Mean Curvature Flow with a Neumann Boundary Condition on Smooth Hypersurfaces

Axel Stahl,
Mathematisches Institut der Universität Tübingen,
Auf der Morgenstelle 10, 72 076 Tübingen, Germany
e-mail: axel@moebius.mathematik.uni-tuebingen.de

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Abstract

In this work we study the behaviour of compact, smooth, immersed manifolds with boundary, which move under the mean curvature flow in Euclidian space. We thereby prescribe the Neumann boundary condition in a purely geometric manner by requiring a vertical contact angle between the unit normal fields of the immersions and a given, smooth hypersurface Σ. We prove a very general existence result and study the formation of the singularity which occurs if Σ is a convex, umbilic hypersurface and the initial hypersurface is strictly convex.

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1 Introduction and Main Results

The behaviour of hypersurfaces which move under various geometric flows has been studied intensively during the last years. One major topic within this research is devoted to the evolution of compact, convex hypersurfaces *without boundary*; here the convergence to single points in finite time has been shown for a wide class of geometric flows [13], [14], [24]. The natural question concerning the "shape" of the developing singularity always has been of special interest. Hereby a classification with respect to the growth rate of the second fundamental form A as t approaches the singular time T_C is used: If there is a constant K such that $|A|^2 \leq K(T_C - t)\ \forall t$, the singularity is said to be *of type I*, otherwise it is called a *type-II*-singularity. *Huisken* proved [13] that convex hypersurfaces moving by mean curvature end up in a type-I-singularity, whereas it was shown by *Angenent* [3] that the curve

shortening flow for plane curves developing a cusp will produce a type-II-singularity. Rescaling the hypersurfaces appropriately often yields a better description of what actually happens as $t \to T_C$: For the mean curvature flow *Huisken* [13] established the convergence of the rescaled hypersurfaces to a "round sphere" in the C^∞-topology. Later [15] he obtained a complete classification of the possible type-I limit surfaces with nonnegative mean curvature. Further results concerning this topic are due to *Andrews* [2], *Hamilton* [8], *Stone* [22], and others.

Another point of interest is the behaviour of compact surfaces *with boundary* which move under mean-curvature-like flows. Here in general the hypersurface is described as graph of a scalar function $w : \Omega \times [0,T] \longrightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and boundary conditions are prescribed on $\partial\Omega \times [0,T]$; thus geometrically the hypersurface moves in an orthogonal cylinder. Both Dirichlet- as well as Neumann boundary conditions have been studied; results concerning the Dirichlet problem are due to *Huisken* [16], *Stone* [21], *Oliker-Ural'ceva* [18], and others. In the Neumann case a solution will typically converge to a hypersurface which moves by translation; the speed of the translation is determined essentially by the angle at which the hypersurface meets the boundary. This result was established in the two-dimensional case by *Altschuler, Wu* [4]; in the special case of an orthogonal contact angle (but in arbitrary dimensions), *Huisken* [16] proved the existence of a unique solution which converges to a minimal surface as $t \longrightarrow \infty$. Further results concerning a wider class of evolution equations are due to *Guan* [7].

Thus the evolution of compact manifolds has been studied extensively in the boundaryless case and for boundary values prescribed on orthogonal cylinders. The scope of this work is to relate these two fields by studying the boundary value problem for given Neumann data on *arbitrary smooth hypersurfaces* in $\mathbb{R}^{n+1}, n \geq 1$: Establishing a sharp local gradient bound on the hypersurfaces, we prove existence and regularity results for arbitrary smooth support and initial hypersurfaces. Furthermore, in a special case, we obtain a complete classification of possible singularities.

The results presented here are essentially the results of the author's Dissertation [23], the research upon which has been done during the years 1992-1994. The author is especially grateful to G. Huisken for his excellent tutorship during this time.

1.1 The Problem

Throughout this work, $\boldsymbol{\Sigma}$ denotes a smooth hypersurface in $\mathbb{R}^{n+1}$, $n \geq 1$, and M^n a compact, smooth, orientable n-manifold with compact, smooth boundary ∂M^n. Furthermore, $\boldsymbol{F_0} : M^n \longrightarrow \mathbb{R}^{n+1}$ is a smooth immersion with $M_0 := \boldsymbol{F_0}(M^n)$ and

$$
\begin{aligned}
\partial M_0 \equiv \boldsymbol{F_0}(\partial M^n) &= M_0 \cap \boldsymbol{\Sigma}, \\
\langle \boldsymbol{\nu_0}, \boldsymbol{\mu} \circ \boldsymbol{F_0} \rangle(x) &= 0 \quad \forall x \in \partial M^n,
\end{aligned} \tag{1.1}
$$

where $\boldsymbol{\nu_0}$ and $\boldsymbol{\mu}$ are unit normal fields to M_0 and $\boldsymbol{\Sigma}$ respectively. We then define

Definition 1.1 *(Mean curvature flow with Neumann boundary condition) Let $\boldsymbol{F}_t : M^n \longrightarrow \mathbb{R}^{n+1}, t \in [0, T_C)$ be a family of smooth immersions. $\boldsymbol{F}_t$ is said to move under the mean curvature flow with homogenious Neumann boundary condition, if*

$$
\begin{cases}
\frac{d}{dt}\boldsymbol{F}(x,t) = \boldsymbol{H}(x,t) & \forall (x,t) \in M^n \times [0, T_C), \\
\boldsymbol{F}(.,0) = \boldsymbol{F_0}, & \\
\boldsymbol{F}(\partial M^n, t) \subset \boldsymbol{\Sigma} & \forall (x,t) \in \partial M^n \times [0, T_C), \\
\langle \boldsymbol{\nu}, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle(x,t) = 0 & \forall (x,t) \in \partial M^n \times [0, T_C),
\end{cases} \tag{1.2}
$$

where $\boldsymbol{H}$ denotes the mean curvature vector of the immersions.

Remark 1.2 *In the sequel we assume that either $\boldsymbol{\Sigma}$ has no boundary or the hypersurfaces M_t "keep away" from $\partial\boldsymbol{\Sigma}$, i.e.*

$$
\exists \varepsilon > 0 \ \textit{ such that } \ dist_{\Sigma}(\boldsymbol{F}(x,t), \partial\boldsymbol{\Sigma}) \geq \varepsilon \quad \forall (x,t) \in \partial M^n \times [0, T_C),
$$

This can be guaranteed à priori by barrier constructions, cf. section 4. Furthermore, we can often show that the solutions of equation 1.2 are contained in a compact domain $\mathcal{K} \subset \mathbb{R}^{n+1}$. In many cases, this implies that $\mathcal{K} \cap \boldsymbol{\Sigma}$ is also compact, hence the second fundamental form ${}^{\Sigma}A$ of $\mathcal{K} \cap \boldsymbol{\Sigma}$ will be uniformly bounded. For the sake of technical simplicity we therefore require $|{}^{\Sigma}A|$ to be uniformly bounded globally.

Given a solution $\boldsymbol{F}$ of equation 1.2, we can locally define an exterior normal field $\boldsymbol{\mu}$ of $\boldsymbol{\Sigma}$ with respect to $\boldsymbol{F}$: Let $c : (-s, 0] \longrightarrow M_t$ be a smooth curve with $c(0) = \boldsymbol{F}(x_0, t) \in \partial M_t$, and choose $\boldsymbol{\mu}$ such that $\langle c'(0), \boldsymbol{\mu} \circ \boldsymbol{F}(x_0, t) \rangle \geq 0$. Thus, if $\boldsymbol{\Sigma}$ is the boundary of a domain $\mathcal{G} \subset \mathbb{R}^{n+1}$ and $M_t \subset \bar{\mathcal{G}}$ (i.e. the surface moves *in* the domain), $\boldsymbol{\mu}$ and the "standard" exterior unit normal field of $\mathcal{G}$ will be parallel; if $M_t \subset \mathbb{R}^{n+1} \backslash \mathcal{G}$, they will be antiparallel. In the sequel, by "exterior" we will always refer to a given solution as described above.

1.2 Main Results

The work is essentially divided into two parts: In sections 2 through 6 we provide a general existence result:

Theorem 1.3 *Let $\boldsymbol{\Sigma}$ be as in Remark 1.2, and let M_0 fulfil equations 1.1. Then there exists a unique solution to equation 1.2 on a maximal time interval $[0, T_C)$. This solution is smooth for $t > 0$ and in the class $C^{2+\alpha,1+\alpha/2}$ (with arbitrary $0 < \alpha < 1$) for $t \geq 0$. Moreover, if $T_C < \infty$, then*

$$sup\{|A|^2(x,t) : x \in M^n\} \to \infty$$

as $t \to T_C$, where $A(.,t)$ denotes the second fundamental form of the hypersurfaces M_t.

The crucial step proving this result is to establish a precise local gradient estimate under the assumption of bounded curvature; hereby we generalize a method of *Ecker* and *Huisken* [5]. Furthermore, in this part we discuss maximum principles and the "C^0-behaviour" of hypersurfaces moving under the mean curvature flow equation 1.2.

The remainder of the work is devoted to the study of the singularity which develops as $t \to T_C$, if $\boldsymbol{\Sigma}$ and M_0 are (strictly) convex hypersurfaces. Under the further restriction that $\boldsymbol{\Sigma}$ is *umbilic*, i.e. the boundary of a ball B_R^{n+1} or a hyperplane, we can prove that convexity of the M_t as well as a pinching condition for the eigenvalues of the second fundamental form A are preserved under the mean curvature flow equation 1.2. Modifying an idea of *Tso* [24], this yields an upper bound on the growth rate of the second fundamental form of the M_t. Using an appropriate rescaling technique and applying results of *Hamilton* [11] (concerning hypersurfaces with pinched second fundamental form) and *Huisken* [13] (on the behaviour of compact hypersurfaces without boundary), we finally arrive at the following

Theorem 1.4 *Let $n \geq 2$, $\mathcal{G} = B_R^{n+1}$ ($R = \infty$ allowed), $\boldsymbol{\Sigma} = \partial\mathcal{G}$, and let $M_0 \subset \mathcal{G}$ be a strictly convex, imbedded hypersurface satisfying equations 1.1. Then the hypersurfaces M_t of the solution to equation 1.2 shrink to a single point on $\boldsymbol{\Sigma}$ as $t \to T_C$, ending up in a type-I-singularity. Moreover, there is a sequence $t_k \to T_C$ such that, rescaling the M_{t_k} appropriately, the rescaled hypersurfaces converge in the C^∞-topology to a hemisphere with boundary on a hyperplane.*

We emphasize that the requirement of $\boldsymbol{\Sigma}$ being umbilic is needed exclusively to establish the conservation of the convexity and pinching condition. Consequently, the Theorem holds for *arbitrary convex support hypersurfaces* $\boldsymbol{\Sigma}$ whenever conservation of convexity and pinching can be guaranteed for the solutions of equation 1.2. Furthermore, all techniques (except for the rescaling procedure, where Hamilton's compactness result requires $n \geq 2$) are valid for plane curves. Since - as we shall see - the conservation of a positive lower bound on H already follows from the convexity of $\boldsymbol{\Sigma}$, we obtain

Proposition 1.5 *Let $\boldsymbol{\Sigma}$ be the smooth boundary of a convex domain $\mathcal{G} \subset \mathbb{R}^2$, and let $\boldsymbol{\Sigma}$ have uniformly bounded curvature. Furthermore, let $M_0 \subset \bar{\mathcal{G}}$ be a convex, imbedded curve satisfying equations 1.1. Then the curves of the solution to equation 1.2 shrink to a single point on $\boldsymbol{\Sigma}$ as $t \to T_C$.*

1.3 Basic Notation and Conventions

In the sequel, $g_{ij} = g_{ij}(x,t)$ will denote the induced metric on M^n with connection Γ_{ij}^k, $A = (h_{ij})$ the second fundamental form and $H = g^{ij}h_{ij}$ the mean curvature of M_t.

To distinguish quantities on M^n and M_t from those on $\boldsymbol{\Sigma}$, we use indices ${}^{\Sigma}_{,\Sigma}$: Thus, g_{ij}^{Σ} is the induced metric on $\boldsymbol{\Sigma}$, ${}^{\Sigma}A$ the second fundamental form and so on. To distinguish summations on M^n, M_t and $\boldsymbol{\Sigma}$, we use the indices $i, j, k...$ on M^n, M_t and $\hat{\imath}, \hat{\jmath}, \hat{k}...$ on $\boldsymbol{\Sigma}$.

We will use the *Einstein* convention for summation of a pair of co- and contravariant indices; here the range of greek indices is from 1 to $n+1$, whereas capital arabic indices range from 1 to $n-1$. Thus greek indices are used in $\mathbb{R}^{n+1}$, capital arabic indices on the boundaries ∂M^n and ∂M_t; the usual small arabics are reserved for calculations on $M^n, M_t, \boldsymbol{\Sigma}$.

$D_\alpha = \frac{\partial}{\partial x^\alpha}$ and ${}^0\nabla_\alpha$ will denote the coordinate and covariant derivative in $\mathbb{R}^{n+1}$; $\langle .,. \rangle$ is the scalar product in $\mathbb{R}^{n+1}$, and $\|.\|$ is the norm of vectors (in $\mathbb{R}^{n+1}$ as well as on M^n). Vector fields in $\mathbb{R}^{n+1}$ and on TM^n will be denoted by $\boldsymbol{\mu}, \boldsymbol{\nu}, ...$ and $\mu, \nu, ...$, respectively.

2 Short Time Existence

The first step in order to obtain an existence result for equation 1.2 is to prove that there exists a unique solution at least for a short time. This can be done by transforming the given problem into an equivalent initial-boundary-value problem for a *scalar* function and using standard results of the parabolic theory. *Ecker* and *Huisken* have applied this method to the case of manifolds without boundary [5]; by using appropriate coordinates which account for the fact that the boundary can be curved, it is possible to adapt this technique to the given problem 1.2. As the calculations are tedious but in general straightforward, we restrict ourselves to presenting the major ideas and leave the details to the reader.

We begin by defining *generalized Gaussian coordinates*: Let $M_0 = \boldsymbol{F_0}(M^n)$ be a smooth immersion of M^n, and let $x = (x^1, ..., x^n)$ be coordinates on M^n. In a tubular neighbourhood $\mathcal{U} \subset \mathbb{R}^{n+1}$ of M_0 we can define a smooth vector field ξ with the following properties:

$$\boldsymbol{\xi}|_{M_0} = \boldsymbol{\nu}_0 \ , \quad \boldsymbol{\xi}|_{\Sigma \cap \mathcal{U}} \in T\boldsymbol{\Sigma} \ , \quad \|\boldsymbol{\xi}\| \equiv 1 \ .$$

The flux lines $\boldsymbol{\Phi} = \boldsymbol{\Phi}(x, .)$ to the corresponding flow are perpendicular to M_0 and tangential to $\boldsymbol{\Sigma}$; hence for any given point $\boldsymbol{p} = \boldsymbol{\Phi}(x,s) \in \mathcal{U}$ we can define $x^{n+1}(\boldsymbol{p}) :=$ length of the flux line through $\boldsymbol{p}$ between $\boldsymbol{p}$ and the intersection point $\boldsymbol{p}_0 = \boldsymbol{\Phi}(x,0)$ on M_0, and $(x^1, ..., x^n)(\boldsymbol{p}) :=$ coordinates of $\boldsymbol{p}_0$ on M^n. These coordinates will be referred to as *generalized Gaussian*

coordinates - note that with respect to these coordinates the part $\boldsymbol{\Sigma} \cap \mathcal{U}$ appears as part of an "orthogonal cylinder".

Now we define a scalar initial-boundary-value problem:

$$(2.1) \qquad \begin{cases} \frac{\partial w}{\partial t}(x,t) &= (-v\,\tilde{H})(x,t) & \forall (x,t) \in M^n \times [0,\delta]\ , \\ w(.,0) &= 0\ , & \\ (\mu^k\, D_k w) &= \mu^{n+1} & \forall (x,t) \in \partial M^n \times [0,\delta]\ . \end{cases}$$

Here $v(x) := \langle \tilde{\boldsymbol{\nu}}, \boldsymbol{\xi} \rangle^{-1}(x, w(x))$, where $\tilde{\boldsymbol{\nu}}$ is the upper unit normal field to $\tilde{M}_t := \tilde{\boldsymbol{F}}(M^n, t) := \mathrm{graph}(w(.,t))$, and $\tilde{H}$ denotes the mean curvature of $\tilde{M}_t$. It can easily be seen that equation 2.1 is a *parabolic* equation with a *transversal* Neumann boundary condition. Hence, applying Hamilton's results concerning the existence of solutions to *linear* BVPs on manifolds ([8], Chapters III and IV), we obtain a $C^\infty(M^n \times (0,\delta]) \cap C^0(M^n \times [0,\delta])$-solution of the linearized problem (Note that, as $\langle \boldsymbol{\nu}_0, \boldsymbol{\mu} \circ \boldsymbol{F_0} \rangle(x) = 0 \quad \forall x \in \partial M^n$, the required compatibility condition is fulfilled). By the standard regularity theory for linear parabolic equations (e.g. [17], chapter IV) it follows that this solution actually is of the class $C^{2+\alpha, 1+\alpha/2}$ for any $\alpha \in (0,1)$. Now the existence of a unique solution in $C^{2+\alpha,1+\alpha/2}(M^n \times [0,\delta])$ as usual is a consequence of the implicit function Theorem.

Next we show that equations 1.2 and 2.1 are equivalent in the following sense: To any solution M_t of 1.2 there exists a unique solution $\tilde{M}_t$ to 2.1 and vice versa such that M_t and $\tilde{M}_t$ are equal as subsets of $\mathbb{R}^{n+1}$. We follow the ideas of Ecker and Huisken and define $\varphi : M^n \times [0,\delta] \longrightarrow M^n$ by the following system of ordinary differential equations:

$$(2.2) \qquad \begin{cases} \frac{d}{dt}\varphi(x,t) &= -(D_x \tilde{\boldsymbol{F}})^{-1} \cdot (\frac{\partial}{\partial t}\tilde{\boldsymbol{F}})^T\,(\varphi(x,t),t)\ , \\ \varphi(x,0) &= x\ . \end{cases}$$

Here, $\boldsymbol{v}^T := \boldsymbol{v} - \langle \boldsymbol{v}, \tilde{\boldsymbol{\nu}} \rangle \tilde{\boldsymbol{\nu}}$ denotes the tangential component of a vector. In view of the work of Ecker and Huisken [5], the only thing which remains to check is that - at least for a short time - φ is a diffeomorphism on M^n. To this end, it suffices to prove that φ is tangential to ∂M^n at the boundary, i.e.

$$x \in \partial M^n \Rightarrow \varphi(x,t) \in \partial M^n \quad \forall t \in [0,\delta].$$

But this is a straightforward consequence of our coordinate choice: At the boundary, we have $\langle \boldsymbol{\xi}, \boldsymbol{\mu} \rangle = 0$ which implies $\langle \frac{\partial}{\partial t}\tilde{\boldsymbol{F}}, \boldsymbol{\mu} \rangle = 0$. The scalar boundary condition yields $\langle \tilde{\boldsymbol{\nu}}, \boldsymbol{\mu} \rangle = 0$; combining these two relations and using equation 2.2 we obtain $\langle (D_x \tilde{\boldsymbol{F}}) \cdot (\frac{d\varphi}{dt}), \boldsymbol{\mu} \rangle = 0$. This means that $(D_x \tilde{\boldsymbol{F}}) \cdot (\frac{d\varphi}{dt})$ is tangential to $\boldsymbol{\Sigma}$; thus $\frac{d\varphi}{dt}$ must be tangential to ∂M^n, for transversality is conserved by the isomorphism $(D_x \tilde{\boldsymbol{F}})$. Recalling the existence of a unique solution to equation 2.1 for a short time, we have thus proved:

Theorem 2.1 *(Short time existence) For any $\alpha \in (0,1)$, equation 1.2 admits a unique solution*

$$\boldsymbol{F} \in C^{\infty}(M^n \times (0,\delta]) \cap C^{2+\alpha,1+\alpha/2}(M^n \times [0,\delta]),$$

where $\delta > 0$ is a small constant. □

3 Maximum Principles

Here we provide maximum principles for scalar functions and symmetric tensors on manifolds which obey a parabolic equation with a Neumann boundary condition. The results of this section are crucial for the following and thus, despite the fact that the main ideas behind the proofs are well known, we will at least highlight the most important points.

