + 4(n+1)\varepsilon\left\langle T^{(n)}, T'\right\rangle^2\\
&+ 2\varepsilon\sum_{i=0}^{n-1} N_i\left\langle T^{(i)}, T^{(n)}\right\rangle + 2\sum_{i=0}^{n-1}\left\langle\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)}), T^{(n)}\right\rangle.
\end{aligned}
$$

We will introduce a tensor norm now in order to be able to obtain estimates for the Riemannian term which appears in all of the evolutions equations.

7.10. Definition. We define the norm of a $\binom{3}{1}$-tensor B with respect to the orthonormal frame $\{e_i\}_{i=0}^n$ as:

$$
\begin{aligned}
|||B|||^2_{\text{op}} =&\sup\sum_{i,j,k,l=0}^{n}\langle B(e_i,e_j)e_k, e_l\rangle^2 \qquad (7.11)\\
=&\sup\sum_{i,j,k}^{n}\sum_{l=1}^{n}(B(e_i,e_j,e_k,e^l))^2 + (-B(e_i,e_j,e_k,e^0))^2,
\end{aligned}
$$

where the supremum is taken over the intersection of the future timecone in $F(a)$ by the past timecone in $F(b)$.
If B has the representation $B = B^i{}_{jkl}e_ie^je^ke^l$, then

$$
|||B|||^2_{\text{op}} = \sup\sum_{i,j,k,l=0}^{n}(B^i{}_{jkl})^2
$$

with respect to the frame $\{e_i\}_{i=0}^n$.
The same argument as explained in the introduction of section 6 reveals the unpleasant characteristic behavior that this norm is not coordinate independant. A basis transformation $\{e_i\}_{i=0}^n \to \{f_i\}_{i=0}^n$ yields the factor (cf. [EH91, eqn. (3)])

$$|||B|||_{\text{op},\{e_i\}} \leq |||B|||_{\text{op},\{f_i\}} \langle e_0, f_0 \rangle^4 . \tag{7.12}$$

7.11. Lemma. *Nevertheless, we obtain the estimate for $\mathcal{M} = \mathcal{M}^{n,1}$:*

$$\left|\left|\left| \sum_{i=0}^{n-1} \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}) \right|\right|\right| \leq |||\text{Riem}|||_{\text{op}} \cdot \left|\left|\left| T^{(n)} \right|\right|\right| \langle e_0, \varepsilon T \rangle^4 + l.o.t., \tag{7.13}$$

where l.o.t. are lower order terms depending on $|||T'|||^2, |||T''|||^2, \ldots, \left|\left|\left|T^{(n-1)}\right|\right|\right|^2$ *and on derivatives of* Riem.

Proof. For all vectors $X = X^i e_i$, $Y = Y^i e_i$, $Z = Z^i e_i$, $W = W^i e_i$, we have:

$$\begin{aligned}
\langle R(X,Y)Z, W \rangle^2 &= \left(\sum_{i,j,k,l=0}^{n} X^i Y^j Z^k W^l \langle R(e_i, e_j) e_k, e_l \rangle \right)^2 \\
&\leq \sum_{i=0}^{n} (X^i)^2 \sum_{j=0}^{n} (Y^j)^2 \sum_{k=0}^{n} (Z^k)^2 \sum_{l=0}^{n} (W^l)^2 \sum_{i,j,k,l=0}^{n} \langle R(e_i, e_j) e_k, e_l \rangle^2 \\
&\qquad \leq \sum_{i=0}^{n} \langle X, e_i \rangle^2 \sum_{j=0}^{n} \langle Y, e_j \rangle^2 \sum_{k=0}^{n} \langle Z, e_k \rangle^2 \sum_{l=0}^{n} \langle W, e_l \rangle^2 \, |||\text{Riem}|||_{\text{op}}^2 ,
\end{aligned}$$

which follows by a generalization of CAUCHY-SCHWARTZ's Inequality to multiple indices: $(\sum_i x_i a_i)^2 \leq (\sum_i x_i^2)(\sum_i a_i^2)$; let $a_i := \sum_j a_{ij} y_j$ — induction gives the result as desired.

Note that $\sum_{i=0}^n \langle X, e_i \rangle^2 \leq |||X|||^2 \langle e_0, T \rangle^2$, which yields

$$\langle R(X,Y)Z, W \rangle \leq |||\text{Riem}|||_{\text{op}} \, |||X||| \, |||Y||| \, |||Z||| \, |||W||| \, \langle e_0, T \rangle^4 .$$

The derivatives of Riem which we are interested in are

$$\frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}) = \frac{\partial^i}{\partial s^i} (R(T', T) T^{(n-1-i)}).$$

Using the product rule for differentiation, we focus on the derivatives of T of largest order that may occur:

- $(\partial^i/\partial s^i)$ applied to $T^{(n-1-i)}$ yields $T^{(n-1)}$.
- $(\partial^i/\partial s^i)$ applied to T yields $T^{(i)}$, at most $(i = n-1)$: $T^{(n-1)}$.
- $(\partial^i/\partial s^i)$ applied to T' yields $T^{(i+1)}$, at most $(i = n-1)$: $T^{(n)}$.

Therefore

$$\begin{aligned}\left|\left|\left|\sum_{i=0}^{n-1} \frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})\right|\right|\right|^2 &\leq \text{l.o.t.} + \left|\left|\left|R(T^{(n)},T)T\right|\right|\right|^2 \\ &\leq \text{l.o.t.} + |||\text{Riem}|||_{\text{op}}^2 \cdot \left|\left|\left|T^{(n)}\right|\right|\right|^2 \langle e_0, T\rangle^8 .\end{aligned}$$

□

Note. **(i)** A corresponding relation holds for Riemannian manifolds $\mathcal{M}^{n+1}$, using the double norm instead of the triple norm. Of course the angle $\langle e_0, T\rangle$ will be bound by 1 then.

(ii) Since

$$\begin{aligned}&\left\langle R(T^{(n)},T)T, R(T^{(n)},T)T\right\rangle = \sum_{i=0}^{n} \left\langle R(T^{(n)},T)T, e_i\right\rangle^2 \\ &= \sum_{i,j,k,l=0}^{n} \left\langle T^{(n)}, e_j\right\rangle^2 \langle T, e_k\rangle^2 \langle T, e_l\rangle^2 \langle R(e_j,e_k)e_l, e_i\rangle^2 \\ &\leq \left|\left|\left|T^{(n)}\right|\right|\right|^2 \langle T, e_0\rangle^6 \, |||\text{Riem}|||_{\text{op}}^2 ,\end{aligned}$$

this special case gives the sixth power of the factor $\langle T, e_0\rangle$ instead of the general case of $\langle T, e_0\rangle^8$. Since this is assumed to be bounded from above, and we are only interested in establishing any finite bound for the terms which include the Riemannian tensor, we do not have to be that precise here.

7.12. Lemma. *The order of the n-th derivatives of the squared curvature can be estimated in terms of the derivatives $T^{(l)}$:*

$$\begin{aligned}(k^2)^{(n)} =&\text{l.o.t.} + \left\langle T^{(n+1)}, Y\right\rangle \leq \text{l.o.t.} + k\,\Pi^{\perp}\left(T^{(n+1)}\right) \\ \leq&\text{l.o.t.} + k\left|\left|\left|T^{(n+1)}\right|\right|\right| ,\end{aligned} \tag{7.14}$$

where l.o.t. are defined as in the previous lemma.

Proof.

$$\begin{aligned}&k^2 = \langle T', T'\rangle \xrightarrow{\partial/\partial s} \langle T'', T'\rangle \xrightarrow{\partial/\partial s} \text{l.o.t.} + \left\langle T^{(3)}, T'\right\rangle \\ &\xrightarrow{\partial^{n-2}/\partial s^{n-2}} \text{l.o.t.} + \left\langle T^{(n+1)}, Y\right\rangle .\end{aligned}$$

□

8 Bounded Rate of Growth of $\left\|T^{(n)}\right\|^2$ in $\mathcal{M}^{n+1}$

In this section, we prove the first part of the long-time existence theorem of the solution F_t to the initial value problem ($\star$) in Riemannian manifolds $\mathcal{M}^{n+1}$. We assume that F_t exists on some time interval $[0, T[$ and we investigate $F(t,x)$, T, and its higher derivatives only for those times t such that F_t is well defined. This remark shall apply for the succeeding two sections as well.

Before proceeding, we state the parabolic maximum principle which will be the major tool for our proofs. We use the terminolgy as has been introduced in [McO96, p. 309].

8.1. The Weak Parabolic Maximum Principle (Exponential Form). *Suppose $u \in \mathcal{C}^{2;1}(U_T) \cap \mathcal{C}(\bar{U}_T)$ satisfies $Lu \geq u_t$ in $U_T := \Omega \times (0,T)$, Ω bounded domain in $\mathbb{R}^n$, with L given by*

$$L = a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t) \qquad (\diamondsuit)$$

such that $(a_{ij})_{ij}$ satisfies the uniform ellipticity condition for all $(x,t) \in U_T$: Then there exists some $\lambda > 0$ such that $a_{ij}\xi^i\xi^j \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^n$ [in the language of linear algebra we can assume that $(a_{ij})_{ij}$ is symmetric, hence diagonalizable, and $(a_{ij})_{ij}$ possesses only positive eigenvalues, i.e. is a positive matrix].

Since we only consider evolution equations for nonnegative functions, we assume in the following that $u : \Omega \times [0,T) \mapsto [0,\infty)$.

- *If $c(x,t) \leq 0$, then the weak parabolic maximum principle holds for the solution u of ($\diamondsuit$):*

  $$\max_{(x,t)\in\bar{U}_T} u(x,t) \leq \max_{(x,t)\in\Gamma_T} u(x,t),$$

  where $\Gamma_T := (\partial\Omega \times [0,T]) \cup (\Omega \times \{0\})$ is the parabolic boundary.

- *If $C = \max_{(x,t)\in\bar{U}_T} c(x,t) > 0$, and if u is a solution of ($\diamondsuit$), then $\tilde{u} := \exp(-Ct)u$ satisfies the weak maximum principle:*
  Let $\tilde{L} := L - C$, then the coefficient $\tilde{c} \leq 0$ and

  $$\begin{aligned}\tilde{L}\tilde{u} = L\tilde{u} - C\tilde{u} &= \exp(-Ct)Lu - C\tilde{u} \\ &\geq \exp(-Ct)u_t - C\exp(-Ct)u = (\tilde{u})_t.\end{aligned}$$

  Therefore the rate of growth of $u(x,t) = \exp(Ct)\tilde{u}(x,t)$ can be estimated by:

- *either* $\tilde{u}(x,t) \leq \sup_{x\in(a,b)} \tilde{u}(x,0)$, *which yields*

$$u(x,t) = \exp(Ct)\tilde{u}(x,t) \leq \exp(CT) \sup_{x\in(a,b)} \tilde{u}(x,0)$$

- *or the maximum principle enables us to conclude that* $\tilde{u}$ *is bounded by some value of* $\tilde{u}$ *at the boundary* $\partial\Omega$,

$$\tilde{u}(x,t) \leq \max_{x\in\{a,b\},t\in[O,T]} \tilde{u}(x,t),\ say,\ = \tilde{u}(a,\tau)$$

which implies that also $u(x,t) = \exp(CT)\tilde{u}(x,t)$ *can be bounded by*

$$\begin{aligned} u(x,t) =& \exp(CT)\tilde{u}(x,t) \leq \exp(C\tau)\exp(C(t-\tau))\tilde{u}(a,\tau) \\ =& u(a,\tau)\exp(C(t-\tau)) \leq u(a,\tau)\exp(CT). \end{aligned}$$

- *The weak parabolic maximum principle in its exponential form also holds for equations of the type*

$$\Delta u + c(x,t)u + d(x,t) \geq u_t, \qquad (\spadesuit)$$

where $d(x,t)$ *is some bounded function. Since this is sufficient for our purposes, we have assumed for simplicity that* $a_{ij} = \delta_{ij}$ *and* $b_i = 0$. *Let* $C := \max\{\max_{(x,t)\in\bar{U}_T} c(x,t), 0\}$ *and* $D := \max\{\max_{(x,t)\in\bar{U}_T} d(x,t), 0\}$. *Assume that* u *solves the evolution equation* ($\spadesuit$). *"Rescale"* u *by setting*

$$\tilde{u} := \exp(-Ct)(u + \frac{1}{2}Dx^2).$$

Then we have:

$$\begin{aligned} \Delta\tilde{u} &= \exp(-Ct)\Delta u + \exp(-Ct)D \\ \tilde{u}_t &= -C\exp(-Ct)u - \frac{1}{2}CD\exp(-Ct)x^2 + \exp(-Ct)u_t. \end{aligned}$$

This yields

$$\begin{aligned} \Delta\tilde{u} - \tilde{u}_t =& \exp(-Ct)(\Delta u + cu + d - u_t) - (cu + d)\exp(-Ct) \\ &+ \exp(-Ct)d + C\exp(-Ct)u + \frac{1}{2}CD\exp(-Ct)x^2 \\ =& (C-c)\exp(-Ct)u + (D-d)\exp(-Ct)u \\ &+ \frac{1}{2}CD\exp(-Ct)x^2 \geq \frac{1}{2}CD\exp(-Ct)x^2 \geq 0 \end{aligned}$$

Thus $\tilde{u}$ satisfies $\Delta\tilde{u} \geq \tilde{u}_t$, therefore the maximum principle holds for $\tilde{u}$. This bounds u by

$$u = \exp(Ct)\tilde{u} - \frac{1}{2}Dx^2 \leq \exp(Ct)\tilde{u} < \infty.$$

For the one-dimensional case where $\Omega = [a, b]$, we consider the normal derivative at the end points, which by the strong maximum pinciple is known to be positive. To be precise here, the strong maximum principle provides us with the following estimates ([McO96, p. 309]):
Suppose $M = \sup_{(x,t)\in[a,b]\times[0,T)} u(x, t) < \infty$.

(i) If $u(x_0, t_0) = M$ for some $(x_0, t_0) \in (a, b) \times (0, T)$, then $u(x_0, t_0) \equiv M$ for all $(x, t) \in (a, b) \times (0, t_0]$.

(ii) If $u(x_0, t_0) = M$ for some $x_0 \in \{a, b\}$ and $0 < t_0 < T$, but $u(x, t) < M$ for all $(x, t) \in (a, b) \times (0, t_0)$, then

$$\frac{\partial u}{\partial \nu}(x_0, t_0) > 0,$$

provided the exterior normal derivative $\partial u/\partial\nu$ exists at (x_0, t_0).

In the setting of an evolution equation (♠), the normal derivative of u in terms of the rescaled function $\tilde{u}$ is:

$$\pm\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial x} = \exp(Ct)\frac{\partial \tilde{u}}{\partial x} - Dx.$$

We can always translate the interval $[a, b]$ by a function of determinant $= +1$ such that the corresponding end point lies at the origin. Then the normal derivatives of u and $\tilde{u}$ equal and case (b) of the strong maximum principle holds for u as well.

8.2. Lemma (The Peter-Paul Inequality). *For the estimates to follow we have to make extensive use of the* Peter-Paul Inequality*:*

$$ab \leq \frac{1}{2\lambda}a^2 + \frac{\lambda}{2}b^2, \quad \text{for all } \lambda > 0, \text{ for all } a, b \in \mathbb{R}$$

Proof. With a, b and $\lambda > 0$:

$$0 \leq \left(\sqrt{\lambda}a - \sqrt{\frac{1}{\lambda}}b\right)^2 = \lambda a^2 + \frac{1}{\lambda}b^2 - 2ab. \qquad \square$$

8.3. Theorem. *If k^2 and $\|\mathrm{Riem}\|_{\mathrm{op}}$ are bounded in $(x,t) \in [a,b] \times [0,T)$, then the weak parabolic maximum principle holds in its exponential form, i.e. there exists a constant C such that for all $t \in [0,T)$ and $x \in [a,b]$,*

$$\|T''(x,t)\|^2 \leq \begin{cases} \exp(CT) \cdot \max_{(x,t=0)} \|T''\|^2 & \textit{or} \\ \exp(CT) \cdot \max_{(x\in\{a,b\},t)} \exp(-Ct)\|T''\|^2. \end{cases}$$

Proof. By 'PPi' and 'CSi' we want to denote the Peter-Paul inequality resp. Cauchy-Schwarz's inequality.

$$\begin{aligned} 4\langle T', T'''\rangle \langle T'', T\rangle &\overset{\text{(CSi)}}{\leq} 2\|T^{(3)}\| \cdot 2k\|T''\| \\ &\overset{\text{(PPi)}}{\leq} 1\left\|T^{(3)}\right\|^2 + 4k^2\left\|T''\right\|^2 \end{aligned}$$

$$\begin{aligned} \text{Since } T \perp T', \quad 0 &= \frac{\partial}{\partial s}(\langle T, T'\rangle) = \langle T, T''\rangle + k^2 \\ &\Rightarrow \quad \langle T, T''\rangle = -k^2. \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left\|T''\right\|^2 \leq& \frac{\partial^2}{\partial s^2}\left\|T''\right\|^2 - 2\left\|T'''\right\|^2 + 4\left\|T''\right\|^2\langle T'', T\rangle \\ &+ 1\left\|T^{(3)}\right\|^2 + 4k^2\left\|T''\right\|^2 + 12k^2\left\|T''\right\|^2 \\ &+ 6k^2\left\|T''\right\|^2 + 2\left\langle T'', R^{(1)}T + 2R^{(0)}T'\right\rangle \\ \leq& \frac{\partial^2}{\partial s^2}\left\|T''\right\|^2 - 4k^2\left\|T''\right\|^2 + 4k^2\left\|T''\right\|^2 \\ &+ 12k^2\left\|T''\right\|^2 + 6k^2\left\|T''\right\|^2 - \left\|T^{(3)}\right\|^2 \\ &+ 3\left\|T''\right\|^2 + \left\|R^{(1)}T\right\|^2 + 2\left\|R^{(0)}T'\right\|^2 \\ \leq& \frac{\partial^2}{\partial s^2}\left\|T''\right\|^2 + (3 + 18k^2)\left\|T''\right\|^2 \\ &+ \sum_{i=0}^{1}\left\|\frac{\partial i}{\partial s^i}(R^{(0)}T^{(1-i)})\right\|^2 - \left\|T^{(3)}\right\|^2 \\ \overset{(7.13)}{\leq}& \frac{\partial^2}{\partial s^2}\left\|T''\right\|^2 + (3 + 18k^2 + \|\mathrm{Riem}\|_{\mathrm{op}})\left\|T''\right\|^2 - \left\|T^{(3)}\right\|^2 \\ &+ \text{lower order terms} \end{aligned}$$

