9 The tensor algebra

9.1 Multi-fold tensor products

Up to this point, we have been considering tensor products of pairs of modules M_1 and M_2 over a ring R. In many applications, the modules involved are free of finite ranks. In this special case, corollary **??** implies that we can identify $M_1 \otimes_R M_2$ with $\mathrm{Hom}_R(M_1^*, M_2)$. So for purely computational purposes, working with matrices would suffice in these cases.

The theory of multilinear algebra unfolds its full strength in the natural generalization to tensor products of several R-modules M_1, M_2, ..., M_p for some $p \geq 2$.

Throughout this section let $(R, +, \cdot)$ always be a commutative ring with a multiplicative identity element.

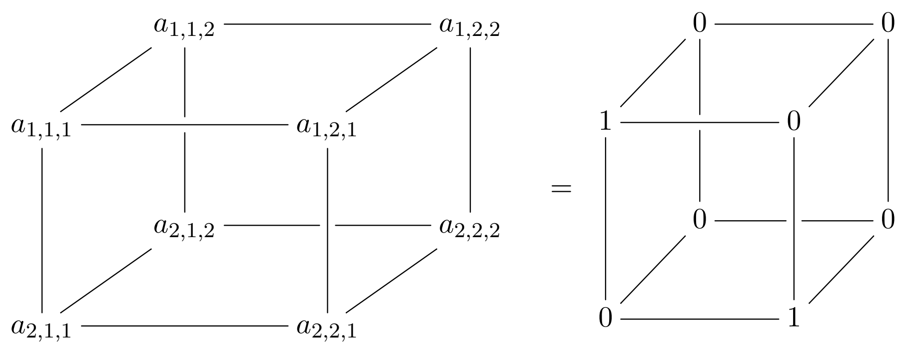
9.1 Example. Consider $M := \mathbb{R}^2$ as an $\mathbb{R}$-module. Let $\{e_1, e_2\} \subseteq \mathbb{R}^2$ denote the standard basis of $\mathbb{R}^2$. The standard inner product $\langle\ ,\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a bilinear map. It is easy to see that the map

$$\begin{array}{rccc} \varphi : & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ & (u, v, w) & \mapsto & \langle u, v\rangle \cdot \langle w, e_1\rangle \end{array}$$

is 3-linear. We claim that all of the information about the map φ can be recovered from the family of real numbers $\{\varphi(e_i, e_j, e_k)\}_{1\leq i,j,k\leq 2}$. Indeed, for an arbitrary element $(u, v, w) = ((u_1, u_2), (v_1, v_2), (w_1, w_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, we compute applying the rules for multilinear maps

$$\begin{array}{rcl} \varphi(u, v, w) & = & \varphi(u_1e_1 + u_2e_2, v_1e_1 + v_2e_2, w_1e_1 + w_2e_2) \\ & = & \sum\limits_{1\leq i,j,k\leq 2} u_iv_jw_k\varphi(e_i, e_j, e_k). \end{array}$$

In analogy to lemma 5.10 **??**, we could think of the family $\{a_{i,j,k}\}_{1\leq i,j,k\leq 2} := \{\varphi(e_i, e_j, e_k)\}_{1\leq i,j,k\leq 2}$ as a “3-dimensional matrix”, or a $2 \times 2 \times 2$ cube with entries in $\mathbb{R}$, which represents the map φ by a suitably defined vector multiplication. In our example we compute



For obvious reasons, this notation has not become standard for writing down multilinear maps.

The first question, which arises when we are adding extra factors to the tensor product, is about the associativity of the construction.

9.2 Proposition. *Let L, M, and N be R-modules. Then there exists a unique isomorphism of R-modules*

$$(L \otimes_R N) \otimes_R N \;\cong\; L \otimes_R (M \otimes_R N)$$

such that for all elements $m \in M$, $n \in N$ and $l \in L$, the decomposable tensor $(l \otimes m) \otimes n$ gets identified with the decomposable tensor $l \otimes (m \otimes n)$.

Proof. The proof follows from comparing the commutative diagrams obtained from the respective universal properties. □

9.3 Definition. Let $p \in \mathbb{N}_{\geq 2}$, and let $M_1, \ldots, M_p$ be R-modules. A *p-fold tensor product of $M_1, \ldots, M_k$ over R* is a p-linear map

$$\tau :\; M_1 \times \ldots \times M_p \to T,$$

where T is an R-module, which satisfies the following *universal property of the tensor product*:

For any R-module Z, and any p-linear map $\varphi : M_1 \times \ldots \times M_p \to Z$, there exists a unique homomorphism of R-modules $\tilde{\varphi} : T \to Z$, such that the diagram

$$\begin{tikzcd} M_1 \times \ldots \times M_p \arrow[r, "\varphi"] \arrow[d, "\tau"'] & Z \\ T \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes.