First, let $\tilde{g}$ be a time independent metric on M^n with connection $\tilde{\nabla}$. We then consider the following linear parabolic differential operator for scalar functions:

$$L(\varphi) := \tfrac{\partial}{\partial t}\varphi - a^{ij}(x,t) \cdot \tilde{\nabla}_i \tilde{\nabla}_j \varphi - b^i(x,t) \cdot \tilde{\nabla}_i \varphi - c(x,t) \cdot \varphi \ ,$$

with $a^{ij}, b^i, c \in L^{\infty}(M^n \times [0,T])$. Furthermore, let $f : M^n \times [0,T] \longrightarrow \mathbb{R}$ be a function with the following properties:

$$\begin{aligned} f(0,x,t) &\geq 0 \ , \\ |f(s,x,t) - f(0,x,t)| &\leq K(x,t) \cdot |s| \quad \text{for} \quad |s| \leq \rho \ ; \end{aligned}$$

here $\rho > 0$ is a (small) positive constant and $K \in L^{\infty}(M^n \times [0,T])$.

Finally, define a vectorfield v on $\partial M^n \times [0,T]$ with $v \in TM^n$ and $\tilde{g}(v,\mu) > 0$, where μ denotes the outward pointing unit normal field of ∂M^n in M^n (i.e. v is transversal to the boundary and outward pointing).

Theorem 3.1 *(Weak maximum principle for scalar functions) Let L, f, v be as above. Let $\varphi : M^n \times [0,T] \longrightarrow \mathbb{R}$ be a continuous function which is of the class $C^{2,1}$ in the neighbourhood of all points $(x,t) \in M^n \times [0,T]$ with $|\varphi(x,t)| < \varepsilon$ ($\varepsilon > 0$ a small constant) and fulfils the following conditions:*

$$\begin{aligned} &(i) && \varphi(.,0) \geq 0 \ , && \\ &(ii) && L(\varphi)(x,t) \geq 0 && \forall (x,t) \in M^n \times (0,T] \text{ with } |\varphi(x,t)| < \rho, \\ &(iii) && \tilde{\nabla}_v \varphi(x,t) = f(\varphi,x,t) && \forall (x,t) \in \partial M^n \times (0,T] \text{ with } |\varphi(x,t)| < \rho. \end{aligned}$$

Then $\varphi(x,t) \geq 0 \quad \forall (x,t) \in M^n \times [0,T]$.

Proof: First, let $\hat{\varphi}$ be a continuous function which is of class $C^{2,1}$ in the neighbourhood of each point (x,t) where $-\rho/10 < \hat{\varphi}(x,t) < \rho$. It can easily be seen that the conditions

$$\begin{aligned} &(i) && \hat{\varphi}(.,0) > 0 \ , && \\ &(ii) && L(\hat{\varphi})(x,t) > 0 && \forall (x,t) \in M^n \times (0,T] \text{ with } \hat{\varphi}(x,t) = 0 \ , \\ &(iii) && \tilde{\nabla}_v \hat{\varphi}(x,t) > 0 && \forall (x,t) \in \partial M^n \times (0,T] \text{ with } \hat{\varphi}(x,t) = 0 \end{aligned}$$

then imply $\hat{\varphi} > 0$ on $M^n \times [0,T]$.

Now let Λ_δ be a small neighbourhood of ∂M^n in M^n and define

$$\hat{\varphi}(x,t) = \begin{cases} \varphi(x,t) + \varepsilon\, \mathrm{e}^{\alpha t} \cdot \Psi(d(x)) & \text{on } \Lambda_\delta\ , \\ \varphi(x,t) + \varepsilon\, \mathrm{e}^{\alpha t} \cdot 1/2 & \text{on } M^n \backslash \Lambda_\delta\ , \end{cases}$$

where

$$\Psi(s) := C \cdot (s-\delta)^4 + 1/2\ .$$

Here $\delta > 0, C > 0$ and $\alpha > 0$ are constants which will be chosen such that (for $\varepsilon > 0$ sufficiently small) $\hat{\varphi}$ fulfils the above conditions: The boundary condition (*iii*) can be guaranteed for sufficiently small $\delta > 0$ and sufficiently large $C > 0$ (depending on $\|K\|_\infty$ and $\Delta := \min_{\partial M^n} \tilde{g}(v,\mu) > 0$). (*ii*) then follows for sufficiently large α, depending on C, δ and the data of the differential operator. The calculations are straightforward and therefore left out. Hence we have $\hat{\varphi} > 0$ for all $\varepsilon > 0$, and $\varepsilon \to 0$ yields the result. □

Corollary 3.2 *(Strong maximum principle for scalar functions) Under the assumptions of Theorem 3.1 let* $\varphi(.,0) \not\equiv 0$ *. Then*

$$\varphi(x,t) > 0 \quad \textit{on} \quad M^n \times (0,T) \cup \overset{\circ}{M}{}^n \times \{T\}\ .$$

Proof: [20], chapter III. □

Next, we turn to symmetric tensors M_{ij} on $M^n \times [0,T]$; here we adapt a Theorem due to *Hamilton* [9] to the situation of compact manifolds with boundary. To this end we let $N_{ij} := p(M_{ij}, g_{ij})$ be a polynomial in M_{ij} which is obtained by contracting M_{ij} with the metric g_{ij}; here, g_{ij} may depend on time. N_{ij} is said to have the *zero eigenvector condition*, if $N_{ij} v^i v^j \geq 0$ for all zero eigenvectors of M_{ij}. We then define a parabolic operator

$$L(M_{ij}) := \tfrac{\partial}{\partial t} M_{ij} - \boldsymbol{\Delta} M_{ij} - u^k \nabla_k M_{ij} - N_{ij}\ ,$$

where $\boldsymbol{\Delta}$ is the *Laplace-Beltrami* operator on M^n with respect to the metric g. If $\mu = \mu(x,t)$ denotes the outward pointing unit normal field of ∂M^n in M^n, we obtain:

Theorem 3.3 *(Weak maximum principle for symmetric tensors) Let L be as above. Let M_{ij} and $N_{ij} := p(M_{ij}, g_{ij})$ be continuous symmetric tensors on $M^n \times [0,T]$, where M_{ij} is of the class $C^{2,1}$ for $t > 0$ and the zero eigenvector condition holds for N_{ij}. Furthermore, let the vector fields u^k, w^k be bounded on $M^n \times [0,T]$ and $\partial M^n \times [0,T]$ respectively with $g(u,\mu) \geq 0$, $g(w,\mu) > 0$ on the boundary. Suppose the following conditions are fulfilled ($\delta > 0$ being an arbitrary, small constant):*

(*i*)	$M_{ij}(.,0)$	≥ 0,	
(*ii*)	$L(M_{ij})$	≥ 0	*on* $M^n \times (0,T]$,
(*iii*)	$(\nabla_w M_{ij}) v^i v^j (x,t)$	≥ 0	$\forall (x,t) \in \partial M^n \times (0,T]$, $\forall$ *minimal eigenvectors v of M_{ij} with eigenvalue* $\in [-\delta, 0]$.

Then $M_{ij} \geq 0$ *on* $M^n \times [0,T]$.

Proof: We let $\hat{M}_{ij} := M_{ij} + \varepsilon\, \mathrm{e}^{\alpha t} g_{ij}$, where $0 < \varepsilon < \delta\, \mathrm{e}^{-\alpha T}$ and $\alpha > 0$ will be chosen later, and show that $\hat{M}_{ij} > 0$ on $M^n \times [0,T]$. The Theorem then follows by $\varepsilon \to 0$.

Suppose there exists a first zero eigenvector v of $\hat{M}_{ij}$ at a point $(x_0, t_0) \in M^n \times (0,T]$. Extend v to a local vector field in a neighbourhood of (x_0, t_0) such that

$$\nabla v(x_0,t_0) = 0\ , \quad \tfrac{\partial}{\partial t} v(x_0,t_0) = 0\ ,$$

and define a scalar function f by

$$f(x,t) := (\hat{M}_{ij} v^i v^j)(x,t).$$

Similarly as in [9], for sufficiently large values of α, f fulfils the inequality

$$(*) \qquad \tfrac{\partial f}{\partial t}(x_0,t_0) > (\boldsymbol{\Delta} f + u^k\, \nabla_k\, f)(x_0,t_0)\ ,$$

and f has a first zero at (x_0, t_0). Now x_0 cannot be an *interior* point, for then $(*)$ yields $\frac{\partial f}{\partial t}(x_0,t_0) > 0$ in contradiction to the minimality of t_0. Hence $x_0 \in \partial M^n$, and $f(x,t_0) > 0$ in the interior, i.e. $f(.,t_0)$ has a local minimum at x_0. Thus for all *tangential* directions $\tau \in T_{x_0}\partial M^n$ we have $0 = \nabla_\tau\, f(x_0,t_0)$, and by assumption (iii) and $g(w,\mu) \geq 0$ we conclude

$$(**) \qquad \nabla_\mu\, M_{ij}(x_0,t_0) v^i v^j \geq 0.$$

Since $g(u,\mu) \geq 0$, this implies $u^k \nabla_k\, M_{ij}(x_0,t_0) v^i v^j \geq 0$ and from $(*)$ we get

$$\tfrac{\partial f}{\partial t}(x_0,t_0) > \boldsymbol{\Delta} f(x_0,t_0)\ .$$

Now $\frac{\partial f}{\partial t}(x_0,t_0) \leq 0$ by the minimality of t_0, and so we have $\boldsymbol{\Delta} f(.,t_0) < 0$ even in a small neighbourhood of x_0 on M^n. Therefore $\nabla_\mu\, f(x_0,t_0) < 0$ by the elliptic Hopf Lemma and in contradiction to $(**)$. □

In applications, the following Lemma will be useful:

Lemma 3.4 *Under the assumptions of Theorem 3.3 suppose that for* $(x,t) \in \partial M^n \times [0,T]$ *the tensor* M_{ij} *decomposes, i.e.*

$$M_{I\mu}(x,t) = M_{\mu I}(x,t) = 0 \quad \forall\ 1 \leq I \leq (n-1).$$

Then condition (iii) *of Theorem 3.3 can be replaced by*

$$\begin{array}{lll} (iii') \quad (\nabla_w\, M_{ij}) v^i v^j(x,t) \geq 0 & \forall \textit{ minimal eigenvectors } v \in T_x\partial M^n \\ & \textit{of } M_{ij} \textit{ with eigenvalue} \in [-\delta, 0]\ , \\ (iii)'' \quad (\nabla_w\, M_{\mu\mu})(x,t) \geq 0 & \textit{if } M_{\mu\mu} \textit{ min. and} - \delta \leq M_{\mu\mu}(x,t) \leq 0\ . \end{array}$$

Proof: If at a boundary point $v := v^i \partial_i$ denotes a minimal eigenvector then, due to the block form of $M_{ij}(x,t)$, the vectors $v' := v^I \partial_I \in T_x\partial M^n$ and $v'' := v^n \mu$ are also minimal eigenvectors; at least one of these two vectors is nonzero, and so working with *this* vector instead of v and conditions $(iii)'$, $(iii)''$ instead of (iii) in the proof of Theorem 3.3, we obtain the desired result. □

4 Barriers

In this section, we study the "C^0-behaviour" of hypersurfaces moving under mean-curvature-like flows. To this end, let $\boldsymbol{F} : M^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ be a solution to equation 1.2; furthermore, let M_B^n be a compact, orientable manifold with boundary, and

$$\boldsymbol{F}_B : M_B^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}, \quad B_t := \boldsymbol{F}_B(M_B^n, t)$$

a family of smooth imbeddings with $\partial B_t \subset \boldsymbol{\Sigma}\ \forall t$ such that B_t and $\boldsymbol{\Sigma}$ bound a domain $\mathcal{G}_t \subset \mathbb{R}^{n+1}$. Assume that $\boldsymbol{F}_B$ obeys the following initial-boundary-value problem:

$$\left\{ \begin{array}{rcl} \langle \frac{d}{dt}\boldsymbol{F}_B, \boldsymbol{\nu}_B \rangle(x,t) & \geq & -H_B(x,t) \quad \forall (x,t) \in M_B^n \times (0,T]\ , \\ \boldsymbol{F}_B(.,0) & = & \boldsymbol{F}_{B,0}\ , \\ \partial B_t & \subset & \boldsymbol{\Sigma}\ , \\ \langle \boldsymbol{\nu}_B, \boldsymbol{\mu} \rangle & \leq & 0 \quad \text{on} \quad \partial M_B^n \times [0,T]\ . \end{array} \right. \tag{4.1}$$

Here $\boldsymbol{\nu}_B$ denotes the exterior unit normal field to B_t with respect to the domain $\mathcal{G}_t$. Thus the equation means that the normal component of the speed which determines the motion of the B_t is not greater than it would be under the mean curvature flow. B_t is called a *barrier* for the M_t, if $M_0 \subset \bar{\mathcal{G}}_0 \backslash B_0$ implies $M_t \subset \bar{\mathcal{G}}_t \backslash B_t \quad \forall t \in [0,T)$.

The main result of this section is

Theorem 4.1 *Let $\boldsymbol{F}_B$ be a family of smooth imbeddings as above, and let $M_0 \subset \bar{\mathcal{G}}_0 \backslash B_0$. Then B_t is a barrier for M_t.*

Before proving and discussing this Theorem in detail, we have to do some preparatory work. We thereby leave the tedious, but nevertheless elementary calculations to the reader.

In the sequel, let $t_0 > 0$ be the first time at which there exists $\boldsymbol{p}_0 \in M_{t_0} \cap B_{t_0}$. Then clearly

$$T_{p_0} M_{t_0} = T_{p_0} B_{t_0}\ ,$$

for a transverse intersection of M_{t_0} and B_{t_0} would contradict the minimality of t_0. (By the boundary conditions for M_t and B_t, this holds equally if $\boldsymbol{p}_0 \in \partial M_{t_0}$.)

We introduce generalized Gaussian coordinates in a tubular neighbourhood $\mathcal{U}_\varepsilon$ of M_{t_0} as in section 2. If we thereby choose $\boldsymbol{\xi}$ and $\boldsymbol{\nu}$ such that $\boldsymbol{\nu}(\boldsymbol{p}_0) = \boldsymbol{\xi}(\boldsymbol{p}_0) = \boldsymbol{\nu}_B(\boldsymbol{p}_0)$, we have $x^{n+1}(\boldsymbol{p}) < 0$ if and only if $\boldsymbol{p} \in \mathcal{G}$. Now, cutting off the hypersurfaces B_t at the boundary of $\mathcal{U}_\varepsilon$ and choosing $\delta > 0$ sufficiently small, we obtain graph representations $u, w : M^n \times [t_-, t_+] \longrightarrow \mathbb{R}$ of the cutted B_t and of M_t on the time interval $[t_-, t_+] := [t_0 - \delta, t_0 + \delta] \cap [0,T]$. To be more precise, we let $u(x,t)$ be the x^{n+1}-coordinate of the intersection

point of the flux line through $\boldsymbol{F}(x,t_0)$ on B_t, if such an intersection point exists in $\mathcal{U}_\varepsilon$; otherwise, we set $u(x,t) := \pm\varepsilon$, choosing the sign such that u is a Lipschitz function.

Next, we define $\varphi := u - w$ and observe that $\varphi > 0$ on $[t_-, t_0)$. Moreover, choosing $\delta > 0$ such that $M_t \subset \mathcal{U}_{\varepsilon/10}$ on $[t_-, t_+]$, the above construction implies that φ is smooth in a neighbourhood of any $(x_1,t_1) \in M^n \times [t_-,t_+]$ with $|\varphi(x_1,t_1)| < \varepsilon/2$.

As in section 2, the geometric boundary condition $\langle \boldsymbol{\nu}_B, \boldsymbol{\mu}\rangle \leq 0$ implies the scalar boundary condition $\mu^k D_k u \geq \mu^{n+1}$ (in the neighbourhood of a boundary point (x_1,t_1) with $|\varphi(x_1,t_1)| < \varepsilon/2$), where $\mu^\alpha = \mu^\alpha(x,t,u)$ are the components of $\boldsymbol{\mu}$ with respect to the chosen coodinates. Analogously, $\mu^k D_k w = \mu^{n+1}$ with $\mu^\alpha = \mu^\alpha(x,t,w)$, and hence

$$\begin{aligned}
0 \;\leq\;& \mu^k(x,t,u)\, D_k u - \mu^k(x,t,w)\, D_k w - \mu^{n+1}(x,t,u) + \mu^{n+1}(x,t,w) \\
=\;& \int_0^1 \tfrac{d}{ds}\Big(\mu^k(x,t,su+(1-s)w)\cdot(sD_ku+(1-s)D_kw)\Big)\,ds \\
-\;& \int_0^1 \tfrac{d}{ds}\Big(\mu^{n+1}(x,t,su+(1-s)w)\Big)\,ds \\
=\;& \left[\int_0^1 \mu^k(x,t,.)\,ds\right]\cdot D_k\varphi \\
+\;& \left[\int_0^1 \Big(D_z\mu^k(x,t,.)\cdot(sD_ku+(1-s)D_kw) - D_z\mu^{n+1}(x,t,.)\Big)\,ds\right]\cdot\varphi \\
=:\;& v^k(x,t)\cdot D_k\varphi + \Psi(x,t)\cdot\varphi\ .
\end{aligned}$$

Hereby the functions v^k and $f(\varphi,x,t) := -\Psi(x,t)\cdot\varphi$ are smooth, and, by the mean value Theorem,

$$v^k(x,t) \;=\; \int_0^1 \mu^k(x,t,su+(1-s)w)\,ds \;=\; \mu^k(x,t,\sigma)\ , \quad |\sigma| < \varepsilon$$

near (x_1,t_1). By the compactness of ∂M^n, v^k will therefore be a *tranversal* vector field for sufficiently small $\varepsilon > 0$. Thus we have proved:

Lemma 4.2 *Let $(x_1,t_1) \in \partial M^n \times [t_-,t_+]$ with $|\varphi(x_1,t_1)| < \varepsilon/4$. Then in a neighbourhood of (x_1,t_1), the boundary condition*

$$D_v\varphi(x,t) = f(\varphi,x,t)$$

is fulfilled for φ. Hereby, f is a smooth function with

$$|f(s,x,t) - f(0,x,t| \leq const\cdot|s|\ , \quad f(0,x,t) \geq 0\ ,$$

and $v \in TM^n$ is a smooth, outward pointing vector field, i.e. $g(v,\mu) > 0$. □

4.1 Proof of the Theorem and Further Results

Proof of Theorem 4.1: Let $t_0 \in (0,T]$ be as above; we have to show that $t_0 = T$. Suppose $t_0 < T$, and choose $\delta > 0$, $\varepsilon > 0$ such that u, w are graph representations of B_t, M_t as before. Then, if $\varphi = u - w$, the choice of t_0 implies $\varphi(.,t) > 0$ on $[t_-, t_0)$, and $\varphi(x_0, t_0) = 0$ for some $x_0 \in M^n$. As in section 2, we have parabolic differential equations for u and w:

$$\begin{aligned} \tfrac{\partial u}{\partial t} &\geq -v_B \cdot H_B &&= \alpha^{ij}(x,t,u,Du) D_i D_j u + \gamma(x,t,u,Du) \ , \\ \tfrac{\partial w}{\partial t} &= -v \cdot H &&= \alpha^{ij}(x,t,w,Dw) D_i D_j w + \gamma(x,t,w,Dw) \ . \end{aligned}$$

By standard methods, this implies a *linear* parabolic differential equation for $\varphi = u - w$:

$$\tfrac{\partial \varphi}{\partial t} \geq a^{ij} D_i D_j \varphi + b^i D_i \varphi + c\varphi \ ,$$

with $a^{ij} = a^{ij}(x,t)$, $b^i = b^i(x,t)$, $c = c(x,t)$. Using Lemma 4.2, we see that φ fulfils the conditions of Theorem 3.1 (*weak* maximum principle); hence $\varphi(.,t) \geq 0$ on $[t_-, t_+]$. By the *strong* maximum principle (Theorem 3.2), $\varphi(.,t) > 0$ on $[t_-, t_+)$ in contradiction to $\varphi(x_0, t_0) = 0$. □

Remark 4.3 *If $M_0 \cap B_0 \neq \emptyset$ and $M_0 \not\equiv B_0$, the strong maximum principle yields that M_t and B_t separate immediately, i.e. $M_t \cap B_t = \emptyset \ \forall t > 0$.*

Theorem 4.1 implies the case of "static" barriers: Let $\frac{d}{dt}\boldsymbol{F}_B \equiv 0$ and $B := \boldsymbol{F}_{B,0}(M_B^n)$. Furthermore, let $H_B \geq 0$. Then $\boldsymbol{F}_B$ fulfils equation 4.1, and Theorem 4.1 yields

Corollary 4.4 *("Static" Barriers) If $H_B \geq 0$ and $\langle \boldsymbol{\nu}_B, \boldsymbol{\mu} \rangle \leq 0$ on the boundary, B is a barrier for M_t.* □

Here we have not made any assumptions on M_0; instead, we have required $H_B \geq 0$ for the mean curvature of the barrier. This condition can be weakened if we impose a condition on the curvatures of M_0 and $\boldsymbol{\Sigma}$: If $H(.,0) \geq C > 0$ and $\boldsymbol{\Sigma}$ is assumed to be convex with respect to the outward unit normal field, it is sufficient to require $H_B \geq -C$ for the mean curvature of B. The proof is very similar to the above techniques; convexity of $\boldsymbol{\Sigma}$ hereby is needed to guarantee that the relation $H \geq C$ is preserved for $t > 0$ (cf. Theorem 8.1).