Since k and T' are assumed to be bounded, the weak parabolic maximum principle in its exponential formulation holds for the evolution of $\|T''\|^2$. $\square$

8.4. Theorem. *For all $n \geq 3$, we obtain the first benchmark to guarantee boundedness of $T^{(n)}$:*
If for all $0 \leq m \leq n$, the curvature k^2, $\|\nabla^m \operatorname{Riem}\|_{\mathrm{op}}$ and, for all $2 \leq m < n$, $\left\|T^{(m)}\right\|^2$ are bounded on the interval $(x,t) \in [a,b] \times [0,T)$, then the weak parabolic maximum principle in its exponential form holds for $\|T^{(n)}\|^2$ on $[a,b] \times [0,T)$: There exists a constant C such that for all $t \in [0,T)$ and $x \in [a,b]$:

$$\|T^{(n)}(x,t)\|^2 \leq \begin{cases} \exp(CT) \cdot \max_{(x,t=0)} \|T^{(n)}\|^2 & \text{or} \\ \exp(CT) \cdot \max_{(x \in \{a,b\},t)} \exp(-Ct)\|T^{(n)}\|^2. \end{cases}$$

Proof.

$$\begin{aligned} 4\left\langle T^{(n+1)}, T'\right\rangle \left\langle T, T^{(n)}\right\rangle &\overset{\text{(CSi)}}{\leq} 4k\|T^{(n+1)}\| \cdot \|T^{(n)}\| \\ &\overset{\text{(PPi)}}{\leq} 1\left\|T^{(n+1)}\right\|^2 + 4k^2\left\|T^{(n)}\right\|^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t}\left\|T^{(n)}\right\|^2 \leq & \frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 - 2\left\|T^{(n+1)}\right\|^2 + \left\|T^{(n+1)}\right\|^2 \\ & + 4k^2\left\|T^{(n)}\right\|^2 + 2(n+1)k^2\left\|T^{(n)}\right\|^2 \\ & + 2n\left\|T''\right\|^2 \cdot \left\|T^{(n)}\right\|^2 + 2n\left\|T^{(n)}\right\|^2 \\ & + 4(n+1)k^2\left\|T^{(n)}\right\|^2 + \sum_{i=0}^{n-1} N_i^2 \left\|T^{(i)}\right\|^2 \\ & + n\left\|T^{(n)}\right\|^2 + n\left\|T^{(n)}\right\|^2 + \sum_{i=0}^{n-1}\left\|\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})\right\|^2 \\ \leq & \frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 + 4n\left\|T^{(n)}\right\|^2 + (10+6n)k^2\left\|T^{(n)}\right\|^2 \\ & - \left\|T^{(n+1)}\right\|^2 + 2n\left\|T''\right\|^2 \cdot \left\|T^{(n)}\right\|^2 \\ & + \|\operatorname{Riem}\|_{\mathrm{op}} \cdot \left\|T^{(n)}\right\|^2 + \text{lower order terms} \end{aligned}$$

Therefore the weak parabolic maximum principle (exponential form) holds for the evolution equation of $\left\|T^{(n)}\right\|^2$. ☐

8.5. Corollary. *If for all $m \in \mathbb{N}$, the curvature k and $\|\nabla^m \operatorname{Riem}\|_{\mathrm{op}}$ are bounded on the time interval $[0,T)$, then for all $2 \leq n \in \mathbb{N}$ the weak parabolic maximum principle in its exponential form holds for the evolution equation of $\|T^{(n)}\|^2$.*

9 Bounded Rate of Growth of $\left|\left|\left|T^{(n)}\right|\right|\right|$ in $\mathcal{M}^{n,1}$

In this section, we consider the case $\mathcal{M} = \mathcal{M}^{n,1}$, where $\varepsilon = -1$. As for the existence of F_t for some small time interval $t \in [0, T)$, we refer to the previous section.
This section is devoted to the first of two main theorems which establish long time existence of the solutions F_t of ($\star\star$). Because we have to consider $|||.|||^2$ instead of $\|.\|^2$, equations tend to get rather lengthy.

9.1. Remark. For any vector field X, the following algebraic relations hold:

$$\begin{aligned}
\frac{\partial}{\partial t}|||X|||^2 &= \frac{\partial}{\partial t}\|X\|^2 + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial t}X, T\right\rangle + 4\langle X,T\rangle\left\langle X, \frac{\partial}{\partial t}T\right\rangle \\
&\overset{(4.5)}{=} \frac{\partial}{\partial t}\|X\|^2 + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial t}X, T\right\rangle + 4\langle X,T\rangle\left\langle X, \frac{\partial^2}{\partial s^2}T - k^2 T\right\rangle
\end{aligned}$$

$$\frac{\partial}{\partial s}|||X|||^2 = \frac{\partial}{\partial s}\|X\|^2 + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial s}X, T\right\rangle + 4\langle X,T\rangle\langle X,T'\rangle$$

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}|||X|||^2 &= \frac{\partial^2}{\partial s^2}\|X\|^2 + 4\left\langle \frac{\partial}{\partial s}X, T\right\rangle^2 + 4\langle X,T'\rangle\left\langle \frac{\partial}{\partial s}X, T\right\rangle \\
&\quad + 4\langle X,T\rangle\left\langle \frac{\partial^2}{\partial s^2}X, T\right\rangle + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial s}X, T'\right\rangle \\
&\quad + 4\left\langle \frac{\partial}{\partial s}X, T\right\rangle\langle X,T'\rangle + 4\langle X,T'\rangle^2 \\
&\quad + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial s}X, T'\right\rangle + 4\langle X,T\rangle\langle X,T''\rangle \\
&= \frac{\partial^2}{\partial s^2}\|X\|^2 + 4\left\langle \frac{\partial}{\partial s}X, T\right\rangle^2 + 8\langle X,T'\rangle\left\langle \frac{\partial}{\partial s}X, T\right\rangle \\
&\quad + 4\langle X,T\rangle\left\langle \frac{\partial^2}{\partial s^2}X, T\right\rangle + 8\langle X,T\rangle\left\langle \frac{\partial}{\partial s}X, T'\right\rangle \\
&\quad + 4\langle X,T'\rangle^2 + 4\langle X,T\rangle\langle X,T''\rangle
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}|||X|||^2 &= \frac{\partial^2}{\partial s^2}|||X|||^2 + \frac{\partial}{\partial t}\|X\|^2 - \frac{\partial^2}{\partial s^2}\|X\|^2 + 4\langle X,T\rangle\left\langle \frac{\partial}{\partial t}X, T\right\rangle \\
&\quad - 4k^2\langle X,T\rangle^2 - 4\left\langle \frac{\partial}{\partial s}X, T\right\rangle^2 - 8\langle X,T'\rangle\left\langle \frac{\partial}{\partial s}X, T\right\rangle \\
&\quad - 4\langle X,T\rangle\left\langle \frac{\partial^2}{\partial s^2}X, T\right\rangle - 8\langle X,T\rangle\left\langle \frac{\partial}{\partial s}X, T'\right\rangle - 4\langle X,T'\rangle^2 \qquad \square
\end{aligned}$$

9.2. Theorem. *If k^2 and $|||\mathrm{Riem}|||_{\mathrm{op}}$ are bounded for all $(x,t) \in [a,b] \times [0,T)$, then the weak parabolic maximum principle holds in its exponential form, i.e. there exists a constant C such that for all $t \in [0,T)$ and $x \in [a,b]$,*

$$|||T''(x,t)|||^2 \leq \begin{cases} \exp(CT) \cdot \max_{(x,t=0)} |||T''|||^2 & \textit{or} \\ \exp(CT) \cdot \max_{(x \in \{a,b\},t)} |||T''|||^2 . \end{cases}$$

Proof.

$$\begin{aligned} \frac{\partial}{\partial t} |||T''|||^2 = & \frac{\partial^2}{\partial s^2} |||T''|||^2 + \frac{\partial}{\partial t} \|T''\|^2 - \frac{\partial^2}{\partial s^2} \|T''\|^2 \\ & + 4 \langle T'', T \rangle \left\langle \frac{\partial}{\partial t} T'' - \frac{\partial^2}{\partial s^2} T'', T \right\rangle - 4k^2 \langle T'', T \rangle^2 \\ & - 4 \left\langle T^{(3)}, T \right\rangle^2 - 8 \langle T'', T' \rangle \left\langle T^{(3)}, T \right\rangle \\ & - 8 \langle T'', T \rangle \left\langle T^{(3)}, T' \right\rangle - 4 \langle T'', T' \rangle^2 \end{aligned}$$

$$\begin{aligned} = & \frac{\partial^2}{\partial s^2} |||T''|||^2 - 2 \|T'''\|^2 - 4 \|T''\|^2 \langle T'', T \rangle \\ & - 4 \langle T', T''' \rangle \langle T'', T \rangle - 12 \langle T', T'' \rangle^2 - 6k^2 \|T''\|^2 \\ & + 2 \left\langle T'', R^{(1)} T + 2R^{(0)} T' \right\rangle - 4 \langle T'', T \rangle \left\langle T, 2 \|T''\|^2 T \right\rangle \\ & - 4 \langle T'', T \rangle \langle T, 2 \langle T', T''' \rangle T \rangle \\ & - 4 \langle T'', T \rangle \langle T, 6 \langle T', T'' \rangle T' \rangle - 4 \langle T'', T \rangle \left\langle T, 3k^2 T'' \right\rangle \\ & + 4 \langle T'', T \rangle \left\langle T, R^{(1)} T + 2R^{(0)} T' \right\rangle - 4k^2 \langle T'', T \rangle^2 \\ & - 4 \left\langle T^{(3)}, T \right\rangle^2 - 8 \langle T'', T' \rangle \left\langle T^{(3)}, T \right\rangle \\ & - 8 \langle T'', T \rangle \left\langle T^{(3)}, T' \right\rangle - 4 \langle T'', T' \rangle^2 \end{aligned}$$

$$\begin{aligned} = & \frac{\partial^2}{\partial s^2} |||T''|||^2 - 2 \|T'''\|^2 + 4k^2 \|T''\|^2 \\ & + 4k^2 \langle T', T''' \rangle - 12 \langle T', T'' \rangle^2 - 6k^2 \|T''\|^2 \\ & + 2 \left\langle T'', R^{(1)} T + 2R^{(0)} T' \right\rangle - 8k^2 \|T''\|^2 - 8k^2 \langle T', T''' \rangle \\ & + 24k^2 \langle T', T'' \rangle \langle T, T' \rangle - 12k^6 - 4k^2 \left\langle T, R^{(1)} T + 2R^{(0)} T' \right\rangle \\ & - 4k^6 - 4 \left\langle T^{(3)}, T \right\rangle^2 \\ & - 8 \langle T'', T' \rangle \left\langle T^{(3)}, T \right\rangle + 8k^2 \left\langle T^{(3)}, T' \right\rangle - 4 \langle T'', T' \rangle^2 \end{aligned}$$

$$
\begin{aligned}
=&\frac{\partial^2}{\partial s^2}\left\||T''\right\||^2 - 2\left\|T'''\right\|^2 + 4k^2\left\langle T', T'''\right\rangle \\
&- 16\left\langle T', T''\right\rangle^2 - 10k^2\left\|T''\right\|^2 + 2\left\langle T'', R^{(1)}T + 2R^{(0)}T'\right\rangle \\
&- 16k^6 - 4k^2\left\langle T, R^{(1)}T + 2R^{(0)}T'\right\rangle \\
&- 4\left\langle T^{(3)}, T\right\rangle^2 - 8\left\langle T'', T'\right\rangle\left\langle T^{(3)}, T\right\rangle
\end{aligned}
$$

Now consider:

$$
\begin{aligned}
\left\|T^{(3)}\right\|^2 + \left\langle T^{(3)}, T\right\rangle^2 &= \left\langle T^{(3)}, T^{(3)} + \left\langle T^{(3)}, T\right\rangle T\right\rangle \\
&= \left\langle T^{(3)}, \Pi^{\perp}\left(T^{(3)}\right)\right\rangle = \left\langle \Pi^{\perp}\left(T^{(3)}\right), \Pi^{\perp}\left(T^{(3)}\right)\right\rangle = \left\|\Pi^{\perp}\left(T^{(3)}\right)\right\|^2
\end{aligned}
$$

The following holds for both $k^2 = 0$ and $k^2 \neq 0$:

$$
\begin{aligned}
k^2\left\langle T^{(3)}, T'\right\rangle = &- \left\langle T^{(3)}, T'\right\rangle\left\langle T, T''\right\rangle \leq |\left\langle T^{(3)}, kN\right\rangle| \cdot |\left\langle T, T''\right\rangle| \\
=&|\left\langle T^{(3)}, N\right\rangle| \cdot |k\left\langle T, T''\right\rangle| \leq \frac{1}{4}|\left\langle T^{(3)}, N\right\rangle|^2 + k^2|\left\langle T, T''\right\rangle|^2 \\
\leq&\frac{1}{4}\left\langle T^{(3)}, \Pi^{\perp}\left(T^{(3)}\right)\right\rangle + k^6 = \frac{1}{4}\left\|\Pi^{\perp}\left(T^{(3)}\right)\right\|^2 + k^6
\end{aligned}
$$

$$
\begin{aligned}
-\left\langle T^{(3)}, T\right\rangle\left\langle T'', T'\right\rangle &\leq |\left\langle T^{(3)}, T\right\rangle| \cdot |\left\langle T'', T'\right\rangle| \\
&\overset{PPi}{\leq} \frac{1}{2\cdot 4}\left\langle T^{(3)}, T\right\rangle^2 + \frac{1}{2\cdot\frac{1}{4}}\left\langle T'', T'\right\rangle^2
\end{aligned}
$$

$$
-k^2\left\|T''\right\|^2 - 2k^6 = -k^2(\left\|T''\right\|^2 + 2\left\langle T, T''\right\rangle^2) = -k^2\left\||T''\right\||^2
$$

$$
\begin{aligned}
\Rightarrow\quad \frac{\partial}{\partial t}\left\||T''\right\||^2 \leq&\frac{\partial^2}{\partial s^2}\left\||T''\right\||^2 - 2\left\|\Pi^{\perp}\left(T^{(3)}\right)\right\|^2 - 2\left\langle T^{(3)}, T\right\rangle^2 \\
&+ 8(\frac{1}{8}\left\langle T^{(3)}, T\right\rangle^2 + 2\left\langle T'', T'\right\rangle^2) \\
&- 16\left\langle T'', T'\right\rangle^2 + 4(\frac{1}{4}\left\|\Pi^{\perp}\left(T^{(3)}\right)\right\|^2 + k^6) \\
&- 10k^2\left\|T''\right\|^2 - 16k^6 \\
&+ \underbrace{2\left\langle T'' + 2\left\langle T, T''\right\rangle T, R^{(1)}T + 2R^{(0)}T'\right\rangle}_{=2\sum_{i=0}^{1}\left\langle \frac{\partial^i}{\partial s^i}(R^{(0)}T^{(1-i)}), \Pi^{\perp}(T'') - \Pi^{\top}(T'')\right\rangle}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial^2}{\partial s^2} \left|\left|\left| T'' \right|\right|\right|^2 - \left\langle T^{(3)}, T \right\rangle^2 - \left|\left|\left| \Pi^\perp \left(T^{(3)}\right) \right|\right|\right|^2 \\
&\quad \underbrace{-4k^2 \left\| T'' \right\|^2 - 6k^2 \left\| T'' \right\|^2 - 12k^6}_{= -4k^2 \|T''\|^2 - 6k^2 |||T'''|||^2 \leq -2k^2 |||T''|||^2} \\
&\quad + \sum_{i=0}^{1} \left|\left|\left| \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(1-i)}) \right|\right|\right|^2 + 2 \left|\left|\left| \Pi^\perp (T'') - \Pi^\top (T'') \right|\right|\right|^2 \\
&\overset{(7.13)}{\leq} \frac{\partial^2}{\partial s^2} \left|\left|\left| T'' \right|\right|\right|^2 - \left|\left|\left| T^{(3)} \right|\right|\right|^2 + 2(1-k^2) \left|\left|\left| T'' \right|\right|\right|^2 \\
&\quad + \langle e_0, T \rangle^8 \left|\left|\left| \mathrm{Riem} \right|\right|\right|_{\mathrm{op}}^2 \cdot \left|\left|\left| T'' \right|\right|\right|^2 + \text{lower order terms}
\end{aligned}
$$

For this kind of evolution equation the weak parabolic maximum principle holds in its exponential formulation. ☐