9.4 Remark. **a)** As in the case of $p = 2$ before, one shows by an explicit construction that a p-fold tensor product always exists. By its universal property, it is again unique up to a unique isomorphism. For any $p \in \mathbb{N}_{\geq 3}$ and any p-tuple of R-modules $M_1, \ldots, M_p$ we choose once and for all a p-fold tensor product, and denote it by

$$\tau : M_1 \times \ldots \times M_p \to M_1 \otimes \ldots \otimes M_p.$$

Note that, as compared to the bilinear case, we omit the subscript of the tensor symbol $\otimes_R$ for better readability. Consequently, in the case $p = 2$ we will write from now on $M_1 \otimes M_2 := M_1 \otimes_R M_2$.

If $p \geq 2$, and $M_1 = \ldots = M_p =: M$, we write for the p-fold tensor product

$$\bigotimes\nolimits^p M := M \otimes \ldots \otimes M.$$

We also put $\bigotimes^1 M := M$ and $\bigotimes^0 M := R$.

b) An element $t \in M_1 \otimes \ldots \otimes M_p$ is called a *tensor of degree* p. Analogously to the case of tensor products of two modules one finds

$$M_1 \otimes \ldots \otimes M_p = \operatorname{im}(\tau) = \operatorname{span}_R(\{\tau(M_1 \times \ldots \times M_p)\}.$$

In particular, for any tensor $t \in M_1 \otimes \ldots \otimes M_p$ there exists an $n \in \mathbb{N}$ and finite families $\{m_j^{(i)}\}_{j=1,\ldots,n}$ with $m_j^{(i)} \in M_i$ for $i = 1, \ldots, p$, such that

$$t = \sum_{j=1}^{n} m_j^{(1)} \otimes \ldots \otimes m_j^{(p)}.$$

The tensor t is called *decomposable*, if such a representation exists with $n = 1$.

c) In particular, if $M_1, \ldots, M_p$ are free R-modules with bases $E_i = \{e_{j_i}^{(i)}\}_{j_i \in I_i}$ for $i = 1, \ldots, p$, then $M_1 \otimes \ldots \otimes M_p$ is again a free R-module with basis $\{e_{j_1}^{(1)} \otimes \ldots \otimes e_{j_p}^{(p)}\}_{(j_1,\ldots,j_p) \in I_1 \times \ldots \times I_p}$. The proof is analogous to **??**. In the special case of modules of finite rank we obtain the formula

$$\operatorname{rank}(M_1 \otimes \ldots \otimes M_p) = \prod_{i=1}^{p} \operatorname{rank}(M_i).$$

9.5 Proposition. *Let $p \in \mathbb{N}_{\geq 2}$ and let $\tau : M_1 \times \ldots \times M_p \to T$ be a p-linear map of R-modules, with $\operatorname{im}(\tau) = T$. Then τ is a p-fold tensor product of $M_1, \ldots, M_p$ over R, if and only if the following property holds:*

For any R-module Z, and any p-linear map $\varphi : M_1 \times \ldots \times M_p \to Z$, there exists a homomorphism of R-modules $\tilde{\varphi} : T \to Z$, such that the diagram

$$\begin{tikzcd} M_1 \times \ldots \times M_p \arrow[r, "\varphi"] \arrow[d, "\tau"'] & Z \\ T \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes.

Proof. The proof is analogous to the proof of the weak universal property of the tensor product of two modules of proposition **??**. □

By now, the kind reader may wonder why we did not introduce in section **??** the tensor product in full generality, rather than now repeating the phrase "analogous to the case $p = 2$" over and over again. Indeed, this text is first of all written for learners. The mathematical ideas should not be obscured by a clutter of indices, which is unavoidable when dealing with multiple modules at the same time. It is a recommendable exercise to expand all of the proofs of this section along the lines given in the case of 2-fold tensor products. You may even discover new or more elegant proofs.