We remark that the above results can be generalized in a straightforward manner to domains which are bounded by $\boldsymbol{\Sigma}$ and *several* barriers.

Corollary 4.5 *(Imbeddings) Let ${}^{\Sigma}H \geq 0$ with respect to the outward unit normal field $\boldsymbol{\mu}$. Then M_0 being an imbedding implies that all M_t are imbeddings as well.*

Proof: We must show that M_t cannot selfintersect. Suppose t_0 is the first time where a selfintersection occurs at points $\boldsymbol{p} = \boldsymbol{F}(x_0, t_0),\ \boldsymbol{q} = \boldsymbol{F}(y_0, t_0)$. By Theorem 4.1 we can exclude the cases $x_0, y_0 \in M^n \backslash \partial M^n$ and $x_0, y_0 \in \partial M^n$; thus it remains to rule out the case $x_0 \in M^n \backslash \partial M^n,\ y_0 \in \partial M^n$. But as ${}^{\Sigma}H \geq 0$, the hypersurface $\boldsymbol{\Sigma}$ is a barrier to M_t (according to Corollary 4.4), and hence

$$x \notin \partial M^n \quad \Leftrightarrow \quad \boldsymbol{F}(x,0) \notin \boldsymbol{\Sigma} \quad \Leftrightarrow \quad \boldsymbol{F}(x,t) \notin \boldsymbol{\Sigma}\ \forall t < T\ .$$

Thus $\boldsymbol{F}(x_0, t_0) \neq \boldsymbol{F}(y_0, t_0)$ by the choice of x_0 and y_0. □

We remark that the condition ${}^{\Sigma}H \geq 0$ is required only to prevent M_t from intersecting $\boldsymbol{\Sigma}$.

5 Distance of Hypersurfaces

In section 4 we considered families B_t of hypersurfaces such that a solution of equation 1.2 could never intersect the B_t. Now we are interested in discussing the behaviour of the *distance* between *two* solutions. To this end, let M_1^n, M_2^n be compact, smooth, orientable manifolds. The main Theorem of this section is

Theorem 5.1 *Let $\mathcal{G}$ be a convex domain in $\mathbb{R}^{n+1}$ with smooth boundary $\boldsymbol{\Sigma} = \partial\mathcal{G}$. Let $\boldsymbol{F}_i : M_i^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ be two families of smooth immersions moving in $\bar{\mathcal{G}}$ under the mean curvature flow equation 1.2; hereby, defining $M_i(t) := \boldsymbol{F}_i(M_i^n, t)$, let $M_1(0) \cap M_2(0) = \emptyset$. Then the function $d(t) := dist_{n+1}(M_1(t), M_2(t))$ is monotonically increasing.*

As in the last section, the main idea of the proof is to consider graph representations of $M_i(t)$ in the neighbourhood of "critical points", i.e. points $\boldsymbol{p}_i \in M_i(t)$ with $d(t) = \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|$ and applying the maximum principles of section 3. Again, we first have to do some preparatory work: Throughout this section, let $\mathcal{G} \subset \mathbb{R}^{n+1}$ be a convex domain with smooth boundary $\boldsymbol{\Sigma}$ and exterior unit normal field $\boldsymbol{\mu}$. Given $\boldsymbol{p}, \boldsymbol{q} \in \bar{\mathcal{G}},\ \boldsymbol{p} \neq \boldsymbol{q}$, define

$$\boldsymbol{\omega}_{pq} := \frac{\boldsymbol{p}-\boldsymbol{q}}{\|\boldsymbol{p}-\boldsymbol{q}\|}\ .$$

The following Lemma is a simple consequence of the convexity of $\mathcal{G}$:

Lemma 5.2 *(Convexity of $\mathcal{G}$)*

(*i*) *Let $\boldsymbol{p} \in \boldsymbol{\Sigma}\,,\ \boldsymbol{q} \in \bar{\mathcal{G}}\,,\ \boldsymbol{p} \neq \boldsymbol{q}$. Then* $\langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\omega}_{pq} \rangle \geq 0$.

(*ii*) *Let $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{\Sigma}\,,\ \boldsymbol{p} \neq \boldsymbol{q}\,,\ \langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\omega}_{pq} \rangle = \langle \boldsymbol{\mu}(\boldsymbol{q}), \boldsymbol{\omega}_{pq} \rangle = 0$.*
Then $\langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\mu}(\boldsymbol{q}) \rangle = 1$, and $\boldsymbol{p}, \boldsymbol{q}$ belong to the same hyperplane. □

We now consider smooth immersions $\boldsymbol{F}_i$ of M_i^n $(i = 1, 2)$ in $\mathbb{R}^{n+1}$ with unit normal fields $\boldsymbol{\nu}_i$. Let $M_i := \boldsymbol{F}_i(M_i^n) \subset \bar{\mathcal{G}}$, $M_1 \cap M_2 = \emptyset$, $\partial M_i \subset \boldsymbol{\Sigma}$ and $\langle \boldsymbol{\nu}_i, \boldsymbol{\mu} \rangle = 0$ on ∂M_i. Moreover, let $\boldsymbol{p} := \boldsymbol{F}_1(x_1)$ and $\boldsymbol{q} := \boldsymbol{F}_2(\tilde{x}_2)$ such that $d := \|\boldsymbol{p} - \boldsymbol{q}\| = \mathrm{dist}_{n+1}(M_1, M_2)$. We then introduce Cartesian coordinates $\{y^1, ..., y^{n+1}\}$ in $\mathbb{R}^{n+1}$ such that $y(\boldsymbol{p}) = (0, ..., 0, d/2)$, $y(\boldsymbol{q}) = (0, ..., 0, -d/2)$. In a neighbourhood of $\boldsymbol{p}, \boldsymbol{q}$, the hypersurfaces M_i can be written as graphs of smooth functions u_i with $u_1(0) = d/2$, $u_2(0) = -d/2$ and $Du_1(0) = Du_2(0) = 0$. This can be seen as follows: By the choice of $\boldsymbol{p}, \boldsymbol{q}$, the function

$$f : M_1^n \longrightarrow \mathbb{R},\ f(x) := \|\boldsymbol{F}_1(x) - \boldsymbol{q}\|^2$$

has a global minimum in x_1. Now, if $\boldsymbol{p} \notin \boldsymbol{\Sigma}$, we have $\nabla f(x_1) = 0$ and thereby $Du_1(0) = 0$. If $\boldsymbol{p} \in \boldsymbol{\Sigma}$, we have $\langle \nabla f(x_1), \boldsymbol{w} \rangle = 0$ for all *tangential* directions $\boldsymbol{w} \in T\partial M_1(\boldsymbol{p})$, and

$$0 \geq \langle \nabla f(x_1), \boldsymbol{\mu} \rangle = 2 \langle \boldsymbol{F}_1(x_1) - \boldsymbol{q}, \boldsymbol{\mu} \rangle = 2 \langle \boldsymbol{p} - \boldsymbol{q}, \boldsymbol{\mu} \rangle \ .$$

Using Lemma 5.2(i), this implies $\langle \nabla f(x_1), \boldsymbol{\mu} \rangle = 0$ and thus $Du_1(0) = 0$ as before.

Lemma 5.3 *The matrix* $(D_i D_j(u_1 - u_2))(0)$ *is positive semidefinite.*

Proof: Let $u_i : \Omega_i \longrightarrow \mathbb{R}$ with $0 \in \overline{\Omega_i}$, and let $\varphi := u_1 - u_2$ on $\Omega_1 \cap \Omega_2$. If $\boldsymbol{p}, \boldsymbol{q} \notin \boldsymbol{\Sigma}$, 0 is in the *interior* of Ω_i, and the result follows by Taylor expansion of φ. The same argument applies if $\boldsymbol{p} \notin \boldsymbol{\Sigma}$ and $\boldsymbol{q} \in \boldsymbol{\Sigma}$ (or vice versa), or if $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{\Sigma}$ and $T_0\partial\Omega_1 = T_0\partial\Omega_2$. Finally the case $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{\Sigma}$ and $T_0\partial\Omega_1 \neq T_0\partial\Omega_2$ can be ruled out as, according to Lemma 5.2(ii), we have $\langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\mu}(\boldsymbol{q}) \rangle = 1$ if $\boldsymbol{p}, \boldsymbol{q} \in \boldsymbol{\Sigma}$. □

Finally, we state a Lemma due to *Hamilton* [10]:

Lemma 5.4 *Let* $g : M^n \times [0, T] \longrightarrow \mathbb{R}$ *be a smooth function, and define* $f(t) := \min_{M^n} g(x, t)$. *Then* f *is a Lipschitz function, and*

$$\tfrac{df}{dt}(t) \geq \min\{\tfrac{\partial g}{\partial t}(\hat{x}, t) \ :\ \hat{x} \in M^n \text{ such that } g(\hat{x}, t) = \min_{x \in M^n} g(x, t)\} \ .$$

5.1 Proof of the Theorem and Further Results

Proof of Theorem 5.1: Let

$$g : M_1^n \times M_2^n \times [0, T] \longrightarrow \mathbb{R}\ ,\ g(x, \tilde{x}, t) := \|\boldsymbol{F}_1(x, t) - \boldsymbol{F}_2(\tilde{x}, t)\|^2 \ ,$$

then $d(t) = \min_{M_1^n \times M_2^n} g(x, \tilde{x}, t)$, and, using the evolution equation 1.2, we have

$$\tfrac{d}{dt} g(x, \tilde{x}, t) = 2 \langle \boldsymbol{F}_1(x, t) - \boldsymbol{F}_2(\tilde{x}, t), \boldsymbol{H}_1(x, t) - \boldsymbol{H}_2(\tilde{x}, t) \rangle \ .$$

Now let $t_0 \in [0,T)$, $\boldsymbol{p} = \boldsymbol{F}_1(x_1,t_0)$, $\boldsymbol{q} = \boldsymbol{F}_2(\tilde{x}_2,t_0)$ be such that $d(t_0) = \|\boldsymbol{p}-\boldsymbol{q}\| > 0$ and use the Cartesian coordinates defined above to conclude

$$\begin{aligned} \boldsymbol{H}_1(x_1,t_0) - \boldsymbol{H}_2(\tilde{x}_2,t_0) &= \Delta(u_1-u_2)(0,t_0)\cdot\boldsymbol{\omega}_{pq}\ , \\ \boldsymbol{F}_1(x_1,t_0) - \boldsymbol{F}_2(\tilde{x}_2,t_0) &= d(t_0)\cdot\boldsymbol{\omega}_{pq}\ . \end{aligned}$$

Lemma 5.3 yields $\Delta(u_1-u_2)(0,t_0) \geq 0$, and thus $\frac{d}{dt}g(x_1,\tilde{x}_2,t_0) \geq 0$. The result follows from Lemma 5.4. □

Let us, for a moment, return to the study of barriers: $\mathcal{G}$ and $\boldsymbol{\Sigma}$ being defined as above, we let $B := \boldsymbol{F}_B(M_B^n) \subset \bar{\mathcal{G}}$ be a smooth imbedding with unit normal field $\boldsymbol{\nu}_B$, $\partial B \subset \boldsymbol{\Sigma}$ and $\langle \boldsymbol{\nu}_B, \boldsymbol{\mu}\rangle = 0$ on ∂B. We assume that $\mathcal{G}$ is divided by B into two disjoint subdomains $\mathcal{G}_1, \mathcal{G}_2$, and that $\boldsymbol{\nu}_B$ is the exterior normal field to $\mathcal{G}_1$. In this situation, we have

Theorem 5.5 *Let $\boldsymbol{F}$ be a family of smooth immersions evolving under the mean curvature flow equation 1.2, and $M_0 \subset \bar{\mathcal{G}}_1$. Let $H_B \geq 0$ with respect to $\boldsymbol{\nu}_B$. Then the function $d(t) := dist_{n+1}(M_t, B)$ is monotonically increasing.*

Proof: Defining $g(x,\tilde{x},t) := \|\boldsymbol{F}_B(x) - \boldsymbol{F}(\tilde{x},t)\|^2$ and $\boldsymbol{p}, \boldsymbol{q}, x_1, \tilde{x}_2$ etc. as before, we have

$$\tfrac{d}{dt}g(x_1,\tilde{x}_2,t_0) = 2\,d(t_0)\,\langle \boldsymbol{\omega}_{pq}, 0 - \boldsymbol{H}(\tilde{x}_2,t_0)\rangle\ .$$

Using $H_B \geq 0$ and $\boldsymbol{\omega}_{pq} = \boldsymbol{\nu}_B(x_1)$,

$$\tfrac{d}{dt}g(x_1,\tilde{x}_2,t_0) \geq 2\,d(t_0)\,\langle \boldsymbol{\omega}_{pq}, \boldsymbol{H}_B(x_1,t_0) - \boldsymbol{H}(\tilde{x}_2,t_0)\rangle,$$

and, proceeding as in the proof of Theorem 5.1, we obtain the result. □

There is a variety of configurations which can be studied using essentially the same techniques as presented here; for example, it is possible to consider imbedded hypersurfaces moving under

$$\langle \tfrac{d}{dt}\boldsymbol{F}, \boldsymbol{\nu}\rangle(x,t) \geq -H(x,t), \quad \langle \boldsymbol{\nu}, \boldsymbol{\mu}\rangle = 0 \ \text{ on } \ \partial M^n\ .$$

Here we must be careful with defining *exterior* normal fields, for the evolution inequality is not invariant under the transformation $\boldsymbol{\nu} \to -\boldsymbol{\nu}$. However, if we assume $\mathcal{G}$ to be a domain bounded by $\boldsymbol{\Sigma}$ and $M_1(0), M_2(0)$ and define $\boldsymbol{\nu}_i$ as exterior normal fields to $\mathcal{G}$, we can conclude that the distance between $M_1(t)$ and $M_2(t)$ once more monotonically increases. A proof of this fact is given in [23].

6 A Local Gradient Bound

In section 2, we saw that locally equation 1.2 is equivalent to the scalar problem 2.1. Now, by a precise control of the gradient Dw, the latter equation will be (locally) *uniformly* parabolic; from there, by standard results of the linear theory the existence Theorem 1.3 follows. In order to establish

the local gradient bound, we assume that the curvature of the M_t already is under control on an arbitrary closed time interval $[0,T] \subset [0,T_C)$. We caution the reader to remark that the crucial point is to obtain a local gradient estimate which is *uniform on some sufficiently large time interval*; this is certainly *not* a straightforward consequence of the underlying curvature bound.

Assumption 6.1 *Throughout this section, let* $\boldsymbol{F} : M^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ *be a solution to equation 1.2, and let the curvature of* M_t *be uniformly bounded on* $[0,T]$ *as follows:*

$$|g^{ij}\, h_{jk}|(x,t) \leq (2\, n^2 \rho_0)^{-1} \quad \forall 1 \leq i,k \leq n\,, \quad \forall (x,t) \in M^n \times [0,T]\,,$$

where ρ_0 *is a positive constant. Furthermore, let* $\boldsymbol{\Sigma}$ *be smooth with a global curvature bound, i.e. there exists* $\rho_\Sigma > 0$ *such that*

$$|{}^\Sigma g^{ij}\, h^\Sigma_{jk}|(\boldsymbol{p}) \leq (2\, n^2 \rho_\Sigma)^{-1} \quad \forall 1 \leq i,k \leq n\,, \quad \forall \boldsymbol{p} \in \boldsymbol{\Sigma}\,.$$

Finally, let $\boldsymbol{\Sigma} \cap B^{n+1}_{\rho_\Sigma}(\boldsymbol{p})$ *be connected for all* $\boldsymbol{p} \in \boldsymbol{\Sigma}$.

We emphasize that in order to obtain the local gradient bound on a small region in spacetime, it is sufficient to require a *local* curvature bound to the M_t on a (slightly larger) region; however, the global curvature bound somewhat simplifies the notation. The global curvature bound on $\boldsymbol{\Sigma}$ is a very slight restriction, as was discussed in Remark 1.2. Finally, the connectedness of $\boldsymbol{\Sigma} \cap B^{n+1}_{\rho_\Sigma}(\boldsymbol{p})$ is included for technical reasons; in general, this requirement can be fulfilled by choosing ρ_Σ sufficiently small, i.e. working in sufficiently small domains.

Now let us fix some notation: By the *standard graph representation (SGR) w of* M_t near a given point $\boldsymbol{p}_0 := \boldsymbol{F}(x_0,t_0)$ we mean the local graph representation around $\boldsymbol{p}_0$ with respect to a Cartesian coordinate system in $\mathbb{R}^{n+1}$ such that $\boldsymbol{p}_0 = (0, w(0,t_0)) = (0,0)$ and $\boldsymbol{\nu}(x_0,t_0) =: \boldsymbol{\omega} = \boldsymbol{e}_{n+1}$ (i.e. $Dw(0,t_0) = 0$). Furthermore, given $\boldsymbol{\omega} \in \mathbb{R}^{n+1}$, we define

$$u : M^n \times [0,T] \longrightarrow \mathbb{R}\,,\ u(x,t) := \langle \boldsymbol{F}(x,t), \boldsymbol{\omega} \rangle\,;$$

thus $u(x,t)$ describes the height of a point $\boldsymbol{F}(x,.)$ over the hyperplane $\mathcal{H}$ through $\boldsymbol{F}(x_0,t_0)$, whereas $w(y,.)$ is the height of a point with fixed coordinates on $\mathcal{H}$.

$\mathcal{Z}_r(\boldsymbol{p}) := B^n_r(\boldsymbol{p}) \times (-\infty, +\infty) \subset \mathbb{R}^{n+1}$ denotes the space cylinder around $\boldsymbol{p}$ with axis $\boldsymbol{\omega}$; analogously $\mathcal{Z}_{r,C}(\boldsymbol{p}) := B^n_r(\boldsymbol{p}) \times (-C\,r, +C\,r)$.

$\boldsymbol{\Sigma}^\Delta_{t_0} := \{\boldsymbol{F}(x,t) : x \in \partial M^n,\ |t - t_0| \leq \Delta\}$ is the subset of $\boldsymbol{\Sigma}$ which coincides with the boundaries ∂M_t in the time interval $[t_0 - \Delta, t_0 + \Delta]$.

6.1 Hypersurfaces with Bounded Curvature

In order to get the right scaling behaviour of the desired estimates, we first have to state some facts concerning hypersurfaces $M := \boldsymbol{F}(M^n) \subset \mathbb{R}^{n+1}$ with a curvature bound

$$|g^{ij}\, h_{jk}|(x) \leq (2\, n^2 \rho)^{-1} \quad \forall 1 \leq i,k \leq n\,, \quad \forall x \in M^n$$

as in Assumption 6.1.