9.3. Theorem. *For all $n \geq 3$, we obtain:*
If for all $0 \leq m \leq n$, the curvature k^2, $|||\nabla^m \mathrm{Riem}|||_{\mathrm{op}}$ and, for all $2 \leq m < n$, $\left|\left|\left| T^{(m)} \right|\right|\right|$ are bounded on the interval $(x,t) \in [a,b] \times [0,T)$, then the weak parabolic maximum principle in its exponential form holds for $\left|\left|\left| T^{(n)} \right|\right|\right|^2$ on $[a,b] \times [0,T)$: There exists a constant C such that for all $t \in [0,T)$ and $x \in [a,b]$:

$$
\left|\left|\left| T^{(n)}(x,t) \right|\right|\right|^2 \leq \begin{cases} \exp(CT) \cdot \max_{(x,t=0)} \left|\left|\left| T^{(n)} \right|\right|\right|^2 & \text{or} \\ \exp(CT) \cdot \max_{(x \in \{a,b\}, t)} \left|\left|\left| T^{(n)} \right|\right|\right|^2 . \end{cases}
$$

Proof.

$$
\begin{aligned}
\frac{\partial}{\partial t} \left|\left|\left| T^{(n)} \right|\right|\right|^2 =& \frac{\partial^2}{\partial s^2} \left|\left|\left| T^{(n)} \right|\right|\right|^2 + \frac{\partial}{\partial t} \left\| T^{(n)} \right\|^2 - \frac{\partial^2}{\partial s^2} \left\| T^{(n)} \right\|^2 \\
&+ 4 \left\langle T^{(n)}, T \right\rangle \left\langle \frac{\partial}{\partial t} T^{(n)} - \frac{\partial^2}{\partial s^2} T^{(n)}, T \right\rangle - 4k^2 \left\langle T^{(n)}, T \right\rangle^2 \\
&- 4 \left\langle T^{(n+1)}, T \right\rangle^2 - 8 \left\langle T^{(n)}, T' \right\rangle \left\langle T^{(n+1)}, T \right\rangle \\
&- 8 \left\langle T^{(n)}, T \right\rangle \left\langle T^{(n+1)}, T' \right\rangle - 4 \left\langle T^{(n)}, T' \right\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{\partial^2}{\partial s^2} \left|\left|\left| T^{(n)} \right|\right|\right|^2 - 2 \left\| T^{(n+1)} \right\|^2 - 4 \left\langle T^{(n+1)}, T' \right\rangle \left\langle T, T^{(n)} \right\rangle \\
&- 2(n+1)k^2 \left\| T^{(n)} \right\|^2 - 4n \left\langle T^{(n)}, T'' \right\rangle \left\langle T, T^{(n)} \right\rangle - 4(n+1) \left\langle T^{(n)}, T' \right\rangle^2 \\
&- 2 \sum_{i=0}^{n-1} N_i \left\langle T^{(i)}, T^{(n)} \right\rangle + 2 \sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T^{(n)} \right\rangle \\
&- 8 \left\langle T^{(n)}, T \right\rangle \left\langle T^{(n+1)}, T' \right\rangle \langle T, T \rangle - 4(n+1)k^2 \left\langle T^{(n)}, T \right\rangle \left\langle T, T^{(n)} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&- 8n \left\langle T^{(n)}, T'' \right\rangle \left\langle T^{(n)}, T \right\rangle \left\langle T, T \right\rangle - 8(n+1) \left\langle T^{(n)}, T' \right\rangle \left\langle T^{(n)}, T \right\rangle \left\langle T, T' \right\rangle \\
&- 4 \left\langle T^{(n)}, T \right\rangle \sum_{i=0}^{n-1} N_i \left\langle T, T^{(i)} \right\rangle + 4 \left\langle T^{(n)}, T \right\rangle \sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T \right\rangle \\
&- 4k^2 \left\langle T^{(n)}, T \right\rangle^2 - 4 \left\langle T^{(n+1)}, T \right\rangle^2 - 8 \left\langle T^{(n)}, T' \right\rangle \left\langle T^{(n+1)}, T \right\rangle \\
&- 8 \left\langle T^{(n)}, T \right\rangle \left\langle T^{(n+1)}, T' \right\rangle - 4 \left\langle T^{(n)}, T' \right\rangle^2
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\partial^2}{\partial s^2} \left\| \left\| T^{(n)} \right\| \right\|^2 - 2 \left\| T^{(n+1)} \right\|^2 - 4 \left\langle T^{(n+1)}, T' \right\rangle \left\langle T, T^{(n)} \right\rangle \\
&- 2(n+1)k^2 \left\| T^{(n)} \right\|^2 + 4n \left\langle T^{(n)}, T'' \right\rangle \left\langle T, T^{(n)} \right\rangle - 4(n+2) \left\langle T^{(n)}, T' \right\rangle^2 \\
&- 4(n+2)k^2 \left\langle T^{(n)}, T \right\rangle^2 \\
&- 4 \left\langle T^{(n+1)}, T \right\rangle^2 - 8 \left\langle T^{(n)}, T' \right\rangle \left\langle T^{(n+1)}, T \right\rangle \\
&+ 4 \left\langle T^{(n)}, T \right\rangle \Big(\sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T \right\rangle - \sum_{i=0}^{n-1} N_i \left\langle T, T^{(i)} \right\rangle \Big) \\
&- 2 \sum_{i=0}^{n-1} N_i \left\langle T^{(i)}, T^{(n)} \right\rangle + 2 \sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T^{(n)} \right\rangle
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\partial^2}{\partial s^2} \left\| \left\| T^{(n)} \right\| \right\|^2 - 2(n+1)k^2 \left\| T^{(n)} \right\|^2 \\
&- \Big(2 \left\langle T^{(n+1)}, T \right\rangle^2 + 2 \left\| T^{(n+1)} \right\|^2\Big) \\
&- 4 \left\langle T^{(n+1)}, T' \right\rangle \left\langle T, T^{(n)} \right\rangle - 4(n+2)k^2 \left\langle T^{(n)}, T \right\rangle^2 \\
&- 8 \left\langle T^{(n)}, T' \right\rangle \left\langle T^{(n+1)}, T \right\rangle - 2 \left\langle T^{(n+1)}, T \right\rangle^2 \\
&- 4(n+2) \left\langle T^{(n)}, T' \right\rangle^2 + 4n \left\langle T^{(n)}, T'' \right\rangle \left\langle T, T^{(n)} \right\rangle \\
&+ 4 \left\langle T^{(n)}, T \right\rangle \Big(\sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T \right\rangle - \sum_{i=0}^{n-1} N_i \left\langle T, T^{(i)} \right\rangle \Big) \\
&- 2 \sum_{i=0}^{n-1} N_i \left\langle T^{(i)}, T^{(n)} \right\rangle + 2 \sum_{i=0}^{n-1} \left\langle \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}), T^{(n)} \right\rangle
\end{aligned}
$$

$$
-2\Big(\left\langle T^{(n+1)}, T \right\rangle^2 + \left\| T^{(n+1)} \right\|^2\Big) = -2 \left\| \Pi^{\perp} \left(T^{(n+1)} \right) \right\|^2
$$

This inequality holds for any k^2 :

$$\begin{aligned}
-4\left\langle T^{(n+1)},T'\right\rangle\left\langle T,T^{(n)}\right\rangle &\leq 4|\left\langle T^{(n+1)},T'\right\rangle|\cdot|\left\langle T,T^{(n)}\right\rangle| \\
&= 4|\left\langle T^{(n+1)},N\right\rangle|\cdot|k\left\langle T,T^{(n)}\right\rangle| \leq 1\left\langle T^{(n+1)},N\right\rangle^2+4k^2\left\langle T,T^{(n)}\right\rangle^2 \\
&\leq \left\|\Pi^{\perp}\left(T^{(n+1)}\right)\right\|^2+4k^2\left\langle T,T^{(n)}\right\rangle^2
\end{aligned}$$

$$\begin{aligned}
-8\left\langle T^{(n)},T'\right\rangle\left\langle T^{(n+1)},T\right\rangle \leq& 8|\left\langle T^{(n)},T'\right\rangle|\cdot|\left\langle T^{(n+1)},T\right\rangle| \\
\leq& 1\left\langle T^{(n+1)},T\right\rangle^2+16\left\langle T^{(n)},T'\right\rangle^2
\end{aligned}$$

$$\begin{aligned}
4n\left\langle T^{(n)},T''\right\rangle\left\langle T^{(n)},T\right\rangle \leq& 2n\left\langle T^{(n)},T''\right\rangle^2+2n\left\langle T^{(n)},T\right\rangle^2 \\
\leq& 2n\,|||T''|||^2\cdot\left|\left|\left|T^{(n)}\right|\right|\right|^2+2n\left\langle T^{(n)},T\right\rangle^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad \frac{\partial}{\partial t}\left|\left|\left|T^{(n)}\right|\right|\right|^2 =& \frac{\partial^2}{\partial s^2}\left|\left|\left|T^{(n)}\right|\right|\right|^2-2(2n+2)k^2\left\langle T^{(n)},T\right\rangle^2+2n\left\langle T^{(n)},T\right\rangle^2 \\
&-4n\left\langle T^{(n)},T'\right\rangle^2+2n\,|||T''|||^2\cdot\left|\left|\left|T^{(n)}\right|\right|\right|^2-\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 \\
&-2(n+1)k^2\left\|T^{(n)}\right\|^2+8\left\langle T^{(n)},T'\right\rangle^2 \\
&+2\sum_{i=0}^{n-1}\left\langle\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)}),\underbrace{T^{(n)}+2\left\langle T^{(n)},T\right\rangle T}_{=\Pi^{\perp}\left(T^{(n)}\right)-\Pi^{\top}\left(T^{(n)}\right)}\right\rangle \\
&-2\sum_{i=0}^{n-1}N_i\left\langle T^{(i)},T^{(n)}+2\left\langle T^{(n)},T\right\rangle T\right\rangle
\end{aligned}$$

$$-2(n+1)k^2\left\|T^{(n)}\right\|^2-(4n+4)k^2(\left\langle T,T^{(n)}\right\rangle)^2=-2(n+1)k^2\left|\left|\left|T^{(n)}\right|\right|\right|^2$$

$$\begin{aligned}
&2\sum_{i=0}^{n-1}\left\langle\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})-N_iT^{(i)},\Pi^{\perp}\left(T^{(n)}\right)-\Pi^{\top}\left(T^{(n)}\right)\right\rangle \\
&\quad\leq 2\sum_{i=0}^{n-1}(\left|\left|\left|\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})\right|\right|\right|\cdot\left|\left|\left|\Pi^{\perp}\left(T^{(n)}\right)-\Pi^{\top}\left(T^{(n)}\right)\right|\right|\right| \\
&\qquad+\left|\left|\left|N_iT^{(i)}\right|\right|\right|\cdot\left|\left|\left|\Pi^{\perp}\left(T^{(n)}\right)-\Pi^{\top}\left(T^{(n)}\right)\right|\right|\right|) \\
&\quad\leq\sum_{i=0}^{n-1}(\left|\left|\left|\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-1-i)})\right|\right|\right|^2+\left|\left|\left|N_iT^{(i)}\right|\right|\right|^2) \\
&\qquad+\underbrace{(n+n)\left|\left|\left|\Pi^{\perp}\left(T^{(n)}\right)-\Pi^{\top}\left(T^{(n)}\right)\right|\right|\right|^2}_{=2n\left|\left|\left|T^{(n)}\right|\right|\right|^2}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad \frac{\partial}{\partial t} \left|\left|\left| T^{(n)} \right|\right|\right|^2 =& \frac{\partial^2}{\partial s^2} \left|\left|\left| T^{(n)} \right|\right|\right|^2 - 2(2n+2)k^2 \left\langle T^{(n)}, T \right\rangle^2 - \left|\left|\left| T^{(n+1)} \right|\right|\right|^2 \\
&+ 2n \left\langle T^{(n)}, T \right\rangle^2 - 4n \left\langle T^{(n)}, T' \right\rangle^2 + 8 \left\langle T^{(n)}, T' \right\rangle^2 \\
&+ 2n \left|\left|\left| T'' \right|\right|\right|^2 \cdot \left|\left|\left| T^{(n)} \right|\right|\right|^2 - 2(n+1)k^2 \left\| T^{(n)} \right\|^2 \\
&+ \sum_{i=0}^{n-1} \left(\left|\left|\left| \frac{\partial^i}{\partial s^i} (R^{(0)} T^{(n-1-i)}) \right|\right|\right|^2 + \left|\left|\left| N_i T^{(i)} \right|\right|\right|^2 \right) + 2n \left|\left|\left| T^{(n)} \right|\right|\right|^2 \\
\leq& \frac{\partial^2}{\partial s^2} \left|\left|\left| T^{(n)} \right|\right|\right|^2 + (-2n-2)k^2 \left|\left|\left| T^{(n)} \right|\right|\right|^2 \\
&+ 2n \left|\left|\left| \Pi^\top \left(T^{(n)} \right) \right|\right|\right|^2 + 4nk^2 \left|\left|\left| \Pi^\perp \left(T^{(n)} \right) \right|\right|\right|^2 \\
&+ 2n(1 + \left|\left|\left| T'' \right|\right|\right|^2) \left|\left|\left| T^{(n)} \right|\right|\right|^2 + 8k^2 \left|\left|\left| \Pi^\perp \left(T^{(n)} \right) \right|\right|\right|^2 \\
&- \left|\left|\left| T^{(n+1)} \right|\right|\right|^2 + \langle e_0, T \rangle^8 \left|\left|\left| \mathrm{Riem} \right|\right|\right|_{\mathrm{op}}^2 \cdot \left|\left|\left| T^{(n)} \right|\right|\right|^2 \\
&+ \sum_{i=0}^{n-1} \left|\left|\left| N_i T^{(i)} \right|\right|\right|^2 + \text{lower order terms} \\
\leq& \frac{\partial^2}{\partial s^2} \left|\left|\left| T^{(n)} \right|\right|\right|^2 + 2(n-1)k^2 \left|\left|\left| T^{(n)} \right|\right|\right|^2 + 8k^2 \left|\left|\left| \Pi^\perp \left(T^{(n)} \right) \right|\right|\right|^2 \\
&+ 2n(2 + \left|\left|\left| T'' \right|\right|\right|^2) \left|\left|\left| T^{(n)} \right|\right|\right|^2 + \langle e_0, T \rangle^8 \left|\left|\left| \mathrm{Riem} \right|\right|\right|_{\mathrm{op}}^2 \left|\left|\left| T^{(n)} \right|\right|\right|^2 \\
&- \left|\left|\left| T^{(n+1)} \right|\right|\right|^2 + \text{lower order terms} \qquad \square
\end{aligned}
$$

9.4. Corollary. *If for all $m \in \mathbb{N}$, k and $|||\nabla^m \mathrm{Riem}|||_{\mathrm{op}}$ are bounded on the time interval $[0, T)$, then for all $2 \leq n \in \mathbb{N}$ the weak parabolic maximum principle in its exponential form holds for the evolution equation of $\left|\left|\left| T^{(n)} \right|\right|\right|^2$.*

10 Boundedness of $\left|\left|\left|T^{(n)}\right|\right|\right|$ at the End Points

If our solutions satisfy an evolution equation of a particular type, these solutions are governed by the maximum principle. Then we can conclude that points of maximal value can occur at the parabolic boundary of $[a,b]\times(0,T_0)$ only, and it remains to set a bound on the derivatives at the spatial boundary points for $t \neq 0$. Then, by induction, we can show that each derivative is uniformly bounded on $[a,b]\times[0,T_0]$, which yields long time existence since we can extend these solutions, by taking the limit $\lim_{t\nearrow T_0} F_t = F_{T_0}$, across any time limit $t = T_0$ to flow some more time.