9.6 Remark. Let $p \in \mathbb{N}_{\geq 2}$. For $i = 1, \ldots, p$ let $\alpha_i : M_i \to M_i'$ be homomorphisms of R-modules. As in **??** one shows that there exists a unique homomorphism $\alpha_1 \otimes \ldots \otimes \alpha_p$ of R-modules, such that the diagram

$$\begin{tikzcd} M_1 \times \ldots \times M_p \arrow[r, "\alpha_1 \times \ldots \times \alpha_p"] \arrow[d, "\tau"'] & M_1' \times \ldots \times M_p' \arrow[d, "\tau'"] \\ M_1 \otimes \ldots \otimes M_p \arrow[r, "\alpha_1 \otimes \ldots \otimes \alpha_p"'] & M_1' \otimes \ldots \otimes M_p' \end{tikzcd}$$

commutes, where for a decomposable tensor $m_1 \otimes \ldots \otimes m_p \in M_1 \otimes \ldots \otimes M_p$ holds $\alpha_1 \otimes \ldots \otimes \alpha_p(m_1 \otimes \ldots \otimes m_p) = \alpha_1(m_1) \otimes \ldots \otimes \alpha_p(m_p)$.

9.7 Proposition. *Let $p, q \in \mathbb{N}_{\geq 1}$, and $r := p + q$. Let $M_1, \ldots, M_r$ be R-modules. Then there is a unique isomorphism of R-modules, which is given on generating elements by*

$$\begin{array}{rrcl} \lambda : & M_1 \otimes \ldots \otimes M_r & \to & (M_1 \otimes \ldots \otimes M_p) \otimes (M_1 \otimes \ldots \otimes M_q) \\ & m_1 \otimes \ldots \otimes m_r & \mapsto & (m_1 \otimes \ldots \otimes m_p) \otimes (m_{p+1} \otimes \ldots \otimes m_q). \end{array}$$

Proof. The claim follows from the universal property of the tensor product, applied to the obvious maps. Compare also proposition 9.2. □

9.8 Corollary. *Let M be an R-module. Then for all $p, q \in \mathbb{N}$ there is a canonical isomorphism*

$$M^{p+q} \cong M^p \otimes M^q.$$

Proof. Directly from proposition 9.7. □

9.2 The tensor algebra of a module

9.9 Definition. Let $(R, +, \cdot)$ be a commutative ring with a multiplicative identity element. An R-*algebra* is a tuple $(A, \mu, \lambda, \sigma)$, where

(1) (A, μ, λ) is an R-module, and
(2) $\sigma : A \times A \to A$ is a bilinear map.

An R-algebra $(A, \mu, \lambda, \sigma)$ is called a *commutative algebra*, if for all $a, b \in A$ holds $\sigma(a, b) = \sigma(b, a)$. It is called *associative*, if for all $a, b, c \in A$ holds $\sigma(\sigma(a, b), c) = \sigma(a, \sigma(b, c))$. An element $e \in A$ is called *unital*, if for all $a \in A$ holds $\sigma(e, a) = a$ and $\sigma(a, e) = a$.

Note that a unital element of an R-algebra, if it exists, is necessarily unique.

9.10 Remark. Let $(A, \mu, \lambda, \sigma)$ be an R-algebra. We then have

$$\begin{array}{lrccc} \text{an Abelian group structure} & \mu : & A \times A & \to & A \\ & & (a, b) & \mapsto & a + b \\ \text{an } R\text{-module multiplication} & \lambda : & R \times A & \to & A \\ & & (r, a) & \mapsto & ra \\ \text{an } R\text{-algebra multiplication} & \sigma : & A \times A & \to & A \\ & & (a, b) & \mapsto & a \cdot b \end{array}$$

together with compatibility conditions coming from the bilinearity of σ: for all $a, a', b, b' \in A$ and $r \in R$ hold $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$, as well as $(a + a') \cdot b = a \cdot b + a' \cdot b$ and $a \cdot (b + b') = a \cdot b + a \cdot b'$.

9.11 Example. The ring $R[X]$ of polynomials over R can be viewed as an R-module. The composition σ is given by the multiplication of polynomials, which is clearly bilinear, and even symmetric. Hence $R[X]$ is an associative commutative R-algebra, with unitary element 1_R.

9.12 Example. Let $n \in \mathbb{N}_{>0}$ be given. The R-module of square matrices $\mathrm{Mat}(n, n, R)$, together with the usual matrix multiplication, is an associative non-commutative R-algebra. The unit matrix is its unique unital element.

9.13 Example. Let $(A, \mu, \lambda, \sigma)$ be an associative commutative R-algebra with a unital 1_A element. Then (A, μ, σ) is a commutative ring with multiplicative identity element 1_A.

Conversely, let $\lambda : (R, +, \cdot) \to