Lemma 6.2 *Let $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0)$ and $w : \mathbb{R}^n \ni \mathcal{U} \longrightarrow \mathbb{R}$ be the SGR of $\boldsymbol{F}(M^n)$ around $\boldsymbol{p}_0$. Let $\boldsymbol{p} = \boldsymbol{F}(x) = (y, w(y))$ with $\|y\| =: r < \rho$ and $\{s\, y : 0 \leq s \leq 1\} \subset \mathcal{U}$. Finally, define $v(x) = \langle \boldsymbol{\nu}(x), \boldsymbol{\omega}\rangle^{-1} = \sqrt{1+\|Dw\|^2}(y)$. Then*

$$v(x) < 1 + (\tfrac{r}{2\,\rho})^2\,, \quad \langle \boldsymbol{\nu}(x), \boldsymbol{\omega}\rangle > 1 - (\tfrac{r}{2\,\rho})^2\,, \quad |\langle \boldsymbol{F}(x) - \boldsymbol{F}(x_0), \boldsymbol{\omega}\rangle| < \tfrac{r^2}{\rho}\,.$$

Proof (sketch): From the curvature bound we get $|h_{ik}| \leq \frac{1}{2\,n\,\rho}\,(1+\|Dw\|^2)$ and thereby

$$|D_i D_k w| \leq (2\,n\rho)^{-1} \sqrt{1+\|Dw\|^2}^{\,3}\,.$$

Now let $e := y/\|y\|$; then, by assumtion, the function $\sigma \longmapsto D_i w(\sigma\, e)$ is defined on $[0,1]$. From Hölder's inequality and the above equation, we obtain

$$|D_i w(s\, e)| \leq \sqrt{\tfrac{r}{n}}\, \tfrac{1}{2\,\rho} \sqrt{\int_0^s (1+\|Dw\|^2)^3 d\sigma}$$

and by summation

$$(1+\|Dw\|^2)(s\, e) \leq 1 + \tfrac{r}{4\,\rho^2} \int_0^s (1+\|Dw\|^2)^3 (\sigma\, e)\, d\sigma\,.$$

Solving this ordinary integral equation for $1+\|Dw\|^2$ and using Taylor expansions, the result follows. □

Note that the Lemma implies vice versa, that the *SGR* of M will be defined *on the entire ball* $B_r^n(0)$, if the $\boldsymbol{p}_0$-component of $\mathcal{Z}_r(\boldsymbol{p}_0) \cap M$ does not have any boundary points in the interior of $\mathcal{Z}_r(\boldsymbol{p}_0)$. Thus, applying the Lemma to $\boldsymbol{\Sigma}$ and recalling Remark 1.2, we arrive at the following

Lemma 6.3 *Let $r \leq \rho_\Sigma$, $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0) \in \boldsymbol{\Sigma}$, $\boldsymbol{\gamma} := \boldsymbol{\mu}(\boldsymbol{p}_0)$ and $v_\Sigma := \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\rangle^{-1}$. Then for any $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap \mathcal{Z}_r(\boldsymbol{p}_0)$, we have*

$$v_\Sigma(\boldsymbol{p}) < 1 + (\tfrac{r}{2\,\rho_\Sigma})^2\,, \quad \langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\gamma}\rangle > 1 - (\tfrac{r}{2\,\rho_\Sigma})^2\,, \quad |\langle \boldsymbol{p} - \boldsymbol{p}_0, \boldsymbol{\gamma}\rangle| < \tfrac{r^2}{\rho_\Sigma}\,. \qquad \square$$

In order to obtain an analogous estimate for the hypersurfaces M_t, we have to be more careful: If $\boldsymbol{p}_0 \in M_{t_0} \backslash \partial M_{t_0}$ we can proceed exactly as before, obtaining a graph representation on an entire ball of radius $\rho_1 := \min\{\rho/4, d/2\}$,

where $d := \mathrm{dist}_{n+1}(\boldsymbol{p}_0, \boldsymbol{\Sigma})$. If $\boldsymbol{p}_0 \in \partial M_{t_0}$, due to the curvature of $\boldsymbol{\Sigma}$, the *SGR* of M_{t_0} will in general *not* be defined along straight lines $\sigma \mapsto \sigma\, e$ as in the proof of Lemma 6.2. However, the curvature of $\boldsymbol{\Sigma}$ being bounded, we have a graph representation for $\boldsymbol{\Sigma}$ if $r \leq \rho_\Sigma$, and it turns out that we can work with curves of length $\leq 3\, r$ to connect various points in the domain of the *SGR* of M_{t_0}. Thus we obtain

Lemma 6.4 *Let* $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$, $d := dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0})$,

$$\rho_1 := \begin{cases} \min\{\rho_0/3\,,\ \rho_\Sigma\} & \text{if } \boldsymbol{p}_0 \in \boldsymbol{\Sigma}\,, \\ \min\{\rho_0/4\,,\ d/2\} & \text{if } \boldsymbol{p}_0 \notin \boldsymbol{\Sigma}\,, \end{cases}$$

and $r \leq \rho_1$. *Then for any* $\boldsymbol{F}(x, t_0) \in \mathcal{Z}_r(\boldsymbol{p}_0)$ *we have:*

$$v(x,t_0) < 1 + (\tfrac{r}{\rho_0})^2\,, \quad \langle \boldsymbol{\nu}(x,t_0), \boldsymbol{\omega}\rangle > 1 - (\tfrac{r}{\rho_0})^2\,, \quad |u(x,t_0) - u(x_0,t_0)| < \tfrac{2\,r^2}{\rho_0}.$$

□

Certainly we could have chosen $\rho_1 := \delta \cdot \min\{\rho_0, \rho_\Sigma\}$ ($\delta > 0$ being sufficiently small) for both interior points and points near the boundary, but as we want to derive the inner and boundary gradient estimates independently, the above definition is somewhat simpler from the technical point of view.

As a further consequence of the evolution equation 1.2 and the curvature bound, we can control the behaviour of points $\boldsymbol{F}(x_0, t)$ for fixed $x_0 \in M^n$: A short calculation yields for any $t_1, t_2 \in [0, T]$:

$$(6.1) \qquad (\|\boldsymbol{F}\|^2 - u^2)(x, t_2) \leq \left(\tfrac{1}{2\,\rho_0} |t_2 - t_1| + \sqrt{\|\boldsymbol{F}\|^2 - u^2}\ (x, t_1) \right)^2 .$$

Combining this equation with the above Lemmata, we can estimate the distance of points $\boldsymbol{F}(x,t)$, $\boldsymbol{F}(x_0,t_0)$, if $\boldsymbol{F}(x,t_0) \subset \mathcal{Z}_r(\boldsymbol{p}_0)$. These points will leave the cylinder $\mathcal{Z}_r(\boldsymbol{p}_0)$ after some time; on the other hand, $\mathcal{Z}_r(\boldsymbol{p}_0)$ contains points $\boldsymbol{F}(x,t)$ which didn't start in the cylinder for $t = t_0$. Now we are interested in controlling the behaviour of $M_t \cap \mathcal{Z}_r(\boldsymbol{p}_0)$ for a short time $|t - t_0| \leq \delta$. Combining the above results, we easily see that this can be done on a *smaller* cylinder $\mathcal{Z}_{r_2}(\boldsymbol{p}_0) \subset \mathcal{Z}_r(\boldsymbol{p}_0)$ as follows:

Lemma 6.5 *Let* $r_2 \leq \frac{\rho_1}{2}$ *and* $\delta \leq 2\,\rho_0\, r_2$. *Then for any* $(x,t) \in M^n \times [0,T]$ *with* $\boldsymbol{F}(x,t) \in \mathcal{Z}_{r_2}(\boldsymbol{p}_0)$ *and* $|t - t_0| \leq \delta$ *the following estimate holds:*

$$|u(x,t) - u(x_0,t_0)| < \tfrac{1}{2\,\rho_0}\,(|t - t_0| + 16\, r_2^2)\ .$$

□

As a last consequence of the curvature bound on the M_t we give an estimate for the distance of a point p from $\Sigma_{t_0}^{\Delta}$. The proof follows directly from equation 1.2.

$$(6.2) \qquad \mathrm{dist}_{n+1}(\boldsymbol{p}, \Sigma_{t_0}^{\Delta}) \ \geq\ \mathrm{dist}_{n+1}(\boldsymbol{p}, \partial M_{t_0}) - \tfrac{1}{2\,\rho_0}\,\Delta\ .$$

6.2 The Interior Estimate

We first consider $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$ with $x_0 \in M^n \backslash \partial M^n$. As before, we thereby choose Cartesian coordinates in $\mathbb{R}^{n+1}$ with origin at $\boldsymbol{p}_0$ and $\boldsymbol{\omega} := \boldsymbol{e}_{n+1} := \boldsymbol{\nu}(x_0, t_0)$. Using ideas of *Ecker* and *Huisken* [5], we define

$$M_{r,\delta} := \sup\{|u(x,t)| : \boldsymbol{F}(x,t) \in \mathcal{Z}_r(\boldsymbol{p}_0)\,,\ 0 \le t - t_0 \le \delta\}\,,$$

$$\varphi(x,t) := \begin{cases} \left(\frac{r}{2\,\beta}(u - M_{r,\delta}) + r^2 - (\|\boldsymbol{F}\|^2 - u^2)\right)_+ (x,t)\,, & \boldsymbol{F}(x,t) \in \mathcal{Z}_r(\boldsymbol{p}_0)\,, \\ 0\,, & \boldsymbol{F}(x,t) \notin \mathcal{Z}_r(\boldsymbol{p}_0)\,, \end{cases}$$

$$\eta(x,t) := \mathrm{e}^{\lambda\,\varphi(x,t)} - 1\,,$$

where $r > 0$, $\delta > 0$, $\beta > 0$ and $\lambda > 0$ will be chosen later. φ and η are Lipschitz functions for $0 \le t - t_0 \le \delta$, and $\mathrm{supp}(\varphi) = \mathrm{supp}(\eta) \subset \mathcal{Z}_r(\boldsymbol{p}_0)$. For the sequel, we fix the following notation:

$$\begin{aligned} \sup_{\mathcal{Z}_r} v(.,t_0) &:= \sup\{v(\xi,t_0) : \boldsymbol{F}(\xi,t_0) \in \mathcal{Z}_r(\boldsymbol{p}_0)\}\,, \\ \sup_{\mathcal{Z}_r,\,\delta} \eta(.,.) &:= \sup\{\eta(\xi,\tau) : \boldsymbol{F}(\xi,\tau) \in \mathcal{Z}_r(\boldsymbol{p}_0), 0 \le t - t_0 \le \delta\}\,. \end{aligned}$$

Now let (x_1, t_1), $x_1 \in M^n \backslash \partial M^n$, $0 \le t_1 - t_0 \le \delta$ be a point where $\eta\, v$ attains a new maximum. Proceeding as in [5], we obtain

$$2\,n \ge \lambda\, r^2 \left(\tfrac{1}{4\,\beta^2}(1 - v^{-2}) - \tfrac{2}{\beta} v^{-1}\right)$$

at this point. Hence, defining $\lambda := \frac{8\,n\,\alpha\,\beta^2}{r^2}$ ($\alpha > 1$ to be chosen), we conclude

$$v(x_1,t_1) < \tfrac{\alpha}{\alpha - 1}\,(8\,\beta + 1)\,.$$

This in turn yields

$$(\eta\, v)(x_1,t_1) < \sup_{\mathcal{Z}_r,\,\delta} \eta(.,.) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}\,(8\,\beta+1)\,,\ (1+\hat{\varepsilon}) \sup_{\mathcal{Z}_r} v(.,t_0) \right\}\,,$$

where $\hat{\varepsilon} > 0$ is an arbitrary constant. Now this inequality can only be broken at a boundary point $x_2 \in \partial M^n$; hence an interior bound for $\eta\, v$ can be obtained by choosing $r > 0$ and $\delta > 0$ such that the $\boldsymbol{p}_0$-components of $M_t \cap \mathcal{Z}_r(\boldsymbol{p}_0)$ do not have any boundary points for $0 \le t - t_0 \le \delta$. Using Lemma 6.5 and equation 6.2, by a short calculation we see that $r \le \min\{\rho_0/8, d/8\}$ and $\delta \le 2\,\rho_0\, r$ are sufficiently small ($d = \mathrm{dist}_{n+1}(\boldsymbol{p}_0, \partial M_{t_0})$). Hence, from

$$\sup_{\mathcal{Z}_r,\,\delta} \eta(.,.) \le \mathrm{e}^{8\,n\,\alpha\,\beta^2} - 1\,, \tag{6.3}$$

we conclude

Lemma 6.6 *Let $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0)$, $x_0 \in M^n \backslash \partial M^n$, $d := dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0}) > 0$ and $\alpha > 1$, $\beta > 0$, $\hat{\varepsilon} > 0$, $r \le \min\{\frac{\rho_0}{8}, \frac{d}{8}\}$, $\delta \le 2\,\rho_0\, r$. Then for any (x,t)*

with $\boldsymbol{F}(x,t) \in \mathcal{Z}_r(\boldsymbol{p}_0)$ *and* $0 \le t - t_0 \le \delta$, *the following estimate holds:*

$$(\eta\, v)(x,t) < (\mathrm{e}^{8\,n\,\alpha\,\beta^2} - 1) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}\,(8\,\beta+1)\,,\ (1+\hat{\varepsilon}) \sup_{\mathcal{Z}_r} v(.,t_0) \right\}\,. \qquad \square$$

Lemma 6.6 immediately yields an estimate for v by bounding η from below. Certainly, this has to be done on a smaller cylinder $\mathcal{Z}_{r_1}(\boldsymbol{p}_0)$, since $\eta \equiv 0$ on $\partial\mathcal{Z}_r(\boldsymbol{p}_0)$. The crucial point is to determine the scaling behaviour of β, and it turns out that a good choice is

$$\beta := \tfrac{1}{2}\sqrt{\tfrac{r}{\rho_0}}\,.$$

If we then let $\kappa \in (0,1)$, $r \le (\frac{\kappa}{32})^2\,\rho_0$, $r_1 \le \sqrt{\frac{\kappa}{4}}\,r$ and $\delta \le 8\,r^2$, we estimate

$$r^2 \ \ge\ \varphi(x,t) \ \ge\ (1-\kappa) r^2$$

in the cylinder $\mathcal{Z}_{r_1}(\boldsymbol{p}_0)$ and for $0 \le t - t_0 \le \delta$. This in turn implies the desired gradient estimate in the interior case on time interval which is comparable to ρ_0^2, d^2 :

Theorem 6.7 *Let* $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0)$, $d = dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0}) > 0$. *Then*

$$\forall\ \varepsilon > 0 \quad \exists\ C_1 > 0\,,\ C_2 > 0\,,\ C_3 > 0\,,\ C_4 > 0 \ \ \textit{depending only on}\ \ n, \varepsilon$$

such that

$$v(x,t) < 1 + \varepsilon \quad \forall (x,t)\ \textit{with}\ \ \boldsymbol{F}(x,t) \in \mathcal{Z}_{r_1}(\boldsymbol{p}_0)\,,\ 0 \le t - t_0 \le \delta\,;$$

hereby $r_1 = \min\{C_1\,\rho_0\,,\ C_2\,d\}$ *and* $\delta = \min\{C_3\,\rho_0^2\,,\ C_4\,d^2\}$.

Proof: Choose

$$r := \tfrac{1}{n}\min\left\{\sqrt{\tfrac{\kappa}{4}}d,\ (\tfrac{\kappa}{32})^2\rho_0\right\}\,,\ r_1 := \sqrt{\tfrac{\kappa}{4}}r\,,\ \delta \le \min\left\{8\,r^2,\ 2\,\rho_0\,r\right\}\,.$$

Furthermore, let $\alpha := 1 + \frac{1}{\kappa}$ and $\hat{\varepsilon} := \kappa$. Then

$$(1+\hat{\varepsilon}) \sup_{\mathcal{Z}_r} v(.,t_0) \le (1+\kappa)^2 \qquad \text{(Lemma 6.4) and } \ \tfrac{\alpha}{\alpha-1}\,(1+8\,\beta) \le (1+\kappa)^2\,.$$

Moreover, for (x,t) as in the Theorem, $\eta(x,t) > \mathrm{e}^{8\,n\,\alpha\,\beta^2\,(1-\kappa)} - 1$. Now we apply the Taylor expansion

$$\tfrac{\mathrm{e}^z - 1}{\mathrm{e}^{z\,(1-\kappa)} - 1} \le 1 + 2\,\kappa + 2\,z\,, \quad z \in (0, \tfrac{1}{32})\,,\ \ \kappa \in (0, \tfrac{1}{2}]$$

to the estimate Lemma 6.6 and conclude

$$v(x,t) < 1 + 10\,\kappa\,.$$

The result follows if we choose $\kappa \le \min\{\frac{1}{2}, \frac{\varepsilon}{10}\}$. $\square$

Remark 6.8 *At a first glance, the estimate Theorem 6.7 depends on the point (x_0, t_0), since r, r_1, δ depend on the distance of $\boldsymbol{F}(x_0, t_0)$ from $\boldsymbol{\Sigma}$. However, for all $(\tilde{x}, \tilde{t})$ with $dist_{n+1}(\boldsymbol{F}(\tilde{x}, \tilde{t}), \boldsymbol{\Sigma}) \geq d_0 > 0$, we obtain the same estimate, since r, r_1, δ depend only on the minimal distance of all $\boldsymbol{F}(\tilde{x}, \tilde{t})$ from $\boldsymbol{\Sigma}$.*

Remark 6.9 *It is clear that Theorem 6.7 immediately implies a local gradient estimate for the SGR of M_t in the vicinity of (x_0, t_0): Given $\varepsilon > 0$, there exist $\hat{r_1} > 0$ and $\hat{\delta} > 0$ such that the SGR of the M_t is defined on $B_{r_1}(0) \times [t_0, t_0 + \hat{\delta}]$ and fulfils the estimate*

$$\sqrt{1 + \|Dw\|^2(y,t)} < 1 + \varepsilon \, ;$$

hereby $\hat{r_1}, \hat{\delta}$ depend only on ε, ρ_0, d and have the "natural" scaling behaviour $\hat{r_1} \sim \rho_0, \hat{\delta} \sim \rho_0^2$. Thus using Lemma 6.5, we obtain a local C^1-estimate $\|w\|_{C^1} \leq C_0$ on $B^n_{\hat{r}_1} \times [t_0, t_0 + \hat{\delta}]$.

6.3 The Boundary Estimate

We now turn our attention to boundary points. Here we have to rule out *boundary maxima* at least for sufficiently large values of $\eta\, v$. This can be done if we guarantee that $\langle \nabla(\eta\, v), \boldsymbol{\mu} \rangle < 0$ for large values of $\eta\, v$. Now the cutoff-function η which we used in the interior case essentially is a paraboloid centered at $\boldsymbol{p}_0$; therefore it does not produce a sufficiently large term $\langle \nabla \eta, \boldsymbol{\mu} \rangle < 0$ that could compensate the (possibly large and positive) term $\langle \nabla v, \boldsymbol{\mu} \rangle$ in the vicinity of $\boldsymbol{p}_0$. The idea is to center the paraboloid not at $\boldsymbol{p}_0$ but at a nearby point $\boldsymbol{p}_m \notin \boldsymbol{\Sigma}$ such that η posesses a sufficiently large gradient near $\boldsymbol{p}_0$.