10.1. Lemma. *At $x \in \{a,b\}$, we have for all times $t > 0$ such that the existence of F_t is guaranteed:*

$$(i) \quad k = 0 \tag{10.1i}$$

$$(ii) \quad T' = 0 \tag{10.1ii}$$

$$(iii) \quad T^{(3)} = 0 \tag{10.1iii}$$

$$(iv) \quad T^{(5)} = 2\,\nabla_T \operatorname{Riem}|_{(T'',T)T} \tag{10.1iv}$$

Proof. (i) and (ii) follow from corollary 5.3. (iii) follows from (ii) and the evolution equation (7.3) of T'. (iv) is a consequence of equation (7.9) □

10.2. Lemma. *For all $l \in \mathbb{N}$, for all points $(x,t) \in [a,b]\times[0,T)$,*

$$\frac{\partial^l}{\partial t^l} T' = T^{(2l+1)} + 2l\varepsilon \left\langle T', T^{(2l)} \right\rangle T + \vec{f}(T,\dots,T^{(2l-1)}), \tag{10.2}$$

where $\vec{f}$ is a vector field depending on $T,\dots,T^{(2l-1)}$ and on Riem *and the bilinear form $\langle\cdot,\cdot\rangle$ only.*

Proof. We use induction by l.
$k = 1$: $\quad (\partial/\partial t)T' = T^{(3)} + 2\varepsilon k^2 T' + 2\varepsilon \langle T', T''\rangle T + R(T',T)T$.
Assume the claim holds true for all $1 \leq l \leq m$. Then we have for $l = m+1$:

$$\begin{aligned}\frac{\partial^{m+1}}{\partial t^{m+1}} T' =& \frac{\partial}{\partial t}\frac{\partial^m}{\partial t^m} T' \overset{\text{assumption}}{=} \frac{\partial}{\partial t}(T^{(2m+1)} + 2m\varepsilon \left\langle T', T^{(2m)} \right\rangle T \\ &+ \vec{f}(T,\dots,T^{(2m-1)})) \overset{(7.10)}{=} T^{(2m+3)} + 2\varepsilon \left\langle T', T^{(2m+2)} \right\rangle T \\ &+ 2m\varepsilon \left\langle T', T^{(2m+2)} \right\rangle T + \vec{f}(T,\dots,T^{(2m+1)}) = T^{(2m+3)} \\ &+ 2(m+1)\varepsilon \left\langle T', T^{(2m+2)} \right\rangle T + \vec{f}(T,\dots,T^{(2m+1)}).\end{aligned}$$

□

10.3. Corollary. *If for all $m \le l$, l odd, $T^{(m)}$ is bounded at $\{a,b\}$, then $T^{(l+2)}$ is also bounded at $\{a,b\}$.*

Proof. Consider equation (10.2) for fixed end points a or b, which yields a function which is only dependent on t.
Since for all $t>0$, $2l\varepsilon \left\langle T'(a,t), T^{(2l)}(a,t)\right\rangle T(a,t) \equiv 0$, and $(\partial^l/\partial t^l)T' = 0$ at $\{a,b\}$, we obtain

$$T^{(2l+1)} = -\vec{f}(T,\dots,T^{(2l-1)}).$$

□

10.4. Lemma. *The following equations hold, where $N_i = N_i(\dots;n-2)$ are the corresponding coefficients for the evolution equation of $T^{(n-2)}$, as introduced in (7.10).*

(i) $$\frac{\partial}{\partial t}T^{(n)} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}T^{(n-1)} \overset{(4.4)}{=} \frac{\partial}{\partial s}\frac{\partial}{\partial t}T^{(n-1)} + \varepsilon k^2 T^{(n)} + R(Y,T)T^{(n-1)}$$

(ii) $$\frac{\partial}{\partial t}T^{(n-1)} = \frac{\partial}{\partial t}\frac{\partial}{\partial s}T^{(n-2)} \overset{(4.4)}{=} \frac{\partial}{\partial s}\frac{\partial}{\partial t}T^{(n-2)} + \varepsilon k^2 T^{(n-1)} + R(Y,T)T^{(n-2)}$$

(iii) $$\begin{aligned}\frac{\partial}{\partial t}T^{(n)} =& \varepsilon k^2 T^{(n)} + R(Y,T)T^{(n-1)} + \frac{\partial}{\partial s}(\frac{\partial}{\partial s}\frac{\partial}{\partial t}T^{(n-2)} + \varepsilon k^2 T^{(n-1)} \\ &+ R(Y,T)T^{(n-2)}) = \varepsilon k^2 T^{(n)} + R(Y,T)T^{(n-1)} + \frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}T^{(n-2)} \\ &+ \varepsilon k^2 T^{(n)} + \varepsilon (k^2)' T^{(n-1)} + \frac{\partial}{\partial s}(R(Y,T)T^{(n-2)})\end{aligned}$$

(iv) $$\begin{aligned}\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}T^{(n-2)} =& \frac{\partial^2}{\partial s^2}(T^{(n)} + 2\varepsilon\left\langle T^{(n-1)}, T'\right\rangle T + (n-1)\varepsilon k^2 T^{(n-2)} \\ &+ 2(n-2)\varepsilon\left\langle T^{(n-2)}, T''\right\rangle T + 2(n-1)\varepsilon\left\langle T^{(n-2)}, T'\right\rangle T' \\ &+ \varepsilon\sum_{i=0}^{n-3}(N_i T^{(i)} + \varepsilon\frac{\partial^i}{\partial s^i}(R^{(0)}T^{(n-3-i)}))) = T^{(n+2)} + 2\varepsilon\left\langle T^{(n+1)}, T'\right\rangle T \\ &+ 4\varepsilon\left\langle T^{(n)}, T''\right\rangle T + 4\varepsilon\left\langle T^{(n)}, T'\right\rangle T' + 2\varepsilon\left\langle T^{(n-1)}, T^{(3)}\right\rangle T \\ &+ 4\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' + 2\varepsilon\left\langle T^{(n-1)}, T'\right\rangle T'' + 2(n-1)\varepsilon\left\langle T'', T''\right\rangle T^{(n-2)} \\ &+ 2(n-1)\varepsilon\left\langle T', T^{(3)}\right\rangle T^{(n-2)} + 4(n-1)\varepsilon\left\langle T', T''\right\rangle T^{(n-1)} \\ &+ (n-1)\varepsilon k^2 T^{(n)} + 2(n-2)\varepsilon\left\langle T^{(n)}, T''\right\rangle T + 4(n-2)\varepsilon\left\langle T^{(n-1)}, T^{(3)}\right\rangle T \\ &+ 4(n-2)\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' + 2(n-2)\varepsilon\left\langle T^{(n-2)}, T^{(4)}\right\rangle T \\ &+ 4(n-2)\varepsilon\left\langle T^{(n-2)}, T^{(3)}\right\rangle T' + 2(n-2)\varepsilon\left\langle T^{(n-2)}, T''\right\rangle T'' \\ &+ 2(n-1)\varepsilon\left\langle T^{(n)}, T'\right\rangle T' + 4(n-1)\varepsilon\left\langle T^{(n-1)}, T''\right\rangle T' \\ &+ 4(n-1)\varepsilon\left\langle T^{(n-1)}, T'\right\rangle T'' + 2(n-1)\varepsilon\left\langle T^{(n-2)}, T^{(3)}\right\rangle T' \\ &+ 4(n-1)\varepsilon\left\langle T^{(n-2)}, T''\right\rangle T'' + 2(n-1)\varepsilon\left\langle T^{(n-2)}, T'\right\rangle T^{(3)} \\ &+ \varepsilon\sum_{i=0}^{n-3} N_i'' T^{(i)} + 2N_i' T^{(i+1)} + N_i T^{(i+2)} + \sum_{i=0}^{n-3}\frac{\partial^{i+2}}{\partial s^{i+2}}(R^{(0)}T^{(n-3-i)})\end{aligned}$$

10.5. Corollary. *The evolution equation of $T^{(n)}$, using the coefficients $N_i(\dots;n-2)$ for $T^{(n-2)}$, is as follows:*

$$\begin{aligned}
\frac{\partial}{\partial t}T^{(n)} =& T^{(n+2)} + 2\varepsilon \left\langle T^{(n+1)}, T'\right\rangle T + (n+1)\varepsilon k^2 T^{(n)} \\
&+ 2n\varepsilon \left\langle T^{(n)}, T''\right\rangle T + 2(n+1)\varepsilon \left\langle T^{(n)}, T'\right\rangle T' \\
&+ \boldsymbol{T}\boldsymbol{\{}2(2n-3)\varepsilon \left\langle T^{(n-1)}, T^{(3)}\right\rangle + 2(n-2)\varepsilon \left\langle T^{(n-2)}, T^{(4)}\right\rangle \boldsymbol{\}} \\
&+ \boldsymbol{T'}\boldsymbol{\{}4(n-1)\varepsilon \left\langle T^{(n-1)}, T''\right\rangle + 2(3n-5)\varepsilon \left\langle T^{(n-2)}, T^{(3)}\right\rangle \\
&\quad + 4(n-1)\varepsilon \left\langle T^{(n-1)}, T''\right\rangle \boldsymbol{\}} \\
&+ \boldsymbol{T''}\boldsymbol{\{}2(2n-1)\varepsilon \left\langle T^{(n-1)}, T'\right\rangle + 2(3n-4)\varepsilon \left\langle T^{(n-2)}, T''\right\rangle \boldsymbol{\}} \\
&+ 2(n-1)\varepsilon \left\langle T^{(n-2)}, T'\right\rangle T^{(3)} + 2(n-1)\varepsilon \left\|T''\right\|^2 T^{(n-2)} \\
&+ 2(n-1)\varepsilon \left\langle T', T^{(3)}\right\rangle T^{(n-2)} + 2(2n-1)\varepsilon \left\langle T', T''\right\rangle T^{(n-1)} \\
&+ \varepsilon \sum_{i=0}^{n-3} N_i'' T^{(i)} + 2N_i' T^{(i+1)} + N_i T^{(i+2)} + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial s^i}(R^{(0)} T^{(n-1-i)})
\end{aligned}$$

Note. Since on $x \in \{a,b\}$, $T' = 0 = T^{(3)}$, we obtain at the end points:

$$\begin{aligned}
\frac{\partial}{\partial t}T^{(n)} =& T^{(n+2)} + 2n\varepsilon \left\langle T^{(n)}, T''\right\rangle T + 2(n-2)\varepsilon \left\langle T^{(n-2)}, T^{(4)}\right\rangle T \\
&+ 2(3n-4)\varepsilon \left\langle T^{(n-2)}, T''\right\rangle T'' + 2(n-1)\varepsilon \left\|T''\right\|^2 T^{(n-2)} \\
&+ \varepsilon \sum_{i=0}^{n-3} N_i'' T^{(i)} + 2N_i' T^{(i+1)} + N_i T^{(i+2)} + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial s^i}(R^{(0)} T^{(n-1-i)})
\end{aligned} \tag{10.3}$$

Also note that if $T'' = 0 = T^{(4)}$, then the evolution equation reads

$$\frac{\partial}{\partial t}T^{(n)} = T^{(n+2)} + \varepsilon \sum_{i=0}^{n-3} N_i'' T^{(i)} + 2N_i' T^{(i+1)} + N_i T^{(i+2)} + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial s^i}(R^{(0)} T^{(n-1-i)})$$

10.6. Lemma. *For all n odd, at $x \in \{a,b\}$, the coefficients $N_i(\dots;n)$ (which appear in the evolution equation* (7.10) *of $T^{(n)}$) are of the following type:*
(i) $N_i \in \{N_i(e,e), N_i(o,o)\}$ for i odd
(ii) $N_i = N_i(e,o)$ for i even.
N_i denotes a sum of bilinear forms $\langle\cdot,\cdot\rangle$ where the entries are derivatives of T of order indicated by e=even and o=odd. E.g., for each i even, N_i is a sum which only contains terms $\left\langle T^{(k)}, T^{(l)}\right\rangle$ such that all (k,l) are pairs where k is odd and l even, or vice versa.

Proof. We use induction by n.
$n = 1$: no such coefficients appear in the evolution equation (7.3).
$n = 3$: $N_0 = 0$, $N_1 = 8\langle T'', T''\rangle$, $N_2 = 12\langle T', T''\rangle$.

The evolution equation (10.3) for $T^{(n)}$ includes terms of correct type and the coefficients $N_i(\dots; n-2)$. Now the previous corollary 10.5 enables us to conclude that if the claim holds for $n-2$, it must also hold for n: If $N_i(\dots; n-2)$ is of the appropriate type for i odd, then $N_i'(\dots; n-2)$ is of the appropriate type for i even, as well as $N_i''(\dots; n-2)$ is of the appropriate type for i odd and vice versa. $\square$

10.7. Theorem. *In the case of a flat manifold $\mathcal{M}$, i.e. a manifold where* Riem $\equiv 0$, *we have that for all $u \in \mathbb{N}$, u odd, for all times $t > 0$ such that F_t exists, that $T^{(u)}(t, x) = 0$ on $x \in \{a, b\}$.*

Proof. We proceed using induction by u, u odd:
$u = 1$ holds; assume, that for n, n odd, the claim holds true.
Using equation (7.10), we have that its left hand side is zero $((\partial/\partial t)T^{(n)} = 0)$. The terms on the right hand side always include at least one derivative of T of odd order or $N_i T^{(i)}$, and derivatives of Riem, the latter being zero because we required a flat space-times.
The previous lemma tells us that N_i has appropriate type if $T^{(i)}$ does not vanish, i.e. if i is even: Then $N_i = N_i(e, o)$ vanishes because its sum always includes derivatives of T of odd order: $lim_{x\to\{a,b\}} N_i T^{(i)} = 0$. $\square$

10.8. Theorem. *If the manifold $T\mathcal{M}$ is flat, and if k^2 is uniformly bounded, then the strong parabolic maximum principle yields a uniform bound on $\left\|T^{(n)}\right\|^2$ resp. $\left|\left\|T^{(n)}\right\|\right|^2$ for all $n \geq 2$, for all time intervals $[0, T_0]$: For all $T_0 > 0$ there exists a constant $\zeta_n(T_0)$ such that*

$$\max_{(x,t)\in[a,b]\times[0,T_0]} \left|\left\|T^{(n)}\right\|\right|^2 \leq \zeta_n(T_0).$$

Proof. Without loss of generality we consider the case $\varepsilon = -1$ only. We proceed by induction on n. Since k^2 is supposed to be bounded, the maximum principle holds for the evolution of $|||T''|||^2$. Therefore $|||T''|||^2$ is bounded either by its initial value at $t = 0$ or its maximum at the boundary $\{a, b\}$ and, due to the strong maximum principle, in this case we obtain $\partial\,|||T''|||^2/\partial\nu\big|_{\{a,b\}} > 0$. Since $|||T'|||^2 = 0 = \left|\left\|T^{(3)}\right\|\right|^2$ at $\{a, b\}$, we get a contradiction:

$$\pm\frac{\partial\,|||T''|||^2}{\partial\nu} = \frac{\partial\,|||T''|||^2}{\partial x} = 2c\left\langle T'', T^{(3)}\right\rangle + 4c\left\langle T'', T'\right\rangle\left\langle T^{(3)}, T\right\rangle,$$

which is zero at $\{a, b\}$.

Now suppose that $\left|\left|\left|T^{(m)}\right|\right|\right|^2$ is bounded for all $2 \leq m < n$. Once again the maximum principle holds for the evolution equation of $\left|\left|\left|T^{(n)}\right|\right|\right|^2$ and a maximum at the boundary $\{a,b\}$ would require that $\partial \left|\left|\left|T^{(n)}\right|\right|\right|^2 / \partial\nu\Big|_{\{a,b\}} > 0$. In the previous theorem we have proven that for all u odd, $T^{(u)} = 0$ at the boundary $\{a,b\}$. Since each term in

$$\pm\frac{\partial \left|\left|\left|T^{(n)}\right|\right|\right|^2}{\partial\nu} = 2c\left\langle T^{(n)}, T^{(n+1)}\right\rangle + 4c\left\langle T^{(n)}, T^{(n+1)}\right\rangle\left\langle T^{(n+1)}, T\right\rangle$$

contains at least one such derivative of odd order, this yields $\partial \left|\left|\left|T^{(n)}\right|\right|\right|^2 / \partial\nu\Big|_{\{a,b\}} = 0$, a contradiction. Therefore the norm of $T^{(n)}$ is bounded by the initial value at $t = 0$ times a factor which increases at most exponentially in time, and therefore we conclude that $\left|\left|\left|T^{(n)}\right|\right|\right|^2$ remains finite. □

We summarize what we have shown so far.

10.9. Remark. We assume that F_t exists on the finite interval $[a,b]\times[0,T_0)$, $T_0 > 0$, and that k^2 is uniformly bounded on $[a,b]\times[0,T_0]$. Assume that one of the following cases holds:

- The curve $F_t(x)$ has no boundary or it is closed, i.e. $F(a) \equiv F(b)$.
- The manifold $\mathcal{M}$ is flat, i.e. $\mathrm{Riem} \equiv 0$.

Then for all $n \geq 2$, $T^{(n)}$ is uniformly bounded on $x \in \{a,b\}$ [with respect to the norm $\|.\|$ resp. $|||.|||$], and therefore $\lim_{t\nearrow T}\left(T^{(n)}\big|_{\{a,b\}}\right)$ is finite.

Proof. We write $|||\cdot|||$ for both Riemannian and Lorentzian manifolds. In the first case, no boundary estimates are needed. In the other case we have shown in the previous theorem that the value of $\left|\left|\left|T^{(n)}\right|\right|\right|$ is either zero at the boundary or — due to the strong maximum principle — that no global strict maximum can occur at the boundary. Therefore the evolution equation of $\left|\left|\left|T^{(n)}\right|\right|\right|$ yields that $\left|\left|\left|T^{(n)}\right|\right|\right|$ can be estimated by its initial value at $t = 0$ times a factor which grows at most exponentially in time. □

10.10. Remark. We expect to be able to establish a bound on the value of the derivatives $\left|\left|\left|T^{(n)}\right|\right|\right|$ at the end points a and b for the general setting of $(\star)$ and $(\star\star)$. Unfortunately there still remains to establish a theorem of this type:

> **(¿...?) Conjecture.** *If for all $m \leq 2n-1 \in \mathbb{N}$, $T^{(m)}$ is uniformly bounded on $[a,b]\times[0,T)$, and if also $T^{(2n+1)}$ is bounded at $\{a,b\}\times[0,T)$, then $T^{(2n)}$ is bounded at $\{a,b\}$.*

Then we could use induction with respect to n, n odd:
We start noting that $T' = 0 = T^{(3)}$ on $\{a, b\}$.
Now proceed iterating the conjecture (¿...?) for $2n$ even and corollary 10.3 for $2n+1$ odd.

We must leave this for further investigations.

Under the conditions of remark 10.9 we state this theorem:

10.11. Theorem (Long-Time Existence of F_t).
If for all $m \in \mathbb{N}$, k^2 and $|||\nabla^m \operatorname{Riem}|||_{\mathrm{op}}$ are bounded on $[a,b]\times[0,T)$, then due to the maximum principle in its exponential form the limit $\lim_{t \nearrow T} F_t = F_T$ exists and is bounded with respect to the norms $\|.\|^2$ resp. $|||.|||^2$ of the arc length derivatives $T^{(i)}$. F_T is an appropriate initial value to the problem $(\star)$ resp. $(\star\star)$ starting at the time $t = T$ [i.e. $F_T(a) = F_0(a)$ and $F_T(b) = F_0(b)$]. Therefore the short-time existence theorem allows F_T to flow for some more time. Since F_t will then be smooth at $t = T$, this process can be continued to yield the result:
The solution F_t to the curvature flow $(\star)$ resp. $(\star\star)$ exists for all times $t \geq 0$.