We choose Cartesian coordinates in $\mathbb{R}^{n+1}$ with origin at $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$ and set $\boldsymbol{\omega} := \boldsymbol{\nu}(x_0, t_0)$, $\boldsymbol{\gamma} := \boldsymbol{\mu}(\boldsymbol{p}_0)$ and

$$\boldsymbol{p}_m := \boldsymbol{p}_0 - \tfrac{1}{K}\, r\, \boldsymbol{\gamma} \, , \quad K > 1 \, , \quad r \leq \tfrac{\rho_\Sigma}{4} \, .$$

Thus $\mathcal{Z}_{r,1}(\boldsymbol{p}_m) \subset B^{n+1}_{\rho_\Sigma}(\boldsymbol{p}_0)$, and hence Lemma 6.3 holds on $\boldsymbol{\Sigma} \cap \mathcal{Z}_{r,1}(\boldsymbol{p}_m)$, and this set is connected due to Assumption 6.1. Now, similar to the interior case, we set (with $\mathcal{Z}_r := \mathcal{Z}_r(\boldsymbol{p}_m)$)

$$M_{r,\delta} := \sup\{|u(x,t)| : \boldsymbol{F}(x,t) \in \mathcal{Z}_r \, , \, 0 \leq t - t_0 \leq \delta\} \, ,$$

$$\varphi(x,t) := \begin{cases} \left(\frac{r}{2\,\beta}(u - M_{r,\delta}) + r^2 - (\|\boldsymbol{F} - \boldsymbol{p}_m\|^2 - u^2)\right)_+ (x,t) \, , & \boldsymbol{F}(x,t) \in \mathcal{Z}_r, \\ 0 \, , & \boldsymbol{F}(x,t) \notin \mathcal{Z}_r, \end{cases}$$

$$\eta(x,t) := \mathrm{e}^{\lambda\, \varphi(x,t)} - 1 \, ,$$

where $0 < r < \frac{\rho_\Sigma}{4}$, $\delta > 0$ and $\alpha > 1$ will be chosen later and

$$\beta := \tfrac{1}{2}\sqrt{\tfrac{r}{\rho_0}} \, , \quad \lambda := \tfrac{8\, n\, \alpha\, \beta^2}{r^2} = \tfrac{2\, n\, \alpha}{\rho_0\, r} \, .$$

Now we estimate the normal component of $\nabla(\eta\, v)$ as follows:

Lemma 6.10 *Let $K \geq 8$, $r \leq \frac{1}{64}\min\{\rho_0, \frac{\rho_0}{K^2}, \frac{\rho_0}{n\,\alpha}, \frac{\rho_\Sigma}{K}, \frac{\rho_\Sigma^2}{K^2\,\rho_0}\}$ and $\delta \leq r^2$. Then for any $(x,t) \in M^n \times [0,T]$ with $\boldsymbol{F}(x,t) \in \boldsymbol{\Sigma} \cap \mathcal{Z}_r(\boldsymbol{p}_m)$, $\varphi(x,t) > 0$ and $0 \leq t - t_0 \leq \delta$, the following estimate holds:*

$$\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle < \tfrac{1}{2\,\rho_0}\, v\, \left((\eta\, v) - \tfrac{4\,n\,\alpha}{K}\right) .$$

Proof: We compute

$$(*) \qquad \langle \nabla\varphi, \boldsymbol{\mu}\rangle \;\leq\; |\langle \boldsymbol{\mu}, \boldsymbol{\omega}\rangle|\, \left(\tfrac{r}{2\,\beta} + 2\,M_{r,\delta}\right) - 2\,\langle \boldsymbol{F} - \boldsymbol{p}_m, \boldsymbol{\mu}\rangle .$$

By the choice of r, δ, we can apply Lemma 6.5 on the cylinder $\mathcal{Z}_{2\,r}(\boldsymbol{p}_0)$; this yields $|u(x,t)| \leq \frac{3}{4}r$ and thus

$$M_{r,\delta} < r .$$

Now a lengthy but nevertheless straightforward calculation involving Lemma 6.3 and the Cauchy-Schwarz-inequality yields

$$\langle \boldsymbol{F} - \boldsymbol{p}_m, \boldsymbol{\mu}\rangle > \tfrac{3}{4}\,\tfrac{r}{K}, \quad \text{and} \quad |\langle \boldsymbol{\mu}, \boldsymbol{\omega}\rangle|^2 < 2(\tfrac{r}{\rho_\Sigma})^2 .$$

Substituting these expressions into equation $(*)$, we obtain

$$(**) \qquad \langle \nabla\varphi, \boldsymbol{\mu}\rangle < -\tfrac{r}{K} .$$

Now we compute $\langle \nabla v, \boldsymbol{\mu}\rangle = -v^2\, A(\boldsymbol{\omega}^T, \boldsymbol{\mu})$; using Assumption 6.1 (bounded curvature), this yields

$$(***) \qquad |\langle \nabla v, \boldsymbol{\mu}\rangle| \;\leq\; v^2\,|A|\,\|\boldsymbol{\omega}^T\| \;\leq\; v^2\tfrac{1}{2\,\rho_0} .$$

Finally,

$$\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle = v\,\langle \nabla\eta, \boldsymbol{\mu}\rangle + \eta\,\langle \nabla v, \boldsymbol{\mu}\rangle = \lambda\, v\,(\eta+1)\,\langle \nabla\varphi, \boldsymbol{\mu}\rangle + \eta\,\langle \nabla v, \boldsymbol{\mu}\rangle ,$$

and the result follows using equations $(**), (***)$. □

Lemma 6.11 *Let $r > 0$, $\delta > 0$ be as in Lemma 6.10 and $\alpha \geq 2$, $\hat{\varepsilon} \leq 1$. Then for any $(x,t) \in M^n \times [0,T]$ with $\boldsymbol{F}(x,t) \in \mathcal{Z}_r(\boldsymbol{p}_m)$, $0 \leq t - t_0 \leq \delta$, the following estimate holds:*

$$(\eta\, v)(x,t) < \sup_{\mathcal{Z}_r,\,\delta} \eta(.,.) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}\,(8\,\beta+1), (1+\hat{\varepsilon})\, \sup_{\mathcal{Z}_r} v(.,t_0) \right\} .$$

Proof: Exactly as in the interior case, this estimate is valid at least until equality holds at a boundary point $\boldsymbol{p} = \boldsymbol{F}(x_2,t_2)$, $x_2 \in \partial M^n$, $0 \leq t_2 - t_0 \leq \delta$. At this point, $\eta\, v$ must have a maximum, hence we have

$$(*) \qquad \langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle(x_2,t_2) \geq 0 .$$

Now, by Lemma 6.4, $\sup_{\mathcal{Z}_r} v(.,t_0) < 2$; furthermore a Taylor expansion of equation 6.3 yields $\sup_{\mathcal{Z}_r,\,\delta} \eta(.,.) \leq 4\,n\,\alpha\frac{r}{\rho_0}$, and thus we have

$$(\eta\, v)(x_2,t_2) < 16\,n\,\alpha\,\tfrac{r}{\rho_0} .$$

Lemma 6.10 now implies $\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle(\boldsymbol{p}) < 0$ in contradiction to $(*)$. □

We proceed as in the interior case by first estimating η from below on a smaller cylinder $\mathcal{Z}_{r_1}(\boldsymbol{p}_0) \subset \mathcal{Z}_r(\boldsymbol{p}_m)$ and then Taylor expanding the quotient of the estimates for η from above (on $\mathcal{Z}_r(\boldsymbol{p}_m)$) and from below (on $\mathcal{Z}_{r_1}(\boldsymbol{p}_0)$). This can be done similarly as before, so we directly state the results:

Lemma 6.12 *Let* $r \leq \frac{1}{64}\min\{\rho_0\,,\, \frac{\rho_0}{K^2}\,,\, \frac{\rho_0}{n\,\alpha}\,,\, \frac{\rho_\Sigma}{K}\,,\, \frac{\rho_\Sigma^2}{K^2\,\rho_0}\}$, $\delta \leq r^2$, $r_1 \leq \frac{r}{K}$ *and* $\alpha \geq 2$, $K \geq 8$, $\hat{\varepsilon} \leq 1$. *Then for any* $(x,t) \in M^n \times [0,T]$ *with* $\boldsymbol{F}(x,t) \in \mathcal{Z}_{r_1}(\boldsymbol{p}_0)$ *and* $0 \leq t - t_0 \leq \delta$ *the following estimate holds:*

$$v(x,t) < \Big(1 + \tfrac{6}{K} + 4\,n\,\alpha\,\tfrac{r}{\rho_0}\Big)\cdot\max\left\{\, \tfrac{\alpha}{\alpha-1}\,(8\,\beta+1)\,,\, (1+\hat{\varepsilon})\sup_{\mathcal{Z}_r} v(.,t_0)\,\right\}\,.$$

□

Theorem 6.13 *Let* $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0) \in \partial M_{t_0}$. *Then*

$$\forall\, \varepsilon > 0 \quad \exists\, C_i > 0\,,\, 1 \leq i \leq 6\,, \quad \textit{depending only on } \; n, \varepsilon$$

such that

$$v(x,t) < 1 + \varepsilon \quad \forall (x,t) \textit{ with } \; \boldsymbol{F}(x,t) \in \mathcal{Z}_{r_1}(\boldsymbol{p}_0)\,,\; 0 \leq t - t_0 \leq \delta\,;$$

hereby
$r_1 = \min\{C_1\,\rho_0\,,\, C_2\,\rho_\Sigma\,,\, C_3\,\frac{\rho_\Sigma^2}{\rho_0}\}$ *and* $\delta = \min\{C_4\,\rho_0^2\,,\, C_5\,\rho_\Sigma^2\,,\, C_6\,\frac{\rho_\Sigma^4}{\rho_0^2}\}$. □

Thus again we have a gradient estimate on a time interval which is comparable to ρ_Σ^2, ρ_0^2. The Theorem immediately provides an estimate for the gradient of the *SGR* near the boundary point $\boldsymbol{p}_0$; but, as $\boldsymbol{\Sigma}$ in general will be curved, this graph representation (which was defined with respect to *Cartesian coordinates in* $\mathbb{R}^{n+1}$) will be defined on a subset $\cup_{t=t_0}^{t_0+\delta}\, \mathcal{U}_t \times \{t\}$ and *not* on a product structure $\mathcal{U} \times [t_0, t_0+\delta]$. A graph representation on a product structure can be obtained by working with generalized Gaussian coordinates as in section 2. The gradient bound with respect to the latter graph representation is equivalent to estimating

$$\tilde{v}(x,t) := \langle \boldsymbol{\nu}, \boldsymbol{\xi} \circ \boldsymbol{F}\rangle^{-1}(x,t)$$

from above. Now a connection between the above estimate for $v = \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle^{-1}$ and $\tilde{v}$ can easily be established by controlling $\langle \boldsymbol{\xi}, \boldsymbol{\omega}\rangle$. This is a simple exercise of analytic geometry, and it turns out that Theorem 6.13 exactly carries over to $\tilde{v}$. Thus we remark:

Remark 6.14 *Theorems 6.7 and 6.13 yield a gradient bound for appropriate local graph representations* $w : \mathcal{U}_{\hat{r}} \times [t_0, t_0+\delta] \longrightarrow \mathbb{R}$ *of the* M_t *around a given point* $\boldsymbol{F}(x_0,t_0)$ *which is independent of* $\boldsymbol{F}(x_0,t_0)$. *Hence, by Lemma 6.5, we obtain a* C^1*-estimate for* w *on a domain* $\mathcal{U}_{\hat{r}} \times [t_0, t_0+\hat{\delta}]$, *whose "dimensions" have the natural scaling behaviour in* ρ_0, *i.e.* $\hat{r} \sim \rho_0$, $\hat{\delta} \sim \rho_0^2$.

6.4 Long Time Existence of Solutions

Proof of Theorem 1.3: According to Theorem 2.1, we have a solution to equation 1.2 on a *maximal* time interval $[0, T_C)$. The only thing that remains to check is that, if

$$(*) \qquad \sup\{\, |A|^2(x,t) \,:\, x \in M^n \,,\, t \in [0,T_C)\,\} \;\leq\; \tfrac{1}{2\,n^2\rho_0} \;<\; \infty \,,$$

we can extend the solution to a larger time interval $[0, T_C + \delta)$. To this end, we start by defining the limit surface

$$\boldsymbol{F}(x,T_C) := \lim_{t\to T_C} \boldsymbol{F}(x,t) \,, \quad M_{T_C} := \boldsymbol{F}(M^n, T_C) \,,$$

which is possible due to equation $(*)$ and Lemma 6.5. By the gradient estimates Theorems 6.7, 6.13, Remark 6.14 and $(*)$ we see that M_{T_C} can be covered by local graph representations of $C^{1+\alpha,(1+\alpha)/2}$-functions. As in chapter 2, each of these functions fulfils a uniformly parabolic differential equation on its domain of definition; on the boundary, in addition we have a transversal Neumann condition. Hence from linear parabolic theory we conclude that M_{T_C} must actually be a *smooth* hypersurface. Now we can apply the short time existence Theorem 2.1 with M_{T_C} as initial hypersurface and extend the given solution to a solution on $[0, T_C + \delta)$. □

7 The Neumann Boundary Condition

Having established a very general existence result, we now consider *convex* hypersurfaces $\boldsymbol{\Sigma}$ and *strictly convex* initial hypersurfaces M_0. We will show that a given solution of equation 1.2 develops a singularity within a finite time interval $[0, T_C)$, and we are interested in the details of this process. To this end, we first analyze the given Neumann boundary condition

$$(7.1) \qquad \langle \boldsymbol{\nu}, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle (x,t) = 0 \quad \forall (x,t) \in \partial M^n \times [0,T] \,.$$

Differentiating this equation with respect to time yields

Proposition 7.1 *Let* $\boldsymbol{p} = \boldsymbol{F}(x,t) \in \partial M^n \times (0,T]$. *Then*

$$\langle \nabla H, \boldsymbol{\mu} \rangle (\boldsymbol{p}) = (H \cdot {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu}))(\boldsymbol{p}) \,.$$

Proof: $\frac{d}{dt}\boldsymbol{\nu} = \nabla H$ for $t > 0$, according to [13], Lemma 3.3. Moreover,

$$\tfrac{d}{dt}\boldsymbol{\mu}(\boldsymbol{F}(x,t)) = d\boldsymbol{\mu}(\tfrac{d}{dt}\boldsymbol{F}) = d\boldsymbol{\mu}(\boldsymbol{H}) = -H\, d\boldsymbol{\mu}(\boldsymbol{\nu}) \,.$$

Hence, from equation 7.1 we obtain

$$0 = \langle \nabla H, \boldsymbol{\mu} \rangle - H \,\langle \boldsymbol{\nu}, d\boldsymbol{\mu}(\boldsymbol{\nu}) \rangle = \langle \nabla H, \boldsymbol{\mu} \rangle - H\, {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu}) \,. \qquad \square$$

Next, we want to differentiate equation 7.1 with respect to a tangential space direction. This is best done using an orthonormal moving frame on M_t for a fixed $t \geq 0$: We define orthonormal vector fields $\boldsymbol{e}_1, ..., \boldsymbol{e}_{n+1}$ in $\mathbb{R}^{n+1}$ such that for $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap M_t$

$$\boldsymbol{e}_1(\boldsymbol{p}), ..., \boldsymbol{e}_{n-1}(\boldsymbol{p}) \in T_p(\boldsymbol{\Sigma} \cap M_t) \,, \quad \boldsymbol{e}_n(\boldsymbol{p}) = \boldsymbol{\mu}(\boldsymbol{p}) \,, \quad \boldsymbol{e}_{n+1}(\boldsymbol{p}) = \boldsymbol{\nu}(\boldsymbol{p}) \,.$$

This means that the $\boldsymbol{e}_\alpha$, $\boldsymbol{e}_i$, $\boldsymbol{e}_I$ are an orthonormal basis for the tangent bundles $T\mathbb{R}^{n+1}$, TM_t, $T(\boldsymbol{\Sigma} \cap M_t)$, respectively[1]. We describe the covariant derivative in $\mathbb{R}^{n+1}$ by

$$^0\nabla_{\boldsymbol{e}_\alpha} \boldsymbol{e}_\beta =: \boldsymbol{\Gamma}^\sigma_{\alpha\beta} \boldsymbol{e}_\sigma \,.$$

Note that the $\boldsymbol{\Gamma}^\sigma_{\alpha\beta}$ are *not* the usual Christoffel symbols which are deduced by using local coordinates; especially they are not symmetric with respect to the lower indices. Instead, using the relations between covariant derivatives and the second fundamental form:

$$\begin{aligned} ^0\nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j &= \nabla_{\boldsymbol{e}_i} \boldsymbol{e}_j - A(\boldsymbol{e}_i, \boldsymbol{e}_j)\boldsymbol{\nu} \,, & ^0\nabla_{\boldsymbol{e}_i} \boldsymbol{\nu} &= A(\boldsymbol{e}_i, \boldsymbol{e}_p)\boldsymbol{e}_p \,, \\ ^0\nabla_{\boldsymbol{e}_{\hat{\imath}}} \boldsymbol{e}_{\hat{\jmath}} &= {}^\Sigma\nabla_{\boldsymbol{e}_{\hat{\imath}}} \boldsymbol{e}_{\hat{\jmath}} - {}^\Sigma A(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{\jmath}})\boldsymbol{\mu} \,, & ^0\nabla_{\boldsymbol{e}_{\hat{\imath}}} \boldsymbol{\mu} &= {}^\Sigma A(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{p}})\, \boldsymbol{e}_{\hat{p}} \,, \end{aligned} \tag{7.2}$$

we obtain the following equations[2]:

$$\boldsymbol{\Gamma}^\sigma_{\alpha\beta} = -\boldsymbol{\Gamma}^\beta_{\alpha\sigma} \,, \quad \boldsymbol{\Gamma}^\nu_{ij} = -A(\boldsymbol{e}_i, \boldsymbol{e}_j) \,, \quad \boldsymbol{\Gamma}^\mu_{\hat{\imath}\hat{\jmath}} = -{}^\Sigma A(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{\jmath}}) \,. \tag{7.3}$$

Finally, a straightforward calculation yields for the covariant derivative of symmetric tensors T, T_Σ on M_t, $\boldsymbol{\Sigma}$, respectively:

$$\begin{aligned} (\nabla_{\boldsymbol{e}_k} T)(\boldsymbol{e}_i, \boldsymbol{e}_j) &= d(T(\boldsymbol{e}_i, \boldsymbol{e}_j))(\boldsymbol{e}_k) \; - \boldsymbol{\Gamma}^p_{ki}\, T(\boldsymbol{e}_p, \boldsymbol{e}_j) \; - \boldsymbol{\Gamma}^q_{kj}\, T(\boldsymbol{e}_i, \boldsymbol{e}_q) \,, \\ ({}^\Sigma\nabla_{\boldsymbol{e}_k} T_\Sigma)(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{\jmath}}) &= d(T_\Sigma(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{\jmath}}))(\boldsymbol{e}_{\hat{k}}) - \boldsymbol{\Gamma}^{\hat{p}}_{k\hat{\imath}}\, T_\Sigma(\boldsymbol{e}_{\hat{p}}, \boldsymbol{e}_{\hat{\jmath}}) - \boldsymbol{\Gamma}^{\hat{q}}_{k\hat{\jmath}}\, T_\Sigma(\boldsymbol{e}_{\hat{\imath}}, \boldsymbol{e}_{\hat{q}}) \,. \end{aligned} \tag{7.4}$$

Using these relations, we can differentiate the Neumann boundary condition in a tangential space direction: From equations 7.1 and 7.2, we derive

$$0 = \langle {}^0\nabla_{\boldsymbol{e}_I} \boldsymbol{\nu}, \boldsymbol{\mu} \rangle + \langle \boldsymbol{\nu}, {}^0\nabla_{\boldsymbol{e}_I} \boldsymbol{\mu} \rangle = A(\boldsymbol{e}_I, \boldsymbol{\mu}) + {}^\Sigma A(\boldsymbol{e}_I, \boldsymbol{\nu}) \,.$$

Since $\boldsymbol{e}_1(\boldsymbol{p}), ..., \boldsymbol{e}_{n-1}(\boldsymbol{p})$ span the tangent space $T_p(\boldsymbol{\Sigma} \cap M_t)$, we obtain

Proposition 7.2 *Let $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap M_t$, $\boldsymbol{v} \in T_p M_t$ and $\boldsymbol{w} := \boldsymbol{v} - \langle \boldsymbol{v}, \boldsymbol{\mu} \rangle \boldsymbol{\mu} \in T_p(M_t \cap \boldsymbol{\Sigma})$ be the projection of $\boldsymbol{v}$ onto $T_p\boldsymbol{\Sigma}$. Then*

$$\begin{aligned} A(\boldsymbol{w}, \boldsymbol{\mu}) &= -{}^\Sigma A(\boldsymbol{w}, \boldsymbol{\nu}) \quad \textit{and} \\ A(\boldsymbol{v}, \boldsymbol{\mu}) &= -{}^\Sigma A(\boldsymbol{w}, \boldsymbol{\nu}) + \langle \boldsymbol{v}, \boldsymbol{\mu} \rangle A(\boldsymbol{\mu}, \boldsymbol{\mu}) \,. \end{aligned}$$

□

Moreover, we have a relation between the covariant derivatives of the second fundamental forms of M_t and $\boldsymbol{\Sigma}$ in tangential directions:

[1] Recall the range of the various summation indices!

[2] Here and in the sequel we replace the indices $_{n,n+1}$ by the somewhat more suggestive notation $_{\mu,\nu}$

Proposition 7.3 *Let $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap M_t$ and $\boldsymbol{v}$, $\boldsymbol{w} \in T_p(\boldsymbol{\Sigma} \cap M_t)$. Then*

$$\begin{aligned}(\nabla_{\boldsymbol{v}} A)(\boldsymbol{\mu}, \boldsymbol{w}) &= -({}^{\Sigma}\nabla_{\boldsymbol{v}}\, {}^{\Sigma}A)(\boldsymbol{\nu}, \boldsymbol{w}) \\ &\quad -A(\boldsymbol{v}, \boldsymbol{e}_P)\, {}^{\Sigma}A(\boldsymbol{e}_P, \boldsymbol{w}) - A(\boldsymbol{w}, \boldsymbol{e}_P)\, {}^{\Sigma}A(\boldsymbol{e}_P, \boldsymbol{v}) \\ &\quad + A(\boldsymbol{\mu}, \boldsymbol{\mu})\, {}^{\Sigma}A(\boldsymbol{v}, \boldsymbol{w}) \quad + \; A(\boldsymbol{v}, \boldsymbol{w})\, {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu}) \; .\end{aligned}$$

Proof: Using equations 7.2 through 7.4 with $T = A$, we obtain

$$\begin{aligned}(\nabla_{\boldsymbol{e}_I} A)(\boldsymbol{\mu}, \boldsymbol{e}_J) &= d(A(\boldsymbol{\mu}, \boldsymbol{e}_J))(\boldsymbol{e}_I) - {}^{\Sigma}A(\boldsymbol{e}_I, \boldsymbol{e}_P)\, A(\boldsymbol{e}_P, \boldsymbol{e}_J) \\ &\quad -\boldsymbol{\Gamma}^Q_{IJ}\, A(\boldsymbol{\mu}, \boldsymbol{e}_Q) + {}^{\Sigma}A(\boldsymbol{e}_I, \boldsymbol{e}_J)\, A(\boldsymbol{\mu}, \boldsymbol{\mu}) \; .\end{aligned}$$

Analogously, for $T_\Sigma = {}^{\Sigma}A$:

$$\begin{aligned}({}^{\Sigma}\nabla_{\boldsymbol{e}_I}\, {}^{\Sigma}A)(\boldsymbol{\nu}, \boldsymbol{e}_J) &= d({}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{e}_J))(\boldsymbol{e}_I) - A(\boldsymbol{e}_I, \boldsymbol{e}_P)\, {}^{\Sigma}A(\boldsymbol{e}_P, \boldsymbol{e}_J) \\ &\quad -\boldsymbol{\Gamma}^Q_{IJ}\, {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{e}_Q) + A(\boldsymbol{e}_I, \boldsymbol{e}_J)\, {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu}) \; .\end{aligned}$$

Now, according to Proposition 7.2, we have $A(\boldsymbol{\mu}, \boldsymbol{e}_J) + {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{e}_J) \equiv 0$ on the boundary and hence $d\left(A(\boldsymbol{\mu}, \boldsymbol{e}_J) + {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{e}_J)\right)(\boldsymbol{e}_I) = 0$. Thus we obtain the result by summing the above equations. □

We finish this section by summarizing the results. If we define

$$h_{ij}(x,t) := A(\boldsymbol{e}_i, \boldsymbol{e}_j)(x,t)\,, \quad \nabla_k\, h_{ij}(x,t) := (\nabla_{\boldsymbol{e}_k} A)(\boldsymbol{e}_i, \boldsymbol{e}_j)(x,t)$$

etc., we have:

Theorem 7.4 *On the boundary $\boldsymbol{\Sigma} \cap M_t$, the components of the second fundamental forms of M_t and $\boldsymbol{\Sigma}$ are related by the equations*

$$\begin{aligned}(i) \qquad h_{\mu I} &= -h^{\Sigma}_{\nu I}\,, \\ (ii) \qquad \nabla_I\, h_{\mu J} &= -{}^{\Sigma}\nabla_I\, h^{\Sigma}_{\nu J} - h_{IP} h^{\Sigma}_{PJ} - h_{JP} h^{\Sigma}_{PI} + h_{\mu\mu} h^{\Sigma}_{IJ} + h_{IJ} h^{\Sigma}_{\nu\nu}\,, \\ (iii) \qquad \nabla_\mu H &= H\, h^{\Sigma}_{\nu\nu}\,.\end{aligned}$$

□

Note that the equations $(i), (ii)$ are valid for $t \geq 0$, whereas equation (iii) requires $t > 0$.

Remark 7.5 *Given a point $\boldsymbol{p} = \boldsymbol{F}(x,t) \in \boldsymbol{\Sigma} \cap M_t$, we can introduce local coordinates around $\boldsymbol{p}$ such that the vector fields $\boldsymbol{e}_I(\boldsymbol{p}) := \frac{\partial}{\partial x^I}\boldsymbol{F}(x,t)$, $\boldsymbol{e}_n(\boldsymbol{p}) := \frac{\partial}{\partial x^n}\boldsymbol{F}(x,t) = \boldsymbol{\mu}(\boldsymbol{p})$ and $\boldsymbol{e}_{n+1} := \boldsymbol{\nu}(x,t)$ are orthonormal at $\boldsymbol{p}$; this relates the formalisms of orthonormal moving frames and local coordinate systems.*

8 Curvature Bounds

This section is devoted to the study of the mean curvature of the hypersurfaces M_t of a given solution to equation 1.2. The evolution equation for H is

$$\frac{\partial}{\partial t} H = \mathbf{\Delta} H + |A|^2 H \ , \tag{8.1}$$

$\mathbf{\Delta}$ being the Laplace-Beltrami-operator of M^n with respect to the metric g (cf. [13], Corollary 3.5).

8.1 Estimates from Below

Theorem 8.1 *Let $H(.,0) \geq 0$. Then $H \geq 0$ on $M^n \times [0,T]$; moreover, if $H(.,0) \not\equiv 0$, we have $H > 0 \ \ \forall t > 0$.*

Proof: According to Proposition 7.1, we have $\langle \nabla H, \boldsymbol{\mu} \rangle(\boldsymbol{p}) = (H \cdot {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu}))(\boldsymbol{p})$. Thus, regarding the evolution equation 8.1, the conditions of the weak maximum principle Theorem 3.1 are fulfilled for H; hence $H \geq 0$ is preserved. The strong maximum principle Theorem 3.2 now yields the result. □

Theorem 8.2 *Let $\boldsymbol{\Sigma}$ be convex with respect to the exterior normal field $\boldsymbol{\mu}$, and let $H(.,0) \geq C_0 > 0$. Then $H \geq C_0$ for all $t \in [0, T_C)$, and the maximal time interval $[0, T_C)$ for the solution is bounded by $T_C \leq t_0 := \frac{n}{2\, C_0^2}$.*

Proof: The solution $u(t)$ of the ordinary initial value problem

$$\frac{d}{dt} u = \frac{1}{n}\, u^3 \ , \quad u(0) = C_0$$

is given by $u(t) = C_0 \cdot (1 - 2\, C_0^2\, t/n)^{-1/2}$. Now, using the simple algebraic identity $|A|^2 \geq \frac{1}{n} H^2$, we see that the function $\varphi(x,t) := H(x,t) - u(t)$ fulfils the following differential equation:

$$(\tfrac{\partial}{\partial t} - \mathbf{\Delta})\varphi \ \geq \ \tfrac{1}{n}(H^3 - u^3) \ = \ \tfrac{1}{n}(H^2 + H\, u + u^2)(H - u) \ =: \ c(x,t)\, \varphi \ .$$

Moreover, by the convexity of $\boldsymbol{\Sigma}$ and since $u > 0$, we have on the boundary

$$\langle \nabla \varphi, \boldsymbol{\mu} \rangle \ = \ {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu})\, H \ \geq \ {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu})\, H - {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu})\, u \ = \ {}^{\Sigma}A(\boldsymbol{\nu}, \boldsymbol{\nu})\, \varphi \ .$$

Thus we can apply the maximum principles Theorems 3.1, 3.2 to φ and obtain the result. □

It can easily be seen that the above estimate on the maximal existence interval is sharp for convex hypersurfaces *without* boundary, since the sphere S_R^n homothetically shrinks to a point in $T_C = \frac{R^2}{2\, n} = \frac{n}{2\, C_0^2} = t_0$. Thus, t_0 can also be reached in the boundary-case: Let $\mathcal{G}$ be a cone, $\boldsymbol{\Sigma} = \partial \mathcal{G}$ and M_0 part of a sphere S_R^n such that the center of the sphere coincides with the vertex of the cone. Note that the boundary condition is fulfilled as M_t shrinks homothetically to the vertex of the cone. Now, the natural question is whether this is the *only* setup with maximal existene time of the solution. We prove:

Theorem 8.3 *Let $\mathcal{G} \subset \mathbb{R}^{n+1}$ be a convex domain with smooth boundary $\boldsymbol{\Sigma}$. Let $\boldsymbol{F}$ be the solution of equation 1.2, where M_0 has strictly positive mean curvature. Then $T_C = t_0$ if and only if $\mathcal{G}$ is a cone with vertex $\boldsymbol{p}$ and $M_0 \subset S_R(\boldsymbol{p})$.*

Proof: We define $U(t) := \min_{M^n} H(x,t)$ and obtain from the proof of Theorem 8.2

$$U(t) \geq \frac{C_0}{\sqrt{1-2\,C_0^2\,t/n}}$$

Now, using Theorem 8.2, we conclude that in order to obtain $T_C = t_0$, actually equality must hold; thus $U(t)$ fulfils the ordinary initial value problem

$$\frac{d}{dt}U = \frac{1}{n}\,U^3\ , \quad U(0) = C_0\ .$$

Now we consider the function $\varphi(x,t) := H(x,t) - U(t)$. Since for every t due to the compactness of M^n there exists a zero of $\varphi(.,t)$, by the strong maximum principle we conclude $\varphi \equiv 0\ \forall t$; thus $H(.,t) \equiv U(t)$ on M_t. This in turn implies $\nabla H \equiv 0$ and Proposition 7.1 yields ${}^{\Sigma}A(\boldsymbol{\nu},\boldsymbol{\nu}) \equiv 0$, since $H \geq C_0 > 0$. Furthermore, from $\boldsymbol{\Delta} H \equiv 0$ and the evolution equations for H and U we conclude $|A|^2 = \frac{1}{n}H^2$ and thereby that the M_t are spheres. Finally the equation $\frac{\partial}{\partial t}\boldsymbol{\nu} = \nabla H$ ([13], Lemma 3.3) implies $\boldsymbol{\nu}(.,t) = const(t)$, and so these spheres must be concentric; the result follows. □

We remark that this proof equally holds for the boundaryless case: Here the only way to obtain the maximal existence interval of a solution which starts with strictly positive mean curvature is M_0 being a sphere.

8.2 Estimate from Above

As we have seen, it is not hard to establish lower bounds on H under reasonably weak conditions. In contrast, it is much harder to control the increasing of H from above; this can be done if the M_t are guaranteed to be *convex*. Now convexity of the M_t is preserved in the special case of $\boldsymbol{\Sigma}$ being umbilic, cf. Theorem 9.3. Thus in the sequel we will require $\boldsymbol{\Sigma}$ to be the boundary of a sphere or a hyperplane, but we emphasize that the following method also applies to *arbitrary* convex hypersurfaces $\boldsymbol{\Sigma}$, if by any means we can guarantee that convexity of the M_t is preserved.

In this section, we apply a technique due to *Tso* [24] who studied closed hypersurfaces moving under the Gauss-Kronecker-flow.

Assumption 8.4 *For the remainder of this section we let $\mathcal{G} = B_R^{n+1}$ or $\mathcal{G}$ be a half space and $\boldsymbol{\Sigma} = \partial\mathcal{G}$. Furthermore, $\boldsymbol{F}$ is the solution of equation 1.2 with $M_0 \subset \bar{\mathcal{G}}$ being an imbedded, convex hypersurface.*

Thus all M_t will be imbedded hypersurfaces due to Corollary 4.5, and since $H(.,0) \geq 0$, this will equally hold for $t > 0$ (Theorem 8.1). Finally, convexity of M_t for $t > 0$ follows from Theorem 9.3.

We start our considerations by establishing evolution equations for some auxiliary functions: First, there exists $\delta_0 > 0$ and a neighbourhood $\mathcal{U} := \{\boldsymbol{p} \in \mathbb{R}^{n+1} : \mathrm{dist}_{n+1}(\boldsymbol{p}, \boldsymbol{\Sigma}) < \delta_0\}$ of $\boldsymbol{\Sigma}$ such that the distance function

$$g : \mathcal{U} \longrightarrow \mathbb{R}\,, \quad g(\boldsymbol{p}) := \mathrm{dist}_{n+1}(\boldsymbol{p}, \boldsymbol{\Sigma})$$

is well defined on $\mathcal{U}$ and smooth on $\mathcal{U} \backslash \boldsymbol{\Sigma}$ and fulfils the estimates

$$\|Dg\| \leq 1\,, \quad |D^2 g| < C_1\,, \tag{8.2}$$

where $C_1 = C_1({}^{\Sigma}A) > 0$ depends only on the curvature of $\boldsymbol{\Sigma}$. Moreover, if $\boldsymbol{p} \in \boldsymbol{\Sigma}$, we have

$$\langle Dg, \boldsymbol{\mu} \rangle(\boldsymbol{p}) = -1\,. \tag{8.3}$$

Lemma 8.5 *(The distance function) The function $d : M^n \times [0,T] \longrightarrow \mathbb{R}$, $d(x,t) := (g \circ \boldsymbol{F})(x,t)$ fulfils the evolution inequality*

$$|(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta}) d(x,t)| \leq n\, C_2\,,$$

where the constant $C_2 > 0$ depends only on ${}^{\Sigma}A$. Furthermore,

$$\langle \nabla d, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle = -1 \quad \textit{on} \quad \partial M^n \times [0,T]\,, \quad \textit{and} \quad \|\nabla d\| \leq 1 \quad \textit{on} \quad \mathcal{U}\,.$$

Proof: A straightforward calculation yields

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta}) d(x,t) = -g^{ij} \cdot (D^2 g \circ \boldsymbol{F})(\tfrac{\partial}{\partial x^i}\boldsymbol{F}, \tfrac{\partial}{\partial x^j}\boldsymbol{F})(x,t)\,;$$

thus the evolution inequality for d follows from equation 8.2. Analogously, the boundary behaviour is a consequence of

$$\langle \nabla d, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle(x,t) = \langle Dg \circ \boldsymbol{F}, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle(x,t)\,.$$

Finally, $\|\nabla d\|^2 = \|Dg\|^2$ on $\mathcal{U}$; again, equation 8.2 yields the result. □

We remark that for convex hypersurfaces $\boldsymbol{\Sigma}$ we have $(\frac{\partial}{\partial t} - \boldsymbol{\Delta}) d \geq 0$; unfortunately, this turns out to be the wrong sign for the desired estimates.

Lemma 8.6 *(The support function) The function $s : M^n \times [0,T] \longrightarrow \mathbb{R}$, $s(x,t) := \langle \boldsymbol{\nu}, \boldsymbol{F} \rangle(x,t)$ fulfils*

$$\begin{aligned} (\tfrac{\partial}{\partial t} - \boldsymbol{\Delta}) s(x,t) &= -|A|^2 + 2\, s\, H \quad \forall t > 0\,, \\ \langle \nabla s, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle &= A(\boldsymbol{F}^T, \boldsymbol{\mu} \circ \boldsymbol{F}) \quad \textit{on the boundary}\,; \end{aligned}$$

hereby $\boldsymbol{F}^T$ is the projection of $\boldsymbol{F}$ onto the tangent hyperplane of M_t.

Proof: The evolution equation was proved by *Ecker* and *Huisken* [6]. The boundary condition is obtained by a straightforward calculation. □

Now we let $\boldsymbol{p} \in \bar{\mathcal{G}}$ be the origin of a coordinate system in $\mathbb{R}^{n+1}$, and we choose $\sigma > 0$ and $T_\sigma > 0$ such that $s(.,t) \geq 2\,\sigma$ on $[0, T_\sigma]$. (This is possible due to the convexity of the imbeddings M_t.) We then consider the function

$$f : M^n \times [0, T_\sigma] \longrightarrow \mathbb{R}\,, \; f(x,t) := H\, \frac{\psi \circ d}{s - \sigma}(x,t)\,,$$

$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ being a C^2-function which will be chosen later.

Lemma 8.7 *For $t > 0$, we have*

$$\begin{aligned}(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f &\leq -\tfrac{\sigma\,(s-\sigma)}{n\,(\psi\circ d)^2}\,f^3 + \left(\tfrac{2}{\psi\circ d} + 2\,\|\boldsymbol{F}\|\,|\tfrac{\psi'}{\psi^2}\circ d|\right) f^2 \\ &\quad + \left(n\,C_2\,|\tfrac{\psi'}{\psi}\circ d| + |\tfrac{2\,\psi'^2-\psi\,\psi''}{\psi^2}\circ d|\right) f \\ &\quad + \langle \nabla Q, \nabla f\rangle\,, \\ \langle \nabla f, \boldsymbol{\mu}\rangle &\leq f\,\left(C_\Sigma - \tfrac{\psi'}{\psi}(0) + \tfrac{1}{s-\sigma}\,C_\Sigma\,\|\boldsymbol{F}\|\right) \quad \textit{on the boundary}\,;\end{aligned}$$

hereby $Q := 2\,\log(\frac{s-\sigma}{\psi\circ d})$ *and* $C_\Sigma := \|{}^\Sigma A\|_\infty = \sup\{\,|{}^\Sigma A|(\boldsymbol{p}) : \boldsymbol{p} \in \boldsymbol{\Sigma}\,\}$.

Proof: Using equation 8.1 and Lemmata 8.5, 8.6, we obtain by a straightforward calculation

$$\begin{aligned}(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f &\leq f\,\tfrac{1}{s-\sigma}\,(-\sigma\,|A|^2 + 2\,H) + f\,|\tfrac{\psi'}{\psi}\circ d|\,n\,C_2 \\ &- f\,\tfrac{2}{s-\sigma}\,\left(\tfrac{\psi'}{\psi}\circ d\right)\langle\nabla s, \nabla d\rangle + f\,\left(\tfrac{2\,\psi'^2-\psi\,\psi''}{\psi^2}\circ d\right)\|\nabla d\|^2 \\ &+ 2\,\tfrac{\psi\circ d}{s-\sigma}\,\langle\nabla\tfrac{s-\sigma}{\psi\circ d}, \nabla f\rangle\,.\end{aligned}$$

Moreover, we compute $\langle\nabla s, \nabla d\rangle = A(\boldsymbol{F}^T, \nabla d\rangle$, and thereby

$$|\langle\nabla s, \nabla d\rangle| \leq |A|\,\|\boldsymbol{F}\|\,\|\nabla d\|\,.$$

Putting this into the above evolution equation for f and using the inequality $\frac{1}{n}H^2 \leq |A|^2 \leq H^2$ (which is valid since the M_t are *convex* hypersurfaces), we obtain

$$\begin{aligned}(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f &\leq f\,\tfrac{1}{s-\sigma}\,(-\tfrac{\sigma}{n}\,H^2 + 2\,H) + f\,|\tfrac{\psi'}{\psi}\circ d|\,n\,C_2 \\ &+ f\,\tfrac{2}{s-\sigma}\,|\tfrac{\psi'}{\psi}\circ d|\,H\,\|\boldsymbol{F}\| + f\,|\tfrac{2\,\psi'^2-\psi\,\psi''}{\psi^2}\circ d| \\ &+ \langle\nabla Q, \nabla f\rangle\,.\end{aligned}$$

Finally, using $H = f\,\frac{s-\sigma}{\psi\circ d}$, the evolution equation follows.

On the boundary, we have

$$\nabla f = \tfrac{\psi(0)}{s-\sigma}\,\nabla H + H\,\left(\tfrac{\psi'(0)}{s-\sigma}\,\nabla d - \tfrac{\psi(0)}{(s-\sigma)^2}\,\nabla s\right)\,.$$

By Proposition 7.1 and Lemmata 8.5, 8.6, we arrive at

$$\langle\nabla f, \boldsymbol{\mu}\rangle = \tfrac{\psi(0)}{s-\sigma}\,{}^\Sigma A(\boldsymbol{\nu}, \boldsymbol{\nu})\,H - H\,\left(\tfrac{\psi'(0)}{s-\sigma} + \tfrac{\psi(0)}{(s-\sigma)^2}\,A(\boldsymbol{F}^T, \boldsymbol{\mu})\right)\,.$$

Corollary 7.2 gives

$$A(\boldsymbol{F}^T, \boldsymbol{\mu}) = \langle\boldsymbol{F}^T, \boldsymbol{\mu}\rangle\,A(\boldsymbol{\mu}, \boldsymbol{\mu}) - {}^\Sigma A(\boldsymbol{F}^T - \langle\boldsymbol{F}^T, \boldsymbol{\mu}\rangle\boldsymbol{\mu}\,, \boldsymbol{\nu})\,;$$

here, $A(\boldsymbol{\mu}, \boldsymbol{\mu}) \geq 0$, as the M_t are convex. Furthermore, $\boldsymbol{\Sigma}$ bounds a convex domain, and thus the boundary condition $\langle\boldsymbol{\mu}, \boldsymbol{\nu}\rangle = 0$ yields [3] $\langle\boldsymbol{F}^T, \boldsymbol{\mu}\rangle = \langle\boldsymbol{F}, \boldsymbol{\mu}\rangle \geq 0$. Hence

$$A(\boldsymbol{F}^T, \boldsymbol{\mu}) \geq -C_\Sigma\,\|\boldsymbol{F}^T - \langle\boldsymbol{F}^T, \boldsymbol{\mu}\rangle\boldsymbol{\mu}\| \geq -C_\Sigma\,\|\boldsymbol{F}\|\,,$$

[3] Recall that the origin of our coordinate system is *in* $\bar{\mathcal{G}}$!

and putting this into the above expression for $\langle \nabla f, \boldsymbol{\mu} \rangle$ leads to the stated boundary condition. □

Before stating the Theorem, we observe that, with respect to the choice of our coordinates, the hypersurfaces M_t "shrink" as t increases; more precisely, we have

$$(8.4) \quad \sup\{\|\boldsymbol{F}(x,t)\| : (x,t) \in M^n \times [0,T_\sigma]\} = \sup\{\|\boldsymbol{F}(x,0)\| : x \in M^n\} .$$

This follows from equation 1.2 by calculating $\frac{d}{dt}\|\boldsymbol{F}\|^2(x,t) = -2\,H\,s \leq 0$.