The following two lemmas which will complete this section restrict the problem of boundary estimates to finding a bound on $\int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds$ *before* knowing the behavior of $\left|\left|\left|T^{(n)}\right|\right|\right|^2$. Therefore we cannot investigate the evolution equation of $\left|\left|\left|T^{(n+1)}\right|\right|\right|^2$ in order to apply the maximum principle, since the boundedness of the coefficients, which also depend on $\left|\left|\left|T^{(n)}\right|\right|\right|^2$, has not yet been established. One method of establishing a bound on $\int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds$ could be through deriving an equation of the type

$$\frac{\partial}{\partial t} \int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds \leq -\text{constant} \cdot \int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds$$

or

$$\frac{\partial}{\partial t} \int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds \leq \text{constant} \cdot \int_0^l k^2 \, ds,$$

which yields convergence to zero for $t \to \infty$ (the latter estimate follows from section 13) . Estimates of this type have frequently been used in section 13, but there it has been assumed that a bound on $\left|\left|\left|T^{(n)}\right|\right|\right|^2$ at the boundary $\{a, b\}$ has been established beforehand.

10.12. Theorem. *Case $\varepsilon = +1$ (consider some fixed time such that F_t exists): If for all $m \leq n \in \mathbb{N}$, $T^{(m)}$ is uniformly bounded for all $x \in [a,b]$, and if $\int_0^l \left\|T^{(n)}\right\|^2 ds < \infty$ and $\int_0^l \left\|T^{(n+2)}\right\|^2 ds < \infty$, then $T^{(n+1)}$ is bounded at $x \in \{a, b\}$ and $\int_0^l \left\|T^{(n+1)}\right\|^2 ds < \infty$.*

Proof. Let z_0 and z_1 be some arbitrary points in $[a, b]$.
By $\sum_{x\in\{z_0,z_1\}} f$ we denote $f(z_0) + f(z_1)$.

$$\begin{aligned}\int_{z_0}^{z_1} \left\|T^{(n+1)}\right\|^2 ds &= \left[\left\langle T^{(n)}, T^{(n+1)}\right\rangle\right]_{z_0}^{z_1} - \int_{z_0}^{z_1} \left\langle T^{(n)}, T^{(n+2)}\right\rangle ds \\ &\le \sum_{x\in\{z_0,z_1\}} \left(\frac{1}{2}\left\|T^{(n)}\right\|^2 + \frac{1}{2}\left\|T^{(n+1)}\right\|^2\right) \\ &\quad + \frac{1}{2}\int_{z_0}^{z_1} \left\|T^{(n)}\right\|^2 + \left\|T^{(n+2)}\right\|^2 ds\end{aligned}$$

$$\begin{aligned}\left|\left[\left\|T^{(n+1)}\right\|^2\right]_{z_0}^{z_1}\right| &= \left|\int_{z_0}^{z_1} \frac{\partial}{\partial s}\left\|T^{(n+1)}\right\|^2 ds\right| \\ &\le \int_{z_0}^{z_1} \left\|T^{(n+1)}\right\|^2 + \left\|T^{(n+2)}\right\|^2 ds\end{aligned}$$

Therefore we have:

$$\begin{aligned}\int_{z_0}^{z_1} \left\|T^{(n+1)}\right\|^2 ds &\le \sum_{x\in\{z_0,z_1\}} \left(\left\|T^{(n)}\right\|^2\right) + 2\left\|T^{(n+1)}\right\|^2 (z_0) \\ &\quad + \int_{z_0}^{z_1} \left\|T^{(n)}\right\|^2 + 2\left\|T^{(n+2)}\right\|^2 ds,\end{aligned}$$

$$\begin{aligned}\int_{z_0}^{z_1} \left\|T^{(n+1)}\right\|^2 ds &\le \sum_{x\in\{z_0,z_1\}} \left(\left\|T^{(n)}\right\|^2\right) + 2\left\|T^{(n+1)}\right\|^2 (z_1) \\ &\quad + \int_{z_0}^{z_1} \left\|T^{(n)}\right\|^2 + 2\left\|T^{(n+2)}\right\|^2 ds\end{aligned}$$

which can be summed up (take $(z_0, z_1) = ((b-a)/2, b)$ resp. $= (a, (b-a)/2)$) as

$$\begin{aligned}\int_{a}^{b} \left\|T^{(n+1)}\right\|^2 ds \le \sum_{x\in\{a,b\}} \left(\left\|T^{(n)}\right\|^2\right) + 2\left\|T^{(n)}\right\|^2 \left(\frac{b-a}{2}\right) + 4\left\|T^{(n+1)}\right\|^2 \left(\frac{b-a}{2}\right) \\ + \int_{a}^{b} \left\|T^{(n)}\right\|^2 + 2\left\|T^{(n+2)}\right\|^2 ds.\end{aligned} \tag{10.4}$$

All quantities on the right hand side are bounded, either by assumption or due to the maximum principle which holds for $T^{(n+1)}$ and therefore yields an inner bound for $T^{(n+1)}$. This gives that $\int_a^b \left\|T^{(n+1)}\right\|^2 \, ds$ can be estimated by some quantities which are already known to be bounded.

In order to set a bound on the values at the end points, we need two estimates.

$$\begin{aligned}\left\|T^{(n+1)}\right\|^2(z_1) \leq & 3\left\|T^{(n+1)}\right\|^2(z_0) + \sum_{x\in\{z_0,z_1\}} \left(\left\|T^{(n)}\right\|^2\right) \\ & + \int_{z_0}^{z_1} \left\|T^{(n)}\right\|^2 + 3\left\|T^{(n+2)}\right\|^2 \, ds\end{aligned} \tag{10.5}$$

and

$$\begin{aligned}\left\|T^{(n+1)}\right\|^2(z_0) \leq & 3\left\|T^{(n+1)}\right\|^2(z_1) + \sum_{x\in\{z_0,z_1\}} \left(\left\|T^{(n)}\right\|^2\right) \\ & + \int_{z_0}^{z_1} \left\|T^{(n)}\right\|^2 + 3\left\|T^{(n+2)}\right\|^2 \, ds.\end{aligned} \tag{10.6}$$

In the case of $z_1 = b$, we estimate $\left\|T^{(n+1)}\right\|^2(b)$ by the first of the last two inequalities, taking $z_0 = (b-a)/2$. In the other case we use the last inequality where we set $z_0 = a$ and $z_1 = (b-a)/2$. $\square$

10.13. Theorem. *Case $\varepsilon = -1$ (consider some fixed time such that F_t exists): If $\max_{x\in[a,b]} k^2$ is finite, and if for all $m \leq n \in \mathbb{N}$, $T^{(m)}$ is uniformly bounded in $x \in [a,b]$, and if $\int_0^l \left|\left|\left|T^{(n)}\right|\right|\right|^2 ds < \infty$ and $\int_0^l \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds < \infty$, then $T^{(n+1)}$ is bounded at $x \in \{a,b\}$ and $\int_0^l \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds < \infty$.*

Proof. Partial differentiation of $\int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds$:

$$\begin{aligned}\int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds = & \left[\left\langle T^{(n)}, T^{(n+1)}\right\rangle\right]_{z_0}^{z_1} - \int_{z_0}^{z_1} \left\langle T^{(n)}, T^{(n+2)}\right\rangle ds \\ & + \left[2\left\langle T^{(n)}, T\right\rangle \left\langle T^{(n+1)}, T\right\rangle\right]_{z_0}^{z_1} - 2\int_{z_0}^{z_1} \left\langle T^{(n)}, T'\right\rangle \left\langle T^{(n+1)}, T\right\rangle \\ & + \left\langle T^{(n)}, T\right\rangle \left\langle T^{(n+2)}, T\right\rangle + \left\langle T^{(n)}, T\right\rangle \left\langle T^{(n+1)}, T'\right\rangle ds \\ \leq & \left[\left\langle T^{(n)}, T^{(n+1)} + 2\left\langle T^{(n+1)}, T\right\rangle T\right\rangle\right]_{z_0}^{z_1} + \frac{1}{2}\int_{z_0}^{z_1} \left|\left|\left|T^{(n)}\right|\right|\right|^2 + \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds \\ & + \int_{z_0}^{z_1} 2k^2 \left|\left|\left|\Pi^{\perp}\left(T^{(n)}\right)\right|\right|\right|^2 + \frac{1}{2}\left|\left|\left|\Pi^{\top}\left(T^{(n+1)}\right)\right|\right|\right|^2 + \left|\left|\left|\Pi^{\top}\left(T^{(n)}\right)\right|\right|\right|^2 \\ & + \left|\left|\left|\Pi^{\top}\left(T^{(n+2)}\right)\right|\right|\right|^2 + 2k^2\left|\left|\left|\Pi^{\top}\left(T^{(n)}\right)\right|\right|\right|^2 + \frac{1}{2}\left|\left|\left|\Pi^{\perp}\left(T^{(n+1)}\right)\right|\right|\right|^2 ds \\ \leq & \left[\left\langle T^{(n)}, T^{(n+1)} + 2\left\langle T^{(n+1)}, T\right\rangle T\right\rangle\right]_{z_0}^{z_1} + \int_{z_0}^{z_1} 2k^2\left|\left|\left|T^{(n)}\right|\right|\right|^2 + \frac{1}{2}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 \\ & + \frac{3}{2}\left|\left|\left|T^{(n)}\right|\right|\right|^2 + \frac{3}{2}\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds.\end{aligned}$$

Thus

$$\int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 \, ds \leq 2\left[\left\langle T^{(n)}, \Pi^{\perp}\left(T^{(n+1)}\right) - \Pi^{\top}\left(T^{(n+1)}\right)\right\rangle\right]_{z_0}^{z_1} + \int_{z_0}^{z_1} (4k^2+3)\left|\left|\left|T^{(n)}\right|\right|\right|^2 + 3\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \, ds. \tag{10.7}$$

We apply the fundamental theorem of calculus:

$$\begin{aligned}\left|\left[\left|\left|\left|T^{(n+1)}\right|\right|\right|^2\right]_{z_0}^{z_1}\right| &= \left|\int_{z_0}^{z_1} \frac{\partial}{\partial s}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 \, ds\right| = \left|\int_{z_0}^{z_1} 2\left\langle T^{(n+1)}, T^{(n+2)}\right\rangle\right. \\ &\quad \left. +4\left\langle T^{(n+1)}, T\right\rangle\left\langle T^{(n+2)}, T\right\rangle + 4\left\langle T^{(n+1)}, T\right\rangle\left\langle T^{(n+1)}, \underbrace{T'}_{kN}\right\rangle ds\right| \\ &\leq \int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \\ &\quad + 2\left|\left|\left|\Pi^{\top}\left(T^{(n+1)}\right)\right|\right|\right|^2 + 2\left|\left|\left|\Pi^{\top}\left(T^{(n+2)}\right)\right|\right|\right|^2 \\ &\quad + 2k^2\left|\left|\left|\Pi^{\top}\left(T^{(n+1)}\right)\right|\right|\right|^2 + 2\left|\left|\left|\Pi^{\perp}\left(T^{(n+1)}\right)\right|\right|\right|^2 \, ds \\ &\leq \int_{z_0}^{z_1} 3\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + 2k^2\left|\left|\left|\Pi^{\top}\left(T^{(n+1)}\right)\right|\right|\right|^2 + 3\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \, ds.\end{aligned}$$

This yields

$$\left|\left[\left|\left|\left|T^{(n+1)}\right|\right|\right|^2\right]_{z_0}^{z_1}\right| \leq \int_{z_0}^{z_1} (3+2k^2)\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + 3\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \, ds. \tag{10.8}$$

Now let $\beta := 3 + 2\max_{x\in[a,b]} k^2(x) \geq 3$, which is finite. Then

$$\begin{aligned}&2\left[\left\langle T^{(n)}, \Pi^{\perp}\left(T^{(n+1)}\right) - \Pi^{\top}\left(T^{(n+1)}\right)\right\rangle\right]_{z_0}^{z_1} \leq 2\sum_{x\in\{z_0,z_1\}}\left(\left|\left|\left|T^{(n)}\right|\right|\right| \cdot \left|\left|\left|T^{(n+1)}\right|\right|\right|\right) \\ &\leq \sum_{x\in\{z_0,z_1\}}\left(2\beta\left|\left|\left|T^{(n)}\right|\right|\right|^2 + \frac{1}{2\beta}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2\right) \overset{(10.8)}{\leq} 2\beta\sum_{x\in\{z_0,z_1\}}\left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) \\ &\quad + \frac{1}{\beta}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2(z_0) + \int_{z_0}^{z_1} \frac{1}{2}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + \frac{3}{2\beta}\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \, ds\end{aligned}$$

and

$$\begin{aligned}&2\left[\left\langle T^{(n)}, \Pi^{\perp}\left(T^{(n+1)}\right) - \Pi^{\top}\left(T^{(n+1)}\right)\right\rangle\right]_{z_0}^{z_1} \overset{(10.8)}{\leq} 2\beta\sum_{x\in\{z_0,z_1\}}\left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) \\ &\quad + \frac{1}{\beta}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2(z_1) + \int_{z_0}^{z_1} \frac{1}{2}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + \frac{3}{2\beta}\left|\left|\left|T^{(n+2)}\right|\right|\right|^2 \, ds.\end{aligned}$$

Therefore we get

$$\begin{aligned}\int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds \leq & 4\beta \sum_{x\in\{z_0,z_1\}} \left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) + \frac{2}{\beta} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_0) \\ & + 2\int_{z_0}^{z_1} (4k^2+3) \left|\left|\left|T^{(n)}\right|\right|\right|^2 + (3+\frac{3}{2\beta}) \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds\end{aligned} \tag{10.9}$$

and

$$\begin{aligned}\int_{z_0}^{z_1} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds \leq & 4\beta \sum_{x\in\{z_0,z_1\}} \left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) + \frac{2}{\beta} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_1) \\ & + 2\int_{z_0}^{z_1} (4k^2+3) \left|\left|\left|T^{(n)}\right|\right|\right|^2 + (3+\frac{3}{2\beta}) \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds.\end{aligned} \tag{10.10}$$

If we add these equations in the special case of $(z_0, z_1) = ((b-a)/2, b)$ for the first inequality and $(z_0, z_1) = (a, (b-a)/2)$ for the second one, we obtain:

$$\begin{aligned}\int_a^b \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds \leq & 4\beta \sum_{x\in\{a,b\}} \left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) + 8\beta \left|\left|\left|T^{(n)}\right|\right|\right|^2 (\frac{b-a}{2}) \\ & + \frac{4}{\beta} \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (\frac{b-a}{2}) + 2\int_a^b (4k^2+3) \left|\left|\left|T^{(n)}\right|\right|\right|^2 \\ & + (3+\frac{3}{2\beta}) \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds.\end{aligned} \tag{10.11}$$

This gives boundedness of $\int_a^b \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds$ (cf. the previous theorem).

In addition, we can also calculate

$$\begin{aligned}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_1) \leq & 3 \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_0) + 4\beta^2 \sum_{x\in\{z_0,z_1\}} \left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) \\ & + \int_{z_0}^{z_1} 2\beta^2 \left|\left|\left|T^{(n)}\right|\right|\right|^2 + 6(\beta+1) \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds\end{aligned} \tag{10.12}$$

and

$$\begin{aligned}\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_0) \leq & 3 \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 (z_1) + 4\beta^2 \sum_{x\in\{z_0,z_1\}} \left(\left|\left|\left|T^{(n)}\right|\right|\right|^2\right) \\ & + \int_{z_0}^{z_1} 2\beta^2 \left|\left|\left|T^{(n)}\right|\right|\right|^2 + 6(\beta+1) \left|\left|\left|T^{(n+2)}\right|\right|\right|^2 ds.\end{aligned} \tag{10.13}$$

As in the previous theorem we can take $(z_0, z_1) = ((b-a)/2, b)$ or $(z_0, z_1) = (a, (b-a)/2)$ in order to prove the boundedness result. □

11 Upper Bound on the Derivatives $T^{(n)}$

In the previous sections we have established that, under particular assumptions on the curvature, the Riemanniann tensor and the "angle", $\left\|T^{(n)}\right\|^2$ resp. $\left|\left|\left|T^{(n)}\right|\right|\right|^2$ $(n \geq 2)$ can at most grow exponentially in time. In particular, we have proven that under these assumptions the curvature flow has a long time solution.

In this section, we show that there is indeed an upper bound which cannot be exceeded by the derivatives of the tangent vector of the curve F_t. Since we already have established such an absolute bound on $\left\|T^{(n)}\right\|^2$ resp. $\left|\left|\left|T^{(n)}\right|\right|\right|^2$ at the end points $x \in [a, b]$, it remains to consider inner points where we can apply the maximum principle without having to be concerned of the boundary.

We use induction by $n \in \mathbb{N}, n \geq 2$, to show that such a bound exists. Notice that the structure of the evolution equations can be sketched as follows: There exist coefficients A_n and c_n such that

$$\frac{\partial}{\partial t}\left\|T^{(n)}\right\|^2 \leq \frac{\partial^2}{\partial s^2}\left\|T^{(n)}\right\|^2 + A_n\left\|T^{(n)}\right\|^2 - \left\|T^{(n+1)}\right\|^2 + c_n$$

for all $(x, t) \in [a, b] \times [0, T)$, $0 < T < \infty$.