We now show that the mean curvature H is bounded from above as long as the support function s is strictly positive; geometrically this latter condition means that the convex body bounded by M_t and $\boldsymbol{\Sigma}$ contains a small ball.

Theorem 8.8 *(An upper bound on the growth of H) Let $\mathcal{G} \subset \mathbb{R}^{n+1}$ be a ball or a half space, $\boldsymbol{\Sigma} := \partial\mathcal{G}$, $C_\Sigma := \|^\Sigma A\|_\infty$ and $M_0 \subset \bar{\mathcal{G}}$ a convex imbedded hypersurface. Let $\boldsymbol{F} : M^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ be a solution of equation 1.2 with $\boldsymbol{F}(M^n,0) = M_0$. Finally, choose [4] $\sigma \in (0\,,\, \sup_{M^n} \|\boldsymbol{F}(.,0)\| \cdot \min\{C_\Sigma\,\delta_0, 1\})$ and $T_\sigma \in (0,T]$ such that $s(.,t) \geq 2\,\sigma \;\; \forall t \in [0,T_\sigma]$. Then*

$$H(x,t) \leq \max\{\, \sup_{M^n} H(.,0)\;,\; C_0\,\tfrac{n}{\sigma^3}\,\} \quad \forall\,(x,t) \in M^n \times [0,T_\sigma]\;,$$

where C_0 is a positive constant depending only on the curvature bound C_Σ and the initial imbedding M_0.

Proof: Define $k_0 := \sup_{M^n} \|\boldsymbol{F}(.,0)\|$, $\delta := \frac{\sigma}{C_\Sigma\,k_0} < \delta_0$, $C := \frac{1}{4\,\delta^3}$ and

$$\psi(z) := \begin{cases} C\,(z-\delta)^3 + 1/2\;, & \text{if } z \leq \delta\;, \\ 1/2\;, & \text{otherwise.} \end{cases}$$

Then $\psi \in C^2(\mathbb{R}_+)$ and $\psi \in [1/4, 1/2]$. We now consider the function f of Lemma 8.7:

(*i*) Using equation 8.4 and $\sigma < k_0$, the boundary behaviour is

$$\langle \nabla f, \boldsymbol{\mu} \rangle \;\leq\; f\,\left(C_\Sigma - \tfrac{3\,C\,\delta^2}{1/2 - C\,\delta^3} + C_\Sigma\,\tfrac{k_0}{\sigma}\right) \;<\; f\,\left(C_\Sigma\,\tfrac{2\,k_0}{\sigma} - \tfrac{3}{\delta}\right) \;<\; 0\;;$$

hence a maximum of f cannot occur on the boundary.

(*ii*) For any point $(x,t) \in [0,T_\sigma]$ with $\mathrm{dist}_{n+1}(\boldsymbol{F}(x,t), \boldsymbol{\Sigma}) > \delta$, we have $\psi \equiv 1/2$ and thus, using $s \geq 2\,\sigma$ on $[0,T_\sigma]$, the evolution equation for f yields

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f \;\leq\; 4\,f^2\,(-\tfrac{\sigma^2}{n} f + 1) \;+\; \langle \nabla Q, \nabla f \rangle\;.$$

Hence f cannot develop a new maximum at (x,t) with $f \geq \frac{n}{\sigma^2}$.

[4] δ_0 is as defined at the beginning of this subsection.

(*iii*) Now let $\mathrm{dist}_{n+1}(\boldsymbol{F}(x,t), \boldsymbol{\Sigma}) \leq \delta$. Here we have $2\,\sigma \leq s \leq \|\boldsymbol{F}\| \leq k_0$ and $1/4 \leq \psi \leq 1/2$, and thereby

$$\begin{aligned}(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f \;\; \leq \;\; -\tfrac{4\sigma^2}{n}\, f^3 \;+\; (8 + 96\, C\, \delta^2\, k_0)\, f^2 \\ + 12\, C\, \delta\, (n\, \delta\, C_2 + 16\, C\, \delta^3 + 4)\, f \\ + \langle \nabla Q, \nabla f \rangle \;.\end{aligned}$$

By *Young*'s inequality,

$$(8 + 96\, C\, \delta^2\, k_0)\, f \;\leq\; \tfrac{2\,\sigma^2}{n} f^2 \;+\; 8\, \tfrac{n}{\sigma^2}\, (1 + 12\, C\, \delta^2\, k_0)^2 \;,$$

and using $\delta = \frac{\sigma}{C_\Sigma\, k_0}$, $\sigma < k_0$ and $C\,\delta^3 = 1/4$, we arrive at

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})f \;\; \leq \;\; f\, \left(-\tfrac{2\,\sigma^2}{n} f^2 + 2\, n\, C_3^2\, \tfrac{1}{\sigma^4}\right) \;+\; \langle \nabla Q, \nabla f \rangle$$

with a constant $C_3 > 0$ depending only on M_0 and C_Σ. Thus, the occurence of new maxima with $f \geq C_3\, \frac{n}{\sigma^3}$ can be ruled out.

(*iv*) From (*i*)-(*iii*) we conclude that H cannot develop a new maximum with

$$H \geq \tfrac{n\, C_3}{\sigma^3}\, 4\, k_0 \;=:\; C_0\, \tfrac{n}{\sigma^3}$$

within the time interval $[0, T_\sigma]$. □

Remark 8.9 *The above proof and thus the Theorem hold for arbitrary convex domains $\mathcal{G}$, if we can guarantee that convexity is preserved for the hypersurfaces M_t.*

Remark 8.10 *Since for convex hypersurfaces $|A|^2 \leq H^2$, by controlling the mean curvature we automatically control the entire second fundamental form of the M_t.*

9 Convexity and Pinching

Here we study the properties of the second fundamental form of a given solution to the mean curvature flow equation 1.2: We show that (under suitable conditions on $\boldsymbol{\Sigma}$) convexity and a pinching estimate are preserved. Hereby we require $\boldsymbol{\Sigma}$ to be an umbilic hypersurface in $\mathbb{R}^{n+1}$, as in this case by the results of section 7 we can completely control the behaviour of the various tensors on the boundary. We will work with the maximum principle Theorem 3.3 and begin by summarizing some evolution equations from [13]:

$$\tfrac{\partial}{\partial t} g_{ij} \;=\; -2\, H\, h_{ij} \;, \tag{9.1}$$

$$\tfrac{\partial}{\partial t} h_{ij} \;=\; \boldsymbol{\Delta} h_{ij} - 2\, H\, h_{ik}\, g^{kl}\, h_{lj} + |A|^2\, h_{ij} \;, \tag{9.2}$$

$$\tfrac{\partial}{\partial t} |A|^2 \;=\; \boldsymbol{\Delta} |A|^2 - 2\, |\nabla A|^2 + 2\, |A|^4 \;. \tag{9.3}$$

Assumption 9.1 *Throughout this section, let $\boldsymbol{\Sigma}$ be the boundary of a ball $B^{n+1}_{1/\alpha}$ or a hyperplane in $\mathbb{R}^{n+1}$. In the former case, let $M_0 \subset \bar{B}^{n+1}_{1/\alpha}$, thereby $\boldsymbol{\Sigma}$ being convex.*

Thus we have

$$ {}^{\Sigma}A(\boldsymbol{v},\boldsymbol{w})(\boldsymbol{p}) = \alpha\, {}^{\Sigma}g(\boldsymbol{v},\boldsymbol{w})(\boldsymbol{p}) \quad \forall \boldsymbol{p} \in \boldsymbol{\Sigma}\,, \quad \forall \boldsymbol{v},\boldsymbol{w} \in T_p\boldsymbol{\Sigma}\,, $$

or, equivalently, for the components with respect to a parametrisation,

$$ h^{\Sigma}_{\hat{\imath}\hat{\jmath}}(\boldsymbol{p}) = \alpha\, {}^{\Sigma}g_{\hat{\imath}\hat{\jmath}}(\boldsymbol{p}) \quad \forall \boldsymbol{p} \in \boldsymbol{\Sigma}\,, \quad \forall \hat{\imath},\hat{\jmath}\,. $$

Thereby $\alpha \geq 0$ due to the convexity of $\boldsymbol{\Sigma}$.

Theorem 9.2 *Let $\boldsymbol{\Sigma}$ be umbilic, $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap M_t$, and choose local coordinates around $\boldsymbol{p}$ as discussed in Remark 7.5. Then the following relations hold:*

$$ \begin{array}{lrcl} (i) & h_{\mu I} & = & 0\,, \\ (ii') & \nabla_\mu\, h_{IJ} & = & \alpha\,(-h_{IJ} + h_{\mu\mu}\,\delta_{IJ})\,, \\ (ii'') & \nabla_\mu\, h_{\mu\mu} & = & \alpha\,(2\,H - n\,h_{\mu\mu})\,, \\ (iii) & \nabla_\mu\, H & = & \alpha\, H\,. \end{array} $$

Proof: (i) follows immediately from $h^{\Sigma}_{\nu I} = \alpha\,\delta_{\nu I} = 0$ and Theorem 7.4,(i). Moreover, by *Ricci*'s Lemma, ${}^{\Sigma}\nabla_{\hat{\imath}}\, {}^{\Sigma}g_{\hat{p}\hat{q}} \equiv 0$; hence ${}^{\Sigma}\nabla_I\, h^{\Sigma}_{\nu J} = \alpha\, {}^{\Sigma}\nabla_I\, {}^{\Sigma}g_{\nu J} = 0$. Theorem 7.4,$(ii)$ thus yields $\nabla_I\, h_{\mu J} = \alpha\,(-h_{IJ} + h_{\mu\mu}\,\delta_{IJ})$, and $(ii)'$ is a consequence of the *Codazzi* equations $\nabla_I\, h_{\mu J} = \nabla_\mu\, h_{IJ}$. (iii) follows from $h^{\Sigma}_{\nu\nu} = \alpha\,\delta_{\nu\nu} = \alpha$, and combining $(ii)'$ and Theorem 7.4,(iii) yields (ii). □

Thus if $\boldsymbol{\Sigma}$ is umbilic, the *normal* derivative of the second fundamental form of M_t can be expressed essentially in terms of the curvature operator itself. Furthermore, in this case the second fundamental form A decomposes on $\boldsymbol{\Sigma}$, cf. Lemma 3.4.

9.1 Convexity

Theorem 9.3 *Let $\boldsymbol{\Sigma}$ be convex and umbilic. Then convexity is preserved for any solution of the mean curvature flow equation 1.2.*

Proof: We want to apply Theorem 3.3. To this end, we set $M_{ij} := h_{ij}$, $N_{ij} := -2\,H\,h_{ik}\,g^{kl}\,h_{lj} + |A|^2\,h_{ij}$, $u^k = 0$ and observe that the required evolution equation and the zero eigenvector condition are fulfilled. Furthermore, $h_{ij}(.,0) \geq 0$ by assumption, and thus in order to apply Theorem 3.3 we only have to check the boundary condition. Let $\boldsymbol{p} := \boldsymbol{F}(x_0,t_0) \in \boldsymbol{\Sigma}$ and choose local coordinates around $\boldsymbol{p}$ as in Remark 7.5. As A decomposes on $\boldsymbol{\Sigma}$ (cf. Theorem 9.2,(i)), we can use Lemma 3.4:

(iv') If $v := v^I \partial_I$ is a minimal eigenvector of h_{ij} with $h_{ij}v^iv^j = h_{IJ}v^Iv^J = -\delta\|v\|^2 \leq 0$, we compute from Theorem 9.2,$(ii)'$ $\nabla_\mu h_{ij}v^iv^j = \alpha\,(\delta + h_{\mu\mu})\,\|v\|^2$. As v is minimal, $h_{\mu\mu} \geq -\delta$ and thus $\nabla_\mu h_{ij}\,v^i\,v^j \geq 0$.

(iv'') Now let $h_{\mu\mu} \leq 0$ be minimal. Then Theorem 9.2,(ii'') yields $\nabla_\mu h_{\mu\mu} \geq 2\,\alpha\,H$. By assumption, $H(.,0) \geq 0$, and by Theorem 8.1, this is preserved; thus $\nabla_\mu h_{\mu\mu} \geq 0$.

Hence the result follows from Theorem 3.3 and Lemma 3.4. □

Proposition 9.4 *Let $\boldsymbol{\Sigma}$ be convex and umbilic and $C \leq \frac{2}{n}\,\min_{M^n} H(.,0)$. Then the estimate*

$$h_{ij} \geq C\,g_{ij}$$

is preserved for any solution of equation 1.2.

Proof: Let $M_{ij} := h_{ij} - C\,g_{ij}$. A straightforward calculation using equations 9.1, 9.2 yields

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})M_{ij} \geq -2\,H\,M_{ik}\,g^{kl}\,M_{lj} + (|A|^2 - 2\,CH)\,M_{ij}\ ,$$

thus a "suitable" evolution equation (with respect to Theoren 3.3) and the zero eigenvector condition are fulfilled. By assumption, $M_{ij}(.,0) \geq 0$; thus again the only thing to check is the boundary condition of Lemma 3.4. Using Theorem 9.2 and *Ricci*'s Lemma, we compute

$$\begin{aligned}\nabla_\mu M_{IJ} &= \alpha\,(-M_{IJ} + M_{\mu\mu})\ ,\\ \nabla_\mu M_{\mu\mu} &= \alpha\,(2\,H - n\,M_{\mu\mu} - n\,C)\ .\end{aligned}$$

Now, if $v = v^I\,\partial_I$ is a minimal eigenvector with $M_{IJ}\,v^I\,v^J = -\delta\,\|v\|^2 \leq 0$, we have $M_{\mu\mu} \geq -\delta$ and thus $\nabla_\mu M_{IJ}\,v^I\,v^J = \alpha\,(\delta + M_{\mu\mu})\|v\|^2 \geq 0$. If, on the other hand, $M_{\mu\mu} \leq 0$ is minimal, we compute $\nabla_\mu M_{\mu\mu} \geq 2\,\alpha\,(H - \min_{M^n} H(.,0))$. As $\boldsymbol{\Sigma}$ is convex, we have $H \geq \min_{M^n} H(.,0)$ by Theorem 8.2; thus $\nabla_\mu M_{\mu\mu} \geq 0$.

Hence we can apply Theorem 3.3 and Lemma 3.4 and obtain the result. □

9.2 Pinching

Now we want to establish various pinching estimates. *Huisken* [13] in the boundaryless case proved that for any $0 < \varepsilon \leq 1/n < \kappa < 1$ the estimate

$$\varepsilon\,H\,g_{ij} \leq h_{ij} \leq \kappa\,H\,g_{ij}$$

is preserved. Note that this estimate is optimal as ε and κ cannot exceed the above stated range for any convex hypersurface. Now, in our case, such an optimal estimate cannot hold: Let M_0 be a subset of S^n_R and $\boldsymbol{\Sigma} = S^n_{1/\alpha}$ such that the Neumann boundary condition is fulfilled. Then $h_{ij} = \frac{1}{n}H\,g_{ij}$, i.e. $\varepsilon = 1/n$; but an elementary geometric consideration shows that under the mean curvature flow M_t will *not* be a sphere for $t > 0$; thus the optimal pinching is not preserved.

Theorem 9.5 *Let $\boldsymbol{\Sigma}$ be convex and umbilic, and let $\kappa \geq \frac{2}{n+1}$. Then equation 1.2 preserves the pinching condition*

$$h_{ij} \leq \kappa\, H\, g_{ij}\ .$$

Proof:

Let $M_{ij} := \kappa\, H\, g_{ij} - h_{ij}$. By a straightforward calculation,

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta}) M_{ij} = 2\, H\, M_{ik}\, g^{kl}\, M_{lj} + (|A|^2 - 2\,\kappa\, H^2)\, M_{ij}\ ,$$

i.e. the evolution equation and the zero eigenvector condition are fulfilled. By assumption $M_{ij}(.,0) \geq 0$, and thus it remains to check the boundary condition. Here, Theorem 9.2 yields

$$\begin{aligned} \nabla_\mu\, M_{IJ} &= \alpha\, (\kappa\, H\, \delta_{IJ} - M_{IJ} + M_{\mu\mu}\, \delta_{IJ})\ , \\ \nabla_\mu\, M_{\mu\mu} &= \alpha\ (-n\, M_{\mu\mu} + ((n{+}1)\kappa - 2)\, H)\ . \end{aligned}$$

Thus, if $v = v^I\, \partial_I$ is a minimal eigenvector with $M_{IJ} v^I v^J = -\delta\, \|v\|^2 \leq 0$, we have $\nabla_\mu\, M_{IJ} v^I v^J = \alpha\, (\kappa\, H + \delta + M_{\mu\mu})\, \|v\|^2 \geq 0$, as $H \geq 0$ is preserved. On the other hand, if $M_{\mu\mu} \leq 0$ is minimal, we obtain $\nabla_\mu\, M_{\mu\mu} \geq \alpha\ ((n{+}1)\kappa - 2)\ H \geq 0$, as $\kappa \geq \frac{2}{n+1}$. Thus Theorem 3.3 and Lemma 3.4 yield the result. □

Now we would like to proceed as in [13] and establish a pinching estimate from below using the function $M_{ij} = h_{ij} - \varepsilon\, H\, g_{ij}$; unfortunately, the normal derivative of this function at the boundary is $\nabla_\mu\, M_{IJ} v^I v^J \geq -\alpha\, \varepsilon\, H\, \|v\|^2$, if v is a minimal tangential eigenvector; this is not sufficient to guarantee for the requirements of Lemma 3.4. It turns out that the negative term occurs from the boundary behaviour of H - thus we must replace H by a function with a "better" boundary behaviour.

Lemma 9.6 *Let $\boldsymbol{\Sigma}$ be convex and umbilic. Furthermore, let M_0 be strictly convex. Then*

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})|A| \leq |A|^3\ ,$$

and on the boundary (with respect to the "usual" coordinates)

$$\nabla_\mu\, |A| = \alpha\, |A|^{-1}\, (3\, H\, h_{\mu\mu} - |A|^2 - n\, h_{\mu\mu}^2)\ .$$

Proof: First note that $|A|^2 > 0$ is preserved (Theorem 9.4). Thus $|A|$ is a smooth function, and we compute

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta})|A| = \tfrac{1}{|A|}\, (\|\nabla |A|\|^2 - |\nabla A|^2) + |A|^3\ .$$

The evolution equation then follows from the *Cauchy-Schwarz* inequality $\|\nabla |A|\|^2 \leq |\nabla A|^2$.

Furthermore, using Theorem 9.2, a simple calculation yields

$$\nabla_\mu\, |A|^2 = 2\, \alpha\, (3\, H\, h_{\mu\mu} - |A|^2 - n\, h_{\mu\mu}^2)$$

on $\boldsymbol{\Sigma}$, and the result follows from $\nabla_\mu\, |A| = \frac{1}{2\, |A|}\, \nabla_\mu\, |A|^2$. □

Theorem 9.7 *Let $\boldsymbol{\Sigma}$ be convex and umbilic, and let M_0 be strictly convex. Then for $\varepsilon \leq \min\{\frac{1}{3\sqrt{n}}, \frac{2}{n}\}$, equation 1.2 preserves the pinching estimate*

$$h_{ij} \geq \varepsilon\, |A|\, g_{ij} \ .$$

Proof: We define $M_{ij} := h_{ij} - \varepsilon\, |A|\, g_{ij}$ and compute, using the above Lemma

$$(\tfrac{\partial}{\partial t} - \boldsymbol{\Delta}) M_{ij} \geq -2\, H\, M_{ik}\, g^{kl}\, M_{lj} + (|A|^2 - 2\,\varepsilon\, H\, |A|)\, M_{ij} \ ,$$

and at a boundary point

$$\begin{aligned} \nabla_\mu\, M_{IJ} &= \alpha\, \Big(-M_{IJ} + M_{\mu\mu}\, (1 - 3\,\varepsilon\, H\, |A|^{-1})\, \delta_{IJ} \\ &\qquad + \varepsilon\, (-3\,\varepsilon\, H + |A| + n\, |A|^{-1}\, h_{\mu\mu}^2)\, \delta_{IJ} \Big) \ , \\ \nabla_\mu\, M_{\mu\mu} &= \alpha\, \Big(-(n + 3\,\varepsilon\, H\, |A|^{-1})\, M_{\mu\mu} + (2 - 3\,\varepsilon^2)\, H \\ &\qquad - \varepsilon\, (n-1)\, |A| + n\, \varepsilon\, |A|^{-1}\, h_{\mu\mu}^2 \Big) \ . \end{aligned}$$

As convexity is preserved, we have $\frac{1}{n} H^2 \leq |A|^2 \leq H^2$; furthermore, by assumption, $\varepsilon \leq \frac{1}{3\sqrt{n}}$ and thus $\lambda := 1 - 3\,\varepsilon\, H\, |A|^{-1} \in [0, 1]$.