For the whole section, we assume that k^2 is uniformly bounded:

$$\text{there exists } k^2_{\max} < \infty: \ k^2(x,t) \leq k^2_{\max} \quad \text{for all } (x,t), x \in [a,b], t > 0,$$

and that the other assumptions which yield long time existence hold as well. In the Lorentzian case, replace double norms $\|.\|$ by triple norms $|||.|||$; we will not further distinguish the two notations.

11.1. Theorem. *For all $2 \leq m \in \mathbb{N}$, there exists a $\gamma_m < \infty$:*

$$\left\|T^{(m)}\right\|^2 \leq \gamma_m \quad \textit{for all } (x,t), x \in [a,b], t > 0.$$

Proof. We use induction by $2 \leq m \in \mathbb{N}$.
$m = 2$: there exist constants A_2 and c_2, independent of the time t, such that for all (x, t), $x \in [a, b]$, $t > 0$, we have

$$\frac{\partial}{\partial t}\left\|T''\right\|^2 \leq \frac{\partial^2}{\partial s^2}\left\|T''\right\|^2 + A_2\left\|T''\right\|^2 + c_2 - \left\|T^{(3)}\right\|^2,$$

since A_2 and c_2 only depend on k^2, which is assumed to be uniformly bounded for all $t \geq 0$. A corresponding relation for $\left\|T'\right\|^2 = k^2$ holds as well, cf. (7.4).

By the same argumentation as will be presented below in detail for general m, we conclude that $\|T''\|^2$ is uniformly bounded by some γ_2 which is independent of t:

$$\|T''\|^2 \leq \gamma_2 \quad \forall (x,t), x \in [a,b], t > 0.$$

Now assume that $m > 2$ and

$$\left\|T^{(l)}\right\|^2 \leq \gamma_l \quad \forall (x,t), x \in [a,b], t > 0, 2 \leq l \leq m.$$

Then for all $2 \leq l \leq m+1$ there exist positive constants A_l and c_l, which are independent of the time t, such that

$$\frac{\partial}{\partial t}\left\|T^{(l)}\right\|^2 \leq \frac{\partial^2}{\partial s^2}\left\|T^{(l)}\right\|^2 + A_l\left\|T^{(l)}\right\|^2 + c_l - \left\|T^{(l+1)}\right\|^2 \quad \forall (x,t), x \in [a,b], t > 0,$$

since A_l and c_l depend on $k^2(x,t) \leq k^2_{\max}$ and $\left\|T^{(j)}\right\|^2 \leq \gamma_j$, $j < l$ only, which by induction have already been bounded.

For all $0 < \delta < 1$ let $\alpha_m := (A_{m+1} + 1)/\delta$ and

$$M_m(\delta) := \max\{c_{m+1}, \frac{1}{1-\delta}(A_m\gamma_m + c_m)\}.$$

Let $\eta > 0$ be a positive constant. We assume that at the point (ξ, τ), $\xi \in (a,b)$, for the first time $t \geq 0$, the maximum of $\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2$ becomes greater or equal than $M_m(\delta) + \alpha_m\gamma_m + \eta$:

$$\begin{aligned}\max_{(x,t)\in[a,b]\times[0,\tau]} & \alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2 \\ &= \left(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2\right)\Big|_{(x,t)=(\xi,\tau)} \geq M_m(\delta) + \alpha_m\gamma_m + \eta.\end{aligned}$$

If we already have that at $t = 0$,

$$\kappa := \max_{(x,0)\in[a,b]\times\{0\}} \alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2 > M_m(\delta) + \alpha_m\gamma_m + \eta,$$

we consider the first time τ when the maximum of $\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2$ becomes greater or equal than $\kappa + \eta$:

$$\begin{aligned}\max_{(x,t)\in[a,b]\times[0,\tau]} \alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2 &= \left(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2\right)\Big|_{(x,t)=(\xi,\tau)} \\ &\geq \kappa + \eta > M_m(\delta) + \alpha_m\gamma_m + \eta.\end{aligned}$$

Therefore $\left\|T^{(m+1)}\right\|^2(\xi,\tau) > M_m(\delta)$, and we obtain the following estimates at $(x,t)=(\xi,\tau)$:

$$\begin{aligned}
(A_m\left\|T^{(m)}\right\|^2 + c_m)\Big|_{(x,t)=(\xi,\tau)} &< (1-\delta)\left\|T^{(m+1)}\right\|^2\Big|_{(x,t)=(\xi,\tau)},\\
(A_{m+1}\left\|T^{(m+1)}\right\|^2 + c_{m+1})\Big|_{(x,t)=(\xi,\tau)} &< (A_{m+1}+1)\left\|T^{(m+1)}\right\|^2\Big|_{(x,t)=(\xi,\tau)}.
\end{aligned}$$

Since (ξ,τ) is the first time when the maximum becomes greater than our bound for the first time:

$$\begin{aligned}
&\frac{\partial}{\partial t}(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2)\Big|_{(x,t)=(\xi,\tau)} \geq 0,\\
&\frac{\partial^2}{\partial s^2}(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2)\Big|_{(x,t)=(\xi,\tau)} \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_m(A_m\left\|T^{(m)}\right\|^2 + c_m - \left\|T^{(m+1)}\right\|^2) &< -\alpha_m\delta\left\|T^{(m+1)}\right\|^2\\
&= -(A_{m+1}+1)\left\|T^{(m+1)}\right\|^2,
\end{aligned}$$

$$A_{m+1}\left\|T^{(m+1)}\right\|^2 + c_{m+1} - \left\|T^{(m+2)}\right\|^2 < (A_{m+1}+1)\left\|T^{(m+1)}\right\|^2.$$

Therefore our evolution equation of $\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2$ yields a contradiction:

$$\begin{aligned}
\frac{\partial}{\partial t}(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2) \leq &\frac{\partial^2}{\partial s^2}(\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2)\\
&+ \alpha_m(A_m\left\|T^{(m)}\right\|^2 + c_m - \left\|T^{(m+1)}\right\|^2)\\
&+ (A_{m+1}\left\|T^{(m+1)}\right\|^2 + c_{m+1} - \left\|T^{(m+2)}\right\|^2),
\end{aligned}$$

since the left hand side is ≥ 0 while the right hand side is negative.

In the previous section we have established that at the end points $\{a,b\}$, there is an upper bound ζ_n on the derivatives $T^{(n)}$:

$$\left\|T^{(n)}(x,t)\right\|^2\Big|_{x\in\{a,b\}} \leq \zeta_n \quad \text{for all } t\in[0,T)$$

for a time interval $[0,T)$ such that a solution F_t exists.

This implies that for all $0<\delta<1$, the maximum of $\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2$ cannot exceed $\max\{M_m(\delta)+\alpha_m\gamma_m, \kappa, \alpha_m\zeta_m+\zeta_{m+1}\}+\eta$:

$$\alpha_m\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2 \leq \max\{M_m(\delta)+\alpha_m\gamma_m, \kappa, \alpha_m\zeta_m+\zeta_{m+1}\}+\eta.$$

Thus we obtain

$$\begin{aligned}
\left\|T^{(m+1)}\right\|^2 - \eta \le \max \quad & \{M_m(\delta) + \alpha_m\gamma_m, \kappa, \alpha_m\zeta_m + \zeta_{m+1}\} \\
= \max \quad & \Bigg\{ \frac{A_m+1}{\delta}\gamma_m + \max\{c_{m+1}, \frac{1}{1-\delta}(A_m\gamma_m + c_m)\}, \\
& \max_{(x,0)\in[a,b]\times\{0\}} \frac{A_m+1}{\delta}\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2, \\
& \frac{A_m+1}{\delta}\zeta_m + \zeta_{m+1} \Bigg\},
\end{aligned}$$

and since $\eta > 0$ was arbitrary

$$\begin{aligned}
\left\|T^{(m+1)}\right\|^2 \le \inf_{0<\delta<1} \max & \Bigg\{ \frac{A_m+1}{\delta}\gamma_m + \max\{c_{m+1}, \frac{1}{1-\delta}(A_m\gamma_m + c_m)\}, \\
& \max_{(x,0)\in[a,b]\times\{0\}} \left(\frac{A_m+1}{\delta}\left\|T^{(m)}\right\|^2 + \left\|T^{(m+1)}\right\|^2 \right), \\
& \frac{A_m+1}{\delta}\zeta_m + \zeta_{m+1} \Bigg\},
\end{aligned}$$

which completes the proof. $\square$

12 Evolution of the Length

Standard theory of curves in manifolds says that a curve is an (unbroken) geodesic iff the curvature $k^2 \equiv 0$ (cf. [O'N83, pp 264–5]).

12.1. Definition. The *length l_t of the curve $F_t(\cdot)$* is defined as follows:

$$l_t = \int_a^b \sqrt{\varepsilon \left\| \frac{\partial}{\partial x} F(\tilde{x}, t) \right\|^2} d\tilde{x} \tag{12.1}$$

12.2. Lemma. *For $\varepsilon = +1$, the length l_t of F_t is decreasing, while for $\varepsilon = -1$, l_t of F_t is increasing.*

Proof. Recall: $c := \sqrt{\varepsilon \left\| \partial/\partial x F(\tilde{x}, t) \right\|^2}$. Therefore (compare section 4)

$$\frac{\partial l}{\partial t} = \int_a^b \frac{\partial c}{\partial t}\, dx = \int_a^b (-\varepsilon k^2 c)\, dx = -\varepsilon \int_a^b k_t^2(x) c\, dx = -\varepsilon \int_0^{l_t} k_t^2(s)\, ds \tag{12.2}$$

□

12.3. Corollary. *For case $\varepsilon = +1$, the length l_t is bounded: $0 \le l_t \le l_0$; if $F(a) \neq F(b)$, $l_t > 0$.*
If $\varepsilon = -1$ ($\mathcal{M} = \mathcal{M}^{n,1}$), $l_t \ge l_0$; an upper bound is set by the fact that:
A curve γ locally maximizes the length between two points p and q over smooth one parameter variations $\iff$ γ is a geodesic with no conjugate points between p and q ([Wal84, p. 229, thm. 9.3.3]).
We therefore need to consider a space $\mathcal{M}^{n,1}$ which, for given end points p and q, guarantees the existence of a geodesic with end points p and q and no conjugate points between p and q.
For instance, a globally hyperbolic spacetime *$\mathcal{M}^{n,1}$ yields such an existence theorem (cf. [Wal84, p. 236, thm. 9.4.4] and [HE73, p. 213]) (this holds true as long as one end point q lies in the causal future $\mathcal{J}^+(p)$ of the other end point $p \in \mathcal{M}$, or vice versa).*
The existence of geodesics (with or without conjugate points) is linked to the connectedness of $\mathcal{M}^{n,1}$:
There is a (broken) geodesic joining any two points in $\mathcal{M}^{n,1}$ $\iff$ $\mathcal{M}^{n,1}$ is connected (cf. [O'N83, p. 72]).

12.4. Theorem. *For $\mathcal{M} = \mathcal{M}^{n+1}$ and $\mathcal{M} = \mathcal{M}^{n,1}$, such that geodesics exist (as explained in the previous corollary), we have:*
If k^2, $|\partial k^2/\partial s|$ and $\int_0^l \partial k^2/\partial t\, ds$ are uniformly bounded $\forall t \ge 0$, then $\lim\limits_{t\to\infty} \max\limits_{x \in [a,b]} k_t^2(x) = 0$.

Proof. Let $M_1 := \sup_{(x,t)} k_t^2(x)$, $M_2 := \sup_{(x,t)} |\partial k^2/\partial s|$ and $M_3 := \sup_{(x,t)} \|\partial k^2/\partial t\|_{L^1}$.

$$\int_{t=0}^{\infty}\int_{s=0}^{l_t} k^2\, ds\, dt = -\varepsilon \int_{t=0}^{\infty} \frac{\partial l}{\partial t}\, dt = -\varepsilon(l_\infty - l_0),$$

which is well defined since $l_\infty := \lim_{t\to\infty} l_t$ is finite.

Assume the claim does not hold true.
Then $\exists \mu > 0$ such that $\forall t > 0\, \exists \tilde{t} > t : \max_{x\in[a,b]} |k_{\tilde{t}}^2| > \mu$.
Therefore there exists a sequence $(t_m)_{m\in\mathbb{N}}$, $t_m < t_{m+1}$,
such that $\max_{x\in[a,b]} |k_{t_m}^2| > \mu \quad \forall m \in \mathbb{N}$.

(i) Let $c_0(t) = \max_{x\,\in\,[a,b]} |(k^2)'(t)| \leq M_2 = \sup_{t\geq 0} c_0(t)$,
and $s_0(t)$ such that $k_t^2(s_0) = \max_{s\in[0,l]} k_t^2(s)$.
Then $k_{t_i}^2(s) \geq \mu/2 \quad \forall s \in |s - s_0| \leq \mu/(2c_0), s \in [0, l_{t_i}]$.

$$\begin{aligned}\int_{F_{t_i}} k^2 ds \geq \frac{\mu}{2}\cdot \min\{\frac{\mu}{2c_0(t_i)}, l_{t_i}\} &= \min\{\frac{\mu^2}{4c_0(t_i)}, \frac{\mu l_{t_i}}{2}\}\\ &\geq \min_t\{\frac{\mu^2}{4M_2}, \frac{\mu l_{t_i}}{2}\} =: c_1\end{aligned}$$

Since $l_t \geq \begin{cases} l_\infty & > 0 \quad \text{for } \mathcal{M}^{n+1} \\ l_0 & > 0 \quad \text{for } \mathcal{M}^{n,1}\end{cases}$, this puts a lower bound c_1 on $\int_{F_{t_i}} k^2 ds$.

$$\begin{aligned}\text{(ii)} \quad \left|\frac{\partial}{\partial t}\int_0^l k^2 ds\right| &= \left|\frac{\partial}{\partial t}\int_a^b k^2 c dx\right| = \left|\int_a^b \frac{\partial k^2}{\partial t} c\, dx + \int_a^b k^2 \frac{\partial c}{\partial t}\, dx\right|\\ &= \left|\int_0^l \frac{\partial k^2}{\partial t}\, ds - \varepsilon\int_0^l k^4\, ds\right| \leq \tilde{M}(M_1, M_3, l_{\max}),\end{aligned}$$

since k^2, $\|\partial k^2/\partial t\|_{L^1}$ and l_t are bounded.

We therefore obtain:

$$\begin{aligned}&\int_{F_t} k^2\, ds \geq \frac{1}{2}c_1 \quad \forall t : |t - t_m| \leq \frac{c_1}{2\tilde{M}} \quad (t_m \text{ large such that } t_m \geq \frac{c_1}{2\tilde{M}})\\ \Rightarrow \quad &\int_{t_m - c_1/(2\tilde{M})}^{t_m + c_1/(2\tilde{M})}\int_{F_t} k^2\, ds\, dt \geq \frac{1}{2}c_1 \cdot 2(\frac{c_1}{2\tilde{M}}) = \frac{c_1^2}{2\tilde{M}}\end{aligned}$$

Since this is independent of the index m, we have a contradiction. □

12.5. Theorem. *Case $\varepsilon = -1$, i.e. $\mathcal{M} = \mathcal{M}^{n,1}$:*
If $|||\mathrm{Riem}|||_{\mathrm{op}}$ is uniformly bounded, then there is a uniform curvature bound for all times $t \geq 0$.

Proof.

$$\frac{\partial k^2}{\partial t} \overset{(7.4)}{=} \frac{\partial^2 k^2}{\partial s^2} - 2 \left|\left|\left|\Pi^{\perp}(T'')\right|\right|\right|^2 - 2k^4 + 2\langle R(Y,T)T,Y\rangle$$

We obtain the estimate $\langle R(Y,T)T,Y\rangle \leq \underbrace{|||\mathrm{Riem}|||_{\mathrm{op}} \langle e_0, T\rangle^4}_{R:=} k^2$ [cf. section 6].

Note that $k^2 = 0$ on the boundary of $[a,b]$, which leaves us to show that no maximum in the inner of the interval (a,b) may become greater than a particular bound independent of time.

Assume that k^2 is unbounded and consider the first time when the maximum of k^2 is larger than some $L > R$. Let τ be the first time such that k^2 attains this maximum $L = \max_{x\in[a.b]} k^2 > R$, and $(\tau,\xi) \in \mathbb{R}_0^+ \times [a,b]$ such that $L = k^2(\tau,\xi)$.

We then have:

$$\frac{\partial}{\partial t}k^2(\tau,\xi) \geq 0,\ \frac{\partial}{\partial s}k^2(\tau,\xi) = 0 \text{ and } \frac{\partial^2}{\partial s^2}k^2(\tau,\xi) \leq 0.$$
$$2Rk^2 - 2k^4 = 2RL - 2L^2 < 0.$$

Therefore the left hand side of the evolution equation is nonnegative, while the right hand side is negative. This yields a contradiction. □

13 Convergence to Geodesics

In this section, we always assume that the Riemannian tensor and its covariant derivative are uniformly bounded and that the angle condition as explained in section 6 holds, i.e. $\langle e_0, \varepsilon T\rangle$ needs to be uniformly bounded on $t \in [0, \infty]$.

We will start with a technical lemma:

13.1. Lemma. *For a function $g \in \mathcal{C}^1([0,l])$:*

$$\text{(i)} \quad \max_{s\in[0,l]} g^2 \le \frac{2}{l_{\min}} \int_0^l g^2 ds + 2l_{\max} \int_0^l \left(\frac{\partial g}{\partial s}\right)^2 ds \tag{13.1}$$

$$\text{(ii)} \quad \max_{s\in[0,l]} g^2 \le \frac{1}{l_{\min}} \int_0^l g^2 ds + \int_0^l \left|\frac{\partial g^2}{\partial s}\right| ds \tag{13.2}$$

Proof. Cf. [Alt, equation (4.10), p. 12](note: substitute l_∞ and l_0 by $l_{\min}$ and $l_{\max}$ respectively).

The inequality

$$\max g^2 \le \max g^2 - \min g^2 + \operatorname{ave} g^2$$

yields

$$\max g^2 \le \int \left|\frac{\partial g^2}{\partial s}\right| ds + \frac{\int g^2\, ds}{\int ds} \le \frac{1}{l_{\min}} \int g^2 ds + \int \left|\frac{\partial g^2}{\partial s}\right| ds.$$

Also consider

$$\max g^2 \le (\max g - \min g + \operatorname{ave} g)^2$$

such that if we apply the Hölder inequality and the Peter-Paul inequality to

$$\begin{aligned}\max g^2 &\le \left(\int \left|\frac{\partial g}{\partial s}\right| ds + \frac{\int g\, ds}{\int ds}\right)^2 \\ &\le \left(\int \left|\frac{\partial g}{\partial s}\right| ds\right)^2 + 2\int \left|\frac{\partial g}{\partial s}\right| ds \cdot \frac{\int g\, ds}{\int ds} + \left(\frac{\int g\, ds}{\int ds}\right)^2,\end{aligned}$$

the result follows. ☐

We will apply this lemma to k^2 and show that both terms on the right hand side converge to zero.

13.2. Lemma. *If k^2 is uniformly bounded, then $-\varepsilon\partial l/\partial t$ grows at most exponentially.*

Proof. We apply the evolution equations (7.4) and (4.3) for k^2 and c, and we use the following notation: $\langle R(Y,T)T,Y\rangle \leq k^2 \langle e_0, \varepsilon T\rangle^4 |||\mathrm{Riem}|||_{\mathrm{op}} =: k^2 R$.

$$\begin{aligned}\frac{\partial}{\partial t}\int k^2\, ds &= \int_0^l \frac{\partial k^2}{\partial t}\, ds + \int_a^b k^2 \frac{\partial c}{\partial t}\, dx \\ &\leq \int_0^l \frac{\partial^2}{\partial s^2} k^2 - 2\left|\left|\left|\Pi^\perp\left(T''\right)\right|\right|\right|^2 + \varepsilon k^4 + 2k^2 R\, ds\end{aligned}$$

Since

$$\int_0^l \frac{\partial^2}{\partial s^2} k^2\, ds = \left[\frac{\partial}{\partial s} k^2\right]_0^l = \left[\langle Y, T''\rangle\right]_0^l = 0$$

and

$$-2\left|\left|\left|\Pi^\perp\left(T''\right)\right|\right|\right|^2 \leq 0,$$

we get

$$\frac{\partial}{\partial t}\int k^2\, ds \leq \{\varepsilon \max k^2 + 2R\}\int k^2\, ds = M\int k^2\, ds,$$

where $M = \varepsilon \max k^2 + 2R$.

If $\varepsilon = -1$, there is no restriction on $\max k^2$ (cf. theorem 12.5: it is the evolution equation of k^2 which keeps k^2 uniformly bounded). For $\varepsilon = +1$, $\max k^2$ has to be bounded.
In any case, this bounds the evolution of $\int k^2\, ds$ by an exponential function. □

13.3. Lemma. *If k^2 is uniformly bounded, we have*

$$\lim_{t\to\infty} \frac{\partial l}{\partial t} \equiv -\varepsilon \lim_{t\to\infty} \int_0^l k^2\, ds = 0 \tag{13.3}$$

i.e. $\lim_{t\to\infty} \|k\|_{L^2} = 0$.

Proof.
(i) l is bounded: $l_{\min} \leq l \leq l_{\max}$
(ii) l is monotonous: $\mathrm{sgn}(\partial l/\partial t) = -\varepsilon$

Since l depends continuously on t, l has a well defined limit $l_\infty = \lim_{t\to\infty} l$. We still have to exclude the possibility that $\partial l/\partial t$ has narrow spikes while shrinking to zero. Since $\partial l/\partial t$ grows at most exponentially, we can show that this is not possible:
Consider the case where $\varepsilon = -1$; the other case works the same way.
The convergence of l implies that the CAUCHY criterion holds:

$$\forall \delta > 0\, \exists \tau : \forall T_1, T_2 > \tau : \int_{T_1}^{T_2} \frac{\partial l}{\partial t}\, dt < \delta.$$

Assume that $\partial l/\partial t$ does not converge to zero. Then there exists a sequence $(T_m)_{m\in\mathbb{N}}$ and some $\delta_0 > 0$ such that

$$\left.\frac{\partial l}{\partial t}\right|_{T_m} > \delta_0\; \forall m \in \mathbb{N}.$$

Without loss of generality we can assume that $\partial l/\partial t$ has a local maximum $\delta_m > \delta_0$ at T_m. Otherwise let $\tilde{T_m} > T_m$ be the smallest parameter t such that $\partial l/\partial t(t)$ has a local maximum at $\tilde{T_m}$. Such a sequence $(\tilde{T_m})_m$ exists, since $\partial l/\partial t$ is continuous and otherwise either $\partial l/\partial t < \delta_0$ for some T large (this contradicts the existence of the sequence $(T_m)_m$) or $\partial l/\partial t > \Delta$ for some $\Delta > 0$, which would not allow l to have a limit.
Then:

$$\frac{\partial l}{\partial t} \geq \frac{\delta_m}{2} \quad \forall |t - T_m| \leq \frac{1}{2M} = \frac{\delta_m/2}{M(\partial l/\partial t|_{T_m})} \left(\leq \frac{\delta_m/2}{\max(\partial l/\partial t)'}\right),$$

where M is the scaling factor which appears in the estimate that bounds the growth of $\partial l/\partial t$ to exponential (cf. the previous lemma). We therefore obtain the estimate

$$\int_{T_m-1/2}^{T_m+1/2} \frac{\partial l}{\partial t}\, dt \geq 2\frac{1}{2M} \min \frac{\partial l}{\partial t} \geq \frac{\delta_m}{2M} \geq \frac{\delta_0}{2M},$$

which yields a contradiction to the CAUCHY criterion for some small $\delta < \delta_0/(2M)$. $\square$

This leaves us to show that $\int_0^l |(\partial k^2/\partial s)|\, ds \to 0$.

We consider the two cases $\varepsilon = \pm 1$ separately. We proceed with the Riemannian case.

The Riemannian Case

13.4. Lemma.

$$\max_{s\in[0,l]} \left\|T^{(n)}\right\|^2 \le \frac{2}{l_{\min}} \int \left\|T^{(n)}\right\|^2 \, ds + 2l_{\max} \int \left\|T^{(n+1)}\right\|^2 \, ds \tag{13.4}$$

Proof. Cf. [Alt, lemma 4.3.] and lemma 13.7 for the proof. □

Due to

$$\left|\frac{\partial k^2}{\partial s}\right| = 2\left|\langle T'', Y\rangle\right| \le 2\|Y\| \cdot \|T''\| \le k^2 + \|T''\|^2 ,$$

we will bound $\int |(\partial k^2/\partial s)| \, ds$ by the estimate

$$\int \left|\frac{\partial k^2}{\partial s}\right| \, ds \le \int k^2 \, ds + \int \|T''\|^2 \, ds.$$

We already know that the first integral converges to zero, which leaves us to show convergence of the second one.

13.5. Theorem. *If k^2 and the norms $\|R\|$ and $\|R'\|$ of* Riem *and* ∇Riem *are uniformly bounded, then we get*

$$\lim_{t\to\infty} \int \|T''\|^2 \, ds = 0. \tag{13.5}$$

Proof. Lemma 7.6 yields:

$$\begin{aligned}\frac{\partial}{\partial t} \|T''\|^2 =& \frac{\partial^2}{\partial s^2} \|T''\|^2 - 2\left\|T^{(3)}\right\|^2 - 4k^2 \|T''\|^2 - 4k^2 \left\langle T', T^{(3)}\right\rangle \\ &+ 12 \langle T', T''\rangle^2 + 6k^2 \|T''\|^2 + 2 \underbrace{\left\langle T'', R^{(1)}T + 2R^{(0)}T'\right\rangle}_{R_1:=} .\end{aligned}$$

Consider these relations:

$$\int \frac{\partial^2}{\partial s^2} \|T''\|^2 \, ds = [\langle T'', T^{(3)}\rangle]_0^l = 0 \text{ , since } T^{(3)} = 0 \text{ on } \{0, l\}$$

$$\|T''\|^2 \frac{1}{c}\frac{\partial c}{\partial t} = -k^2 \|T''\|^2$$

For each point (t, s):

$$\begin{aligned}\left|-4k^2 \left\langle T', T^{(3)}\right\rangle\right| \le& 4k^2 \left|k \left\langle N, T^{(3)}\right\rangle\right| \le 2(6k^2)k^4 + 2(6k^2)^{-1}k^2 \left\langle N, T^{(3)}\right\rangle^2 \\ =& 12k^6 + \frac{1}{3}\left\langle N, T^{(3)}\right\rangle^2 \le 12k^6 + \frac{1}{3}\left\|T^{(3)}\right\|^2\end{aligned}$$

$$k^6 + k^2 \langle N, T''\rangle^2 = k^2(\langle T, T''\rangle^2 + \langle N, T''\rangle^2) \le k^2 \|T''\|^2$$

$$\begin{aligned} R_1 =& \langle T'', R'(Y,T)T + R(T'',T)T + R(Y,Y)T + 2R(Y,T)T'\rangle \\ \le& \|T''\|(2k^2\|R\| + \|R'\|k + \|R\|\|T''\|) \\ \le& 2k^2\|R\|\|T''\| + \|R\|\|T''\|^2 + \|R'\|k\|T''\| \end{aligned}$$

Since $\left|2\|R\|k^2\|T''\|\right| = |-2\|R\| \langle T, T''\rangle \|T''\|| \le 2\|R\| \|T''\|^2$,

we obtain

$$R_1 \le 3\|R\| \|T''\|^2 + \|R'\|k\|T''\|.$$

Therefore the evolution equation reads:

$$\frac{\partial}{\partial t}\int \|T''\|^2 \, ds \le \int -\frac{5}{3}\left\|T^{(3)}\right\|^2 + 13k^2\|T''\|^2 + 6\|R\|\|T''\|^2 + 2\|R'\|k\|T''\|\, ds.$$

There are two cases which are to be distinguished:
(i) $\int \|T''\|^2 \, ds <$ constant $\cdot \int k^2 \, ds$ *or*
(ii) $\int \|T''\|^2 \, ds \ge$ constant $\cdot \int k^2 \, ds$.
In the first case, $\int \|T''\|^2 \, ds$ is bound by an integral which eventually converges to zero. For times t when the second case holds, we will show that $\int \|T''\|^2 \, ds$ decays at least exponentially.

Lemma. *If for some constant $c > 0$, $\int \|T''\|^2 \, ds \ge c \int k^2 \, ds$, then*

$$\int \|T''\|^2 \, ds \le \frac{1}{c}\int \left\|T^{(3)}\right\|^2 \, ds.$$

Proof. Partial integration, Cauchy-Schwarz's inequality and the Hölder inequality yield (notice: $k|_{\{0,l\}} = 0$):

$$\int \langle T'', T''\rangle \, ds = [\langle T'', T'\rangle]_0^l - \int \left\langle T', T^{(3)}\right\rangle ds \le \sqrt{\int k^2 \, ds}\sqrt{\int \|T^{(3)}\|^2 \, ds}.$$

Since we assumed that $\int \|T''\|^2 \, ds \ge c \int k^2 \, ds$, the claim follows. □

If $\int \|T''\|^2 \, ds \ge 18\|R\| \int k^2 \, ds$, we get

$$6\|R\| \int \|T''\|^2 \, ds \le \frac{1}{3}\int \left\|T^{(3)}\right\|^2 \, ds.$$

Furthermore, if $\int \|T''\|^2 \, ds \geq \max\{1, 36\,\|R'\|^2\} \int k^2 \, ds$:

$$\int 2\|R'\|k\|T''\| \, ds \leq 6\,\|R'\|^2 \int k^2 \, ds + \frac{1}{6}\int \|T''\|^2 \, ds \leq \frac{1}{3}\int \left\|T^{(3)}\right\|^2 \, ds.$$

This yields

$$\frac{\partial}{\partial t}\int \|T''\|^2 \, ds \leq \int -\left\|T^{(3)}\right\|^2 + 13k^2\,\|T''\|^2 \, ds$$

if $\int \|T''\|^2 \, ds \geq M := \max\{1, 18\|R\|, 36\,\|R'\|^2\} \int k^2 \, ds$.

Since $\int k^2 \, ds \to 0$, there exists for all $\delta > 0$ a time T_0 such that $\forall t > T_0$: $\int k^2 \, ds \leq \delta$. For these $t > T_0$, we obtain:

$$\begin{aligned} 13\int k^2\,\|T''\|^2 \, ds \leq & 13 \max_{s\in[0,l]} \|T''\|^2 \int k^2 \, ds \leq 13\delta \max_{s\in[0,l]} \|T''\|^2 \\ & \stackrel{(13.1)}{\leq} 13\delta\Big(\frac{2}{l_{\min}}\int \|T''\|^2 \, ds + 2l_{\max}\int \left\|T^{(3)}\right\|^2 \, ds\Big), \end{aligned}$$

and, since $\int \|T''\|^2 \, ds \geq \int k^2 \, ds$,

$$\leq \delta\Big(\frac{26}{l_{\min}} + 26l_{\max}\Big)\int \left\|T^{(3)}\right\|^2 \, ds.$$

Therefore if we choose δ small such that $\delta \leq (52/l_{\min} + 52l_{\max})^{-1}$:

$$\frac{\partial}{\partial t}\int \|T''\|^2 \, ds \leq \int -\frac{1}{2}\left\|T^{(3)}\right\|^2 \leq -\frac{1}{2}\int \|T''\|^2 \, ds.$$

In summary, we get the following result:
For all $t > T_0$, where T_0 large such that $\forall t > T_0$: $\int k^2 \, ds \leq 52/l_{\min} + 52l_{\max})^{-1}$, we have:

- Either $\int \|T''\|^2 \, ds < M \int k^2 \, ds$, which means that $\int \|T''\|^2 \, ds$ gets arbitrarily small, *or*
- $\int \|T''\|^2 \, ds \geq M \int k^2 \, ds$; in this event we have shown that $(\partial/\partial t)\int \|T''\|^2 \, ds \leq -1/2 \int \|T''\|^2 \, ds$, i.e. $\int \|T''\|^2 \, ds$ decays at least exponentially.

In both cases, $\lim_{t\to\infty} \int \|T''\|^2 \, ds = 0$. □

13.6. Corollary. *If k^2 and the norms $\|R\|$ and $\|R'\|$ of* Riem *and* ∇Riem *are uniformly bounded, then*

$$\lim_{t\to\infty} \max_{x\in[a,b]} k^2 = 0$$

which implies that the curve F_t evolving by $(\star)$ eventually becomes a geodesic.

Proof. We apply lemma 13.4 (or, equivalently, inequality (13.2)) to the case $T^{(n)} = T'$ and use lemma 13.3 and lemma 13.5 to show that both integrals on the right hand side of (13.4) converge to zero. $\square$

Now we have to deal with the case $\varepsilon = -1$:

The Lorentzian Case

The following lemma is a an adaptation of lemma 13.4 to the Lorentzian case. It will be used instead of inequality (13.2) and is particularly useful to show $\mathcal{C}^\infty$-convergence of the higher derivatives $T^{(n)}$, which we do not carry out here (compare [Gra89, section 7] for a reference on the induction argument which yields convergence of the higher-order derivatives).