Now, if $v = v^I\, \partial_I$ is a minimal eigenvector with $M_{IJ}\, v^I\, v^J = -\delta\, \|v\|^2 \leq 0$,

$$\nabla_\mu\, M_{ij} v^i v^j \geq \alpha\, (\delta + M_{\mu\mu}\, \lambda)\, \|v\|^2 \ ;$$

and $M_{\mu\mu} \geq -\delta$ (by minimality of $-\delta$) and $0 \leq \lambda \leq 1$ yield $\nabla_\mu\, M_{ij} v^i v^j \geq 0$.

If $M_{\mu\mu} = -\delta \leq 0$ is minimal, we have

$$\nabla_\mu\, M_{\mu\mu} \geq \alpha\, |A|\, \Big(2 - 3\,\varepsilon^2 - \varepsilon\, (n-1) \Big) \geq 0 \ ,$$

by the assumption on ε.

Again the result follows from Theorem 3.3 and Lemma 3.4. □

Corollary 9.8 *Let $\boldsymbol{\Sigma}$ be convex and umbilic, M_0 strictly convex and $\varepsilon \leq \min\{\frac{1}{3\sqrt{n}}, \frac{2}{n}\}$. Then the estimate $h_{ij}(.,0) \geq \varepsilon\, (H\, g_{ij})(.,0)$ implies*

$$h_{ij} \geq \tfrac{\varepsilon}{\sqrt{n}} H\, g_{ij} \quad \text{on} \quad M^n \times [0, T] \ .$$

Proof: This follows directly from Theorem 9.7 and the estimate $\frac{1}{n} H^2 \leq |A|^2 \leq H^2$ for convex hypersurfaces. □

10 Convergence

We finally turn to the study of the singularity which develops as $t \to T_C$. The main results have been stated in Theorem 1.4 and Proposition 1.5; thus it essentially remains to give the proofs. As before, we let $\boldsymbol{\Sigma}$ be the boundary of a ball or a half space in $\mathbb{R}^{n+1}$; however, in this section we will *not* use the fact that $\boldsymbol{\Sigma}$ is umbilic. Thus this latter property is needed only to preserve convexity and pinching of the solutions M_t, cf. section 9. We begin with summarizing the assumptions we will need throughout this section:

Assumption 10.1 *Let $\boldsymbol{\Sigma}$ be convex and umbilic and M_0 a strictly convex, imbedded hypersurface. Furthermore, let $\boldsymbol{F} : [0, T_C) \longrightarrow \mathbb{R}^{n+1}$ be a solution to equation 1.2 on a maximal time interval and $\boldsymbol{F}(M^n, 0) = M_0$.*

10.1 Convergence to a Point

Here we generalize an idea of *Tso* [24] to the boundary case. Note that, by Theorem 4.5 and Proposition 9.4, the M_t will be imbedded and strictly convex with a uniform lower bound on the second fundamental form, i.e. there exists $\varepsilon > 0$ such that

$$h_{ij} \geq \varepsilon\, g_{ij} \quad \forall (x,t) \in M^n \times [0, T_C)\ . \tag{10.1}$$

Now for all t the hypersurfaces M_t and $\boldsymbol{\Sigma}$ bound a convex domain $\mathcal{K}_t \subset \mathbb{R}^{n+1}$, and from equation 8.4 we immediately conclude

$$\mathcal{K}_{t_2} \subset \mathcal{K}_{t_1} \quad \text{if} \quad t_2 \geq t_1\ .$$

We set

$$\mathcal{K} := \bigcap \{\mathcal{K}_t : 0 \leq t < T_C\}$$

and observe that $\mathcal{K}$ is convex as this holds for each $\mathcal{K}_t$. Now suppose that $\sigma > 0$ and $\boldsymbol{p} \in \mathcal{K}$ such that the ball $B^{n+1}_{2\,\sigma}(\boldsymbol{p}) \subset \mathbb{R}^{n+1}$ is contained in $\mathcal{K}$. Then, choosing coordinates in $\mathbb{R}^{n+1}$ with origin in $\boldsymbol{p}$, we obtain $s \geq 2\,\sigma$ on $M^n \times [0, T_C)$ for the support function of our solution $\boldsymbol{F}$. According to Theorem 8.8 this implies an upper bound on H and thus, by convextiy, on $|A|$. Hence, by the results of section 6, we can extend $\boldsymbol{F}$ to a solution on $[0, T_C + \delta)$ which contradicts the maximaltiy of T_C.

We therefore conclude that $\mathcal{K}$ cannot contain any ball B^{n+1}_r. But this implies that $\mathcal{K}$ has zero volume, $V(\mathcal{K}) = 0$, as $\mathcal{K}$ is a bounded, convex domain. Now we can prove:

Theorem 10.2 *(Convergence to a point) Under the assumptions 10.1 the hypersurfaces M_t converge to a point on $\boldsymbol{\Sigma}$ as $t \to T_C$.*

Proof: (cf. [24], Theorem 4.2) Suppose $\mathcal{K}$ is not a point, i.e. $\operatorname{diam}(\mathcal{K}) > 0$. Let $\boldsymbol{p}, \boldsymbol{q}$ be such that $d := \operatorname{dist}_{n+1}(\boldsymbol{p}, \boldsymbol{q}) = \operatorname{diam}(\mathcal{K})$.

We have $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{K}_t \ \ \forall t$, and as $V(\mathcal{K}) = 0$ there is a (two dimensional) plane E through $\boldsymbol{p}, \boldsymbol{q}$ such that the area of the convex subset $C_t := \mathcal{K}_t \cap E \subset \mathbb{R}^2$ of E fulfils

$$(*) \qquad A(C_t) \overset{t \to T_C}{\longrightarrow} 0 \ .$$

As $\mathcal{K}_{t_2} \subset \mathcal{K}_{t_1}$ whenever $t_2 \geq t_1$, we have

$$(**) \qquad C_{t_2} \subset C_{t_1} \ \ \forall \, t_2 \geq t_1 \ .$$

The convex curve $\gamma_t := \partial C_t$ can be decomposed into $\gamma_t = \gamma_t^{'} \cup \gamma_t^{''}$ such that $\gamma_t^{'} \subset M_t$, $\gamma_t^{''} \subset \boldsymbol{\Sigma}$ (hereby $\gamma_t^{''} = \emptyset$ being allowed) and (from $(**)$):

$$(***) \qquad \gamma_{t_2}^{''} \subset \gamma_{t_1}^{''} \ \ \forall \, t_2 \geq t_1 \ .$$

This means that as t increases, no "new" boundary points of $\boldsymbol{\Sigma}$ can occur on γ_t. As γ_t is convex, we conclude from $(*)$ that there must exist points on γ_t with smaller and smaller curvature as t increases. From $(***)$ it follows that these points must belong to $\gamma_t^{'} \subset M_t$ which is a contradiction to the uniform convexity equation 10.1. Thus $\mathcal{K}$ must be a point, and the result follows. □

Proof of Proposition 1.5: In the case $n = 1$ strict convexity of the M_t is equivalent to $H \geq C_0 > 0$, and by Theorem 8.2 this condition is preserved for arbitrary convex hypersurfaces $\boldsymbol{\Sigma}$. □

10.2 Convergence to a Hemisphere

Now we want to study the *shape* of the singularity; hereby, we follow an idea due to *Hamilton* [12] and rescale the entire flow. A similar rescaling for curves was carried out by *Altschuler* in [1]. In addition to Assumption 10.1, we will require $n \geq 2$ in the sequel. Note that by Theorem 9.7 and Corollary 9.8, a pinching condition

$$h_{ij} \geq \varepsilon \, H \, g_{ij}$$

will hold for $t \in [0, T_C)$.

Definition 10.3 *Let (x_k, t_k) be a sequence in $M^n \times [0, T_C)$ such that $t_k \nearrow T_C$. If*

$$H(x_k, t_k) = \max\{H(x, t_k) : x \in M^n\} \quad \forall k \ ,$$

(x_k, t_k) is called a blowup-sequence.

In the sequel we will rescale the mean curvature flow equation 1.2 along a given blowup-sequence. To this end, let

$$\lambda_k \ := \ H(x_k, t_k) \ = \ \max_{M^n} H(., t_k) \ ,$$

$$\begin{aligned}
\alpha_k &:= -\lambda_k^2\, t_k\ , \\
\omega_k &:= \lambda_k^2\,(T_C - t_k)\ , \\
\tau &:= \lambda_k^2\,(t - t_k)\ , \\
\tilde{\boldsymbol{F}}_k(.,\tau) &:= \lambda_k\,(A_k \cdot \boldsymbol{F}(.,t) + \boldsymbol{b}_k)\ ,\ A_k \in SO_{n+1}\mathbb{R}\ ,\ \boldsymbol{b}_k \in \mathbb{R}^{n+1}\ , \\
\tilde{M}_\tau^{(k)} &:= \tilde{\boldsymbol{F}}_k(M^n,\tau)\ .
\end{aligned}$$

Thereby let A_k and $\boldsymbol{b}_k$ be such that $\tilde{\boldsymbol{F}}_k(x_k,0) = \mathbf{o}$ and $\tilde{\boldsymbol{\nu}}_k(x_k,0) = \boldsymbol{e}_{n+1}$ for any k. A straightforward calculation then yields that each $\tilde{\boldsymbol{F}}_k$ is itself a mean curvature flow, and

$$\tilde{H}_k(x_x,0) = 1 \quad \forall k\ .$$

As usual, a singularity will be referred to as of *type I* if $\max_{M^n} |A|^2(.,t) \leq K\,(T_C - t)^{-1}$ as $t \to T_C$ with a given constant $K > 0$; otherwise it will be called a *type-II*-singularity. Now, given a blowup-sequence, $\alpha_k \searrow -\infty$ as $t \to T_C$. Moreover, by taking a subsequence, we can obtain $\omega_k \nearrow \omega < \infty$ in the case of a type-I-singularity, and $\omega_k \nearrow \infty$, otherwise. Finally, as $\lambda_k \to \infty$, we can assume $\lambda_k > 1\ \forall k$.

Now the crucial step is to control the curvature of the rescaled hypersurfaces:

Proposition 10.4 *(Curvature estimates for the rescaled hypersurfaces)*

$$\begin{aligned}
&\textit{Type I:}\quad \forall \delta > 0\ \exists k_0 = k_0(\delta)\ \textit{ and }\ C = C(\delta)\ \textit{ such that} \\
&\qquad \tilde{H}_k \leq C\ \textit{ on }\ M^n \times [\alpha_{k_0}, \omega - \delta]\ ,\ \ \forall k \geq k_0\ . \\
&\textit{Type II:}\quad \forall\, \bar{\omega} > 0,\ \forall \varepsilon > 0\ \exists k_0 = k_0(\bar{\omega},\varepsilon)\ \textit{ such that} \\
&\qquad \tilde{H}_k \leq 1 + \varepsilon\ \textit{ on }\ M^n \times [\alpha_{k_0}, \bar{\omega}]\ ,\ \ \forall k \geq k_0\ .
\end{aligned}$$

Hence on compact domains of spacetime, the curvature of the rescaled flows is bounded uniformly in k.

Proof:

I: As $\omega_k \to \omega$, we can choose k_0 such that $[\alpha_{k_0}, \omega - \delta] \subset [\alpha_k, \omega_k - \frac{\delta}{2}]$ $\forall k \geq k_0$. Now suppose the statement does not hold. Then there is a sequence $(y_k, \tau_k) \in M^n \times [\alpha_{k_0}, \omega - \delta]$ such that $\tilde{H}_k(y_k,\tau_k) \to \infty$. As for $k \geq k_0$, $\omega_k \geq \omega - \frac{\delta}{2}$ we can choose a subsequence with

$$\tilde{H}_{\hat{k}}(y_{\hat{k}},\tau_{\hat{k}}) \geq C_{\hat{k}}\ ,\quad y_{\hat{k}} \in M^n\ ,\quad \tau_{\hat{k}} \leq \omega_{\hat{k}} - \tfrac{\delta}{2}\ ,\quad C_{\hat{k}} \nearrow \infty$$

and thereby, defining $\tilde{t}_{\hat{k}} := \lambda_{\hat{k}}^{-2}\,\tau_{\hat{k}} + t_{\hat{k}}$:

$$(*) \qquad H(y_{\hat{k}},\tilde{t}_{\hat{k}}) \geq \lambda_{\hat{k}}\, C_{\hat{k}}\ ,\quad y_{\hat{k}} \in M^n\ ,\quad 0 \leq \tilde{t}_{\hat{k}} \leq T_C - \lambda_{\hat{k}}^{-2}\,\tfrac{\delta}{2}\ .$$

But the singularity being of type I implies

$$H(y_{\hat{k}},\tilde{t}_{\hat{k}}) \leq \sqrt{\tfrac{2K}{\delta}}\,\lambda_{\hat{k}}\ ;$$

hence $\hat{k} \to \infty$ yields a contradiction to $(*)$.

II: This has been proved by *Hamilton* [12] as follows: Given the sequence (x_k, t_k), choose $(\tilde{x}_k, \tilde{t}_k)$ such that

$$(*) \qquad (t_k - \tilde{t}_k)\, H^2(\tilde{x}_k, \tilde{t}_k) = \max_{M^n \times [0,t_k]} (t_k - t)\, H^2(., t)\ .$$

Defining $\tilde{\omega}_k := \lambda_k^2\,(t_k - \tilde{t}_k)$, we obtain $\tilde{H}_k(\tilde{x}_k, 0) = 1\ \ \forall k$ and $\tilde{\omega}_k \to +\infty$ as $k \to \infty$. Furthermore, $(*)$ yields $\tilde{H}_k^2(., \tau) \leq \frac{\omega_k}{\omega_k - \tau} \quad \forall\, \tau \in [\alpha_k, \omega_k)$ and thereby the result. □

Now the curvature bounds immediately imply local gradient bounds which are *uniform in k*: Let $\boldsymbol{p}_0^{(k)} := \tilde{\boldsymbol{F}}_k(x_0, \tau_0)\,,\ (x_0, \tau_0) \in M^n \times [\alpha_k, \omega_k]$ and $\mathcal{Z}_r(\boldsymbol{p}_0^{(k)}) \subset \mathbb{R}^{n+1}$ denote the cylinder through $\boldsymbol{p}_0^{(k)}$ with axis $\boldsymbol{\omega}_k := \tilde{\boldsymbol{\nu}}_k(x_0, \tau_0)$ as in section 6. Furthermore, let

$$\tilde{v}_k(x, \tau) := \langle\ \boldsymbol{\omega}_k\ ,\ \tilde{\boldsymbol{\nu}}_k(x, \tau)\ \rangle^{-1}\ .$$

Then we can apply Theorems 6.7 and 6.13 to the rescaled flows $\tilde{\boldsymbol{F}}_k$ and obtain local gradient bounds on domains $\mathcal{U}_{r_k} \times [\tau_0, \tau_0 + \Delta_k]$ with r_k and Δ_k depending only on the maximum of the curvatures $|\tilde{A}_k|\,, |^{\Sigma}\tilde{A}_k|$ of the rescaled hypersurfaces $\tilde{M}_\tau$, $\tilde{\boldsymbol{\Sigma}}_k$. But as the $\tilde{M}_\tau$ are convex, the above Proposition yields a bound on $|\tilde{A}_k|$ which is uniform in k. Moreover, as we assumed $\lambda_k \geq 1$, the curvatures of the $\tilde{\boldsymbol{\Sigma}}_k$ will be uniformly bounded in k as well. Hence we obtain a local gradient estimate which is *uniform in k*. Now we can state the first Theorem concerning a limit flow:

Theorem 10.5 *A subsequence of the rescaled flows $\tilde{\boldsymbol{F}}_k$ converges to a limit flow $\tilde{\boldsymbol{F}}_\infty : M^n \times (\alpha, \omega) \longrightarrow \mathbb{R}^{n+1}$ ($\omega = \infty$ allowed) such that*

(*i*) *$\tilde{\boldsymbol{F}}$ is a mean curvature flow;*

(*ii*) *If the M_τ^∞ possess a boundary, then $\partial M_\tau^\infty \subset \tilde{\boldsymbol{\Sigma}}_\infty$, $\tilde{\boldsymbol{\Sigma}}_\infty$ is a hyperplane and $\langle \tilde{\boldsymbol{\nu}}_\infty, \tilde{\boldsymbol{\mu}}_\infty \rangle = 0$;*

(*iii*) *The hypersurfaces $\tilde{M}_\tau^\infty$ are pinched as the original hypersurfaces, i.e. $\varepsilon\, H\, g_{ij} \leq h_{ij} \leq \kappa\, H\, g_{ij}$ implies $\varepsilon\, \tilde{H}_\infty\, \tilde{g}_{ij}^\infty \leq \tilde{h}_{ij}^\infty \leq \kappa\, \tilde{H}_\infty\, \tilde{g}_{ij}^\infty$.*

Proof: In view of the above considerations, the only thing to check is the convergence of a subsequence. This can be done by standard methods involving the *Arzelà-Ascoli* Theorem: We first let $A : B_R^{n+1}(\mathbf{o}) \times [\tau_-, \tau_+]$. Using the above k-uniform local gradient estimate , we can cover A by finitely many spacetime-cylinders $C_{r,\Delta}(\boldsymbol{p}_m, \tau_m)$ such that for each $(\boldsymbol{p}_m, \tau_m)$ *and every k* the intersection of the hypersurfaces $\tilde{M}_\tau^k$ with $C_{r,\Delta}(\boldsymbol{p}_m, \tau_m)$ can be written as a graph with bounded gradient. Hence the *Arzelà-Ascoli* Theorem yields the convergence of a subsequence on A with the desired properties. Doing this for a whole *sequence* $A_{R^l, [\tau_-^l, \tau_+^l]}$ and choosing a diagonal sequence completes the proof. □

Remark 10.6 *$\mathcal{H} := \tilde{\boldsymbol{\Sigma}}_\infty$ is a hyperplane if the hypersurfaces of the limit flow do have a boundary. Thus, in this case we can by reflection at $\mathcal{H}$ extend the hypersurfaces symmetrically to closed, convex and pinched hypersurfaces without boundary which again move under the mean curvature flow. Thus, in any case we can in the sequel assume the $\tilde{M}_\tau^\infty$ to be complete, strictly convex, pinched hypersurfaces.*

The limit flow can now be described by means of a result due to *Hamilton* ([11], Main Theorem):

Theorem 10.7 *Let M be a smooth, strictly convex, complete hypersurface bounding a region in $\mathbb{R}^{n+1}, n \geq 2$. Let M be ε-pinched, i.e. let $h_{ij} \geq \varepsilon H g_{ij}$. Then M is compact.*

We remark that this is the point where we need $n \geq 2$. An immediate consequence of this Theorem is

Theorem 10.8 *Under Assumption 10.1 the singularity at $t = T_C$ is a type-I-singularity.*

Proof: By Theorem 10.5, $\tilde{\boldsymbol{F}}_\infty$ moves under the mean curvature flow. Thus the singularity being of Type *II* would imply that the solution would be defined for $\tau \in (-\infty, \infty)$. But then all the $\tilde{M}_\tau$ would have to be noncompact, as compact hypersurfaces are contained in a ball $B_R \subset \mathbb{R}^{n+1}$ and therefore cannot "live" eternally. Now the $\tilde{M}_\tau$ are pinched by Theorem 10.5, and thus we run into a contradiction to Theorem 10.7. □

Hence the boundary of the $\tilde{M}_k$ cannot be "shot away" to infinity as $k \to \infty$. But, up to now, it could shrink away such that the $\tilde{M}_k$ would be compact, *closed* hypersurfaces. In order to rule out this possibility we observe that the image of the Gauss map of the "original" hypersurfaces M_t is contained in a hemisphere. Hence this holds for each of the rescaled flows, the various hemispheres depending on k. Again by the *Arzelà-Ascoli* Theorem, we obtain the C^∞-convergence of a subsequence to a limit flow $\tilde{\boldsymbol{F}}_\infty$ such that in addition to the above properties the images of all $\tilde{M}_\tau^\infty$ are contained in a *unique* hemisphere. According to Theorem 10.5 the boundary of these hypersurfaces will therefore lie on a hyperplane. From there, the proof of Theorem 1.4 is an immediate consequence of Remark 10.6 and a Theorem of *Huisken* [13] which states that closed, strictly convex hypersurfaces moving by mean curvature shrink to a single point within finite time, the (appropriately) rescaled hypersurfaces thereby converging to a sphere in the C^∞-topology.

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