13.7. Lemma.

$$\max \left|\left|\left|T^{(n)}\right|\right|\right|^2 \leq \int (\frac{2}{l_{\text{min}}} + 4l_{\text{max}}k^2) \left|\left|\left|T^{(n)}\right|\right|\right|^2 \, ds + 4l_{\text{max}} \int \left|\left|\left|T^{(n+1)}\right|\right|\right|^2 \, ds \tag{13.6}$$

Proof. Cf. [Alt, lemma 4.3.]. Let $\eta > 0$ be any positive number. The inequality (13.1) gives

$$\begin{aligned} \max \left|\left|\left|T^{(n)}\right|\right|\right|^2 &= \max(\left|\left|\left|T^{(n)}\right|\right|\right|^2 + \eta) - \eta \\ &\leq \frac{2}{l_{\text{min}}} \int (\left|\left|\left|T^{(n)}\right|\right|\right|^2 + \eta) \, ds + 2l_{\text{max}} \int \left(\frac{\partial}{\partial s}\sqrt{|||T^{(n)}|||^2 + \eta}\right)^2 ds \end{aligned}$$

and

$$\begin{aligned} &\int \left(\frac{\partial}{\partial s}\sqrt{|||T^{(n)}|||^2 + \eta}\right)^2 ds \\ &\quad = \int \left(\frac{2\left\langle T^{(n)}, T^{(n+1)}\right\rangle + 4\left\langle T^{(n)}, T\right\rangle (\left\langle T^{(n+1)}, T\right\rangle + \left\langle T^{(n)}, T'\right\rangle)}{2\sqrt{|||T^{(n)}|||^2 + \eta}}\right)^2 ds. \end{aligned}$$

Furthermore

$$\left\langle T^{(n)}, T^{(n+1)}\right\rangle + 2\left\langle T^{(n)}, T\right\rangle\left\langle T^{(n+1)}, T\right\rangle \le \left|\left|\left|T^{(n)}\right|\right|\right| \cdot \left|\left|\left|T^{(n+1)}\right|\right|\right|$$

and

$$4k(\left\langle T^{(n)}, T\right\rangle\left\langle T^{(n)}, N\right\rangle) \le 2k(\left\langle T^{(n)}, T\right\rangle^2 + \left\langle T^{(n)}, N\right\rangle^2) \le 2k\left|\left|\left|T^{(n)}\right|\right|\right|^2.$$

Therefore the numerator of the above right hand side can be estimated by:

$$\begin{aligned}
&\le \left(2\left|\left|\left|T^{(n)}\right|\right|\right| \cdot \left|\left|\left|T^{(n+1)}\right|\right|\right| + 2k\left|\left|\left|T^{(n)}\right|\right|\right|^2\right)^2 \\
&\le 4\left|\left|\left|T^{(n)}\right|\right|\right|^2 \left(\left|\left|\left|T^{(n+1)}\right|\right|\right| + k\left|\left|\left|T^{(n)}\right|\right|\right|\right)^2 \\
&\le 4\left|\left|\left|T^{(n)}\right|\right|\right|^2 \left((1+1)\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 + (1+1)k^2\left|\left|\left|T^{(n+1)}\right|\right|\right|^2\right)^2.
\end{aligned}$$

Consequently, we obtain

$$\int\left(\frac{\partial}{\partial s}\sqrt{|||T^{(n)}|||^2+\eta}\right)^2 ds \le \int 2\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds + \int 2k^2\left|\left|\left|T^{(n)}\right|\right|\right|^2 ds.$$

Thus we have

$$\begin{aligned}
\max\left|\left|\left|T^{(n)}\right|\right|\right|^2 \le& \frac{2}{l_{\min}}\int\left|\left|\left|T^{(n)}\right|\right|\right|^2 ds + 2\frac{l_{\max}}{l_{\min}}\eta \\
&+ 2l_{\max}\left(\int 2\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds + \int 2k^2\left|\left|\left|T^{(n)}\right|\right|\right|^2 ds\right) \\
=& \int\left(\frac{2}{l_{\min}} + 4l_{\max}k^2\right)\left|\left|\left|T^{(n)}\right|\right|\right|^2 ds \\
&+ 2\frac{l_{\max}}{l_{\min}}\eta + 4l_{\max}\int\left|\left|\left|T^{(n+1)}\right|\right|\right|^2 ds.
\end{aligned}$$

Since this inequality holds for any η, the result follows. □

Since

$$\int\left|\frac{\partial k^2}{\partial s}\right| ds \le \int k^2\, ds + \int |||T''|||^2\, ds,$$

we have to consider $\int |||T''|||^2\, ds$ now.

13.8. Lemma. *If* $\int |||T''|||^2\, ds \ge (c + 2k_{\max}^2)\int k^2\, ds$ *for some* $c > 0$*, then* $\int \|T''\|^2\, ds \le 1/c\int\left|\left|\left|T^{(3)}\right|\right|\right|^2 ds$*, which implies that*

$$\begin{aligned}
\int |||T''|||^2\, ds \le& \frac{1}{c}\int\left|\left|\left|T^{(3)}\right|\right|\right|^2 ds + 2\int k^4\, ds \\
\le& \frac{1}{c}\int\left|\left|\left|T^{(3)}\right|\right|\right|^2 ds + 2k_{\max}^2\int k^2\, ds.
\end{aligned}$$

Proof. We have that $|||T''|||^2 = \|T''\|^2 + 2k^4$. The assumptions yield

$$0 < \int \|T''\|^2 \; ds \leq \frac{1}{c} \int \left|\left|\left|T^{(3)}\right|\right|\right|^2 \; ds,$$

which is being proven analogously to the lemma of theorem 13.5. □

We state the necessary conditions for this theorem explicitly:

13.9. Theorem. *If k^2, the norms $|||\mathrm{Riem}|||_{\mathrm{op}}$ and $|||\nabla\mathrm{Riem}|||_{\mathrm{op}}$, and the angle $\langle e_0, -T\rangle$ are uniformly bounded, then*

$$\lim_{t\to\infty} \int |||T''|||^2 \; ds = 0. \tag{13.7}$$

Proof. We use the calculations of theorem 9.2:

$$\begin{aligned}
\frac{\partial}{\partial t} |||T''|||^2 \leq & \frac{\partial^2}{\partial s^2} |||T''|||^2 - 2\left\|\Pi^\perp\left(T^{(3)}\right)\right\|^2 - 2\left\langle T^{(3)}, T\right\rangle^2 \\
& + 8(\frac{1}{8}\left\langle T^{(3)}, T\right\rangle^2 + 2\left\langle T'', T'\right\rangle^2) \\
& - 16\left\langle T'', T'\right\rangle^2 + 4(\frac{1}{4}\left\|\Pi^\perp\left(T^{(3)}\right)\right\|^2 + k^6) \\
& - 10k^2\|T''\|^2 - 16k^6 \\
& + \underbrace{2\left\langle T'' + 2\left\langle T, T''\right\rangle T, R^{(1)}T + 2R^{(0)}T'\right\rangle}_{2R_1:=} \\
\leq & \frac{\partial^2}{\partial s^2} |||T''|||^2 - \left|\left|\left|T^{(3)}\right|\right|\right|^2 - 12k^6 - 10k^2\|T''\|^2 + 2R_1
\end{aligned}$$

We let $R := |||\mathrm{Riem}|||_{\mathrm{op}} \langle U, -T\rangle^4$ and $R' := |||\nabla\mathrm{Riem}|||_{\mathrm{op}} \langle U, -T\rangle^4$. As in the proof of 13.5, we obtain the estimate

$$R_1 \leq 3R\,|||T''|||^2 + R'k\,|||T''|||\,.$$

Also:

$$\int \frac{\partial^2}{\partial s^2} |||T''|||^2 \; ds = \left[2\left\langle T'', T^{(3)}\right\rangle + 4\left\langle T, T''\right\rangle \left\{\left\langle T', T''\right\rangle + \left\langle T, T^{(3)}\right\rangle\right\}\right]_0^l = 0$$

and

$$\begin{aligned}
& -2k^6 - k^2\|T''\|^2 = -k^2(\|T''\|^2 + 2\left\langle T, T''\right\rangle^2) = -k^2\,|||T''|||^2 \\
\Rightarrow \quad & -12k^6 - 10k^2\|T''\|^2 = -6k^2\,|||T''|||^2 - 4k^2\|T''\|^2 \leq -2k^2\,|||T''|||^2\,.
\end{aligned}$$

Noting again that $1/c\,|||T''|||^2\,\partial c/\partial t = k^2\,|||T''|||^2$, the evolution equation gets

$$\frac{\partial}{\partial t}\int |||T''|||^2\ ds \leq \int -\left|||T^{(3)}|||\right|^2 - k^2\,|||T''|||^2 + 6R\,|||T''|||^2 + 2R'k\,|||T''|||\ ds.$$

We apply the previous lemma now:
If $\int |||T''|||^2\ ds \geq (36R + 2k_{\max}^2)\int k^2\,ds$, then

$$6R\int |||T''|||^2\ ds \leq \frac{1}{6}\int \left|||T^{(3)}|||\right|^2\ ds + 12R\int k^4\,ds.$$

Furthermore, if $\int |||T''|||^2\ ds \geq \max\{2 + 2k_{\max}^2, 36R'^2\}\int k^2\,ds$:

$$\begin{aligned}\int 2R'k\,|||T''|||\ ds \leq& 6R'^2\int k^2\,ds + \frac{1}{6}\int |||T''|||^2\ ds\\ \leq& \frac{1}{3}\int |||T''|||^2\ ds \leq \frac{1}{6}\int \left|||T^{(3)}|||\right|^2\ ds + \frac{2}{3}\int k^4\,ds.\end{aligned}$$

This implies that if $\int |||T''|||^2\ ds \geq \max\{2{+}2k_{\max}^2, 36R{+}2k_{\max}^2, 36R'^2\}\int k^2\,ds$,

$$\frac{\partial}{\partial t}\int |||T''|||^2\ ds \leq \int -\frac{2}{3}\int \left|||T^{(3)}|||\right|^2 - k^2\,|||T''|||^2 + (12R + \frac{2}{3})k^4\,ds.$$

We also have:

$$\int |||T''|||^2\ ds \leq \frac{1}{2}\int \left|||T^{(3)}|||\right|^2\ ds + 2\int k^4\,ds$$

$$\Longleftrightarrow$$

$$-\frac{2}{3}\int \left|||T^{(3)}|||\right|^2\ ds \leq \frac{8}{3}\int k^4\,ds - \frac{4}{3}\int |||T''|||^2\ ds$$

If $\int |||T''|||^2\ ds \geq \max\{20k_{\max}^2, 72Rk_{\max}^2\}\int k^2\,ds$:

$$\begin{aligned}12R\int k^4\,ds &\leq 12Rk_{\max}^2\int k^2\,ds \leq \frac{1}{6}\int |||T''|||^2\ ds\\ \frac{10}{3}\int k^4\,ds &\leq \frac{10}{3}k_{\max}^2\int k^2\,ds \leq \frac{1}{6}\int |||T''|||^2\ ds\end{aligned}$$

Summarizing our calculations, we obtain:
If $\int |||T''|||^2\ ds \geq \underbrace{\max\{2 + 2k_{\max}^2, 36R + 2k_{\max}^2, 36R'^2, 20k_{\max}^2, 72Rk_{\max}^2\}}_{M:=}\int k^2\,ds$,

we have:

$$\frac{\partial}{\partial t}\int |||T''|||^2\ ds \leq -(\int |||T''|||^2 + k^2\,|||T''|||^2\ ds) \leq -\int |||T''|||^2\ ds.$$

Therefore either $\int |||T''|||^2\,ds < M \int k^2\,ds$ or $\int |||T''|||^2\,ds \geq M \int k^2\,ds$, which implies that the evolution behaves like $\partial/\partial t \int |||T''|||^2\,ds \leq -\int |||T''|||^2\,ds$; this allows us to conclude that $\int |||T''|||^2\,ds$ decreases at least exponentially. □

13.10. Corollary. *If k^2, the norms $|||\mathrm{Riem}|||_{\mathrm{op}}$ and $|||\nabla\mathrm{Riem}|||_{\mathrm{op}}$, and the angle $\langle e_0, -T\rangle$ are uniformly bounded, then*

$$\lim_{t\to\infty} \max_{x\in[a,b]} k^2 = 0,$$

which implies that the curve F_t evolving by $(\star\star)$ eventually becomes a geodesic.

Proof. We apply lemma 13.7 (or, equivalently, inequality (13.2)) to the case $T^{(n)} = T'$ and use lemma 13.3 and lemma 13.9 to show that both integrals on the right hand side of (13.6) converge to zero. □

We summarize the results in the following theorems.

Conclusion

For all these theorems we impose the natural condition that the Riemannian tensor and its higher covariant derivatives are bounded on the tangent bundle $T\mathcal{M}$, and that there is a uniform bound for the Riemannian tensor and its first covariant derivative on the whole $T\mathcal{M}$.

Note. We have only established a bound for the derivatives at the boundary $\{a, b\}$ for the cases of closed curves or flat manifolds (cf. remark 10.9). Therefore a part of our argumentation is being left open, but we expect that this could be proven to hold true for the most general setting.

13.11. Theorem (Parallel Unit Vector Fields). *If the Riemannian manifold $\mathcal{M}^{n+1}$ or the Lorentzian manifold $\mathcal{M}^{n,1}$ admit a parallel unit vector field and if the sectional curvature $K(A,B) \geq 0$ for all timelike vector fields A and spacelike vector fields B (in the Riemannian case: for all vector fields A and B), then the curve length extremizing flow $(\star)$ resp. $(\star\star)$ has a solution for all times $t \geq 0$ and this solution converges to a geodesic for $t \to \infty$.*

In particular, this holds for all Euclidian and Minkowskian manifolds.

Proof. In theorem 6.7 we have shown that the curvature remains uniformly bounded, and in 6.4 we have established a uniform bound on the angle $\langle U, \varepsilon T\rangle$. Therefore the assumptions of 10.11 are satisfied which yield long-time existence and by 13.6 resp. 13.10 we conclude that these solutions will become geodesics. □

13.12. Theorem (Warped Product Spaces). *If the Riemannian manifold $\mathcal{M}^{n+1}$ or the Lorentzian manifold $\mathcal{M}^{n,1}$ are warped product spaces with a one-dimensional base $\langle\langle U\rangle\rangle_{\mathbb{R}}$, and if the warping function $f = f(u)$ satisfies $U(U(f)/f) = (f'/f)' \leq 0$ [which is a natural assumption for most cosmological theories, cf. remark 6.14], and if, in the Riemannian case, the curvature k^2 is uniformly bounded [in the Lorentzian case, this follows from the evolution equation of k^2, cf. theorem 12.5], then the curve length extremizing flow $(\star)$ resp. $(\star\star)$ has a solution for all times $t \geq 0$ and this solution converges to a geodesic for $t \to \infty$.*

13.13. Theorem. *If we have a Riemannian manifold $\mathcal{M}^{n+1}$ and if we can show that the curvature flow $(\star)$ yields a uniform bound on the curvature k^2 for all times $t \geq 0$, then we have a solution to this flow for all times $t \geq 0$ and this solution converges to a geodesic for $t \to \infty$.*

13.14. Theorem. *If we have a Lorentzian manifold $\mathcal{M}^{n,1}$ and unit vector field $U \in T\mathcal{M}$ such that we can establish a uniform upper bound on the angle $|\langle U, T\rangle|$ for all times $t \geq 0$, and if furthermore the sectional curvature $K(A, B) \geq 0$ for all timelike vector fields A and spacelike vector fields B, then, for all times $t \geq 0$, there is a solution to the curve length maximizing flow $(\star\star)$ and this solution converges to a geodesic for $t \to \infty$.*

Appendix

Proposition. *Let v and w be two spacelike vectors in some tangent space $T_p\mathcal{M}^{n,1}$, $p \in \mathcal{M}^{n,1}$. Consider the decomposition of v with respect to w:*

$$v = aw + v^{\perp} + v^{\top},$$

such that $a \in \mathbb{R}$, $v^{\top}$ timelike, $v^{\perp}$ spacelike and $\left\langle w, v^{\top} \right\rangle = \left\langle w, v^{\perp} \right\rangle = \left\langle v^{\top}, v^{\perp} \right\rangle = 0$, which indeed is possible to achieve [there exists some $v^{\perp} \in V^{\perp} \leq T\mathcal{M}^{n,1}$ such that $T\mathcal{M}^{n,1} = \left\langle \left\langle v^{\top}, w \right\rangle \right\rangle \oplus V^{\perp}$]. Then we obtain the following results:

(i) If v is purely spacelike, *which we define by requiring that the timelike part of v vanishes [$v^{\top} = 0$], then we get:*

$$\langle w, v \rangle^2 \leq \|w\|^2 \|v\|^2$$

with equality iff $v = aw$.

(ii) If the entire spacelike part of v is in $\langle \langle w \rangle \rangle$, i.e. if $v^{\perp} = 0$, then we get:

$$\langle w, v \rangle^2 \geq \|w\|^2 \|v\|^2$$

with equality iff $v = aw$.

In the "mixed case" $v^{\perp} \neq 0 \neq v^{\top}$ we have no knowledge on which type of inequality holds.

Proof.

$$\langle v, v \rangle = a^2 \langle w, w \rangle + \left\| v^{\top} \right\|^2 + \left\| v^{\perp} \right\|^2 \geq 0$$

Therefore

$$\langle w, v \rangle^2 = a^2 \langle w, w \rangle^2 = \langle w, w \rangle \left(\langle v, v \rangle - \left\| v^{\top} \right\|^2 - \left\| v^{\perp} \right\|^2 \right)$$

- if $v^{\top} = 0 : \langle w, v \rangle^2 = \|w\|^2 \|v\|^2 - \|w\|^2 \left\| v^{\perp} \right\|^2 \leq \|w\|^2 \|v\|^2$
- if $v^{\perp} = 0 : \langle w, v \rangle^2 = \|w\|^2 \|v\|^2 - \|w\|^2 \left\| v^{\top} \right\|^2 \geq \|w\|^2 \|v\|^2$

with equality iff $v = aw$. $\square$

Corollary. *Lemma 2.1 holds for Lorentzian manifolds as well if such variation vector fields are considered only, which are perpendicular to the tangent vector T.*

Proof. We have to show that the gradient flow in length is maximal for a variation vector field $V \parallel Y$ [we therefore only have to consider Y if $k \neq 0$] . We choose an orthonormal frame $\{e_i\}_{0 \leq i \leq n}$ such that $e_0 = T$ is its timelike vector, $e_1 \parallel Y$ and $v^\perp \in \langle\langle e_2, \ldots, e_n \rangle\rangle$. Since we only considered variations perpendicular to the tangent vector T, in terms of the notation of the preceding proposition, $V^\top = 0$ [let $v = V$ and $w = Y$]. Therefore the first inequality of the previous proposition applies, which yields the result. $\square$

List of Main Theorems

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Erklärung nach §17 (5) der Prüfungsordnung der Universität Tübingen für den Diplomstudiengang Mathematik in der Fassung vom 29. August 1994

Hiermit erkläre ich, die vorliegende Diplomarbeit mit dem Titel „*The Curve Length Extremizing Flow in Riemannian and Lorentzian Manifolds*“ selbständig unter der Betreuung von Prof. Huisken angefertigt und keine nicht-trivialen Quellen und Hilfsmittel als angegeben benutzt zu haben.

Oliver Stoll   Tübingen, im Juli 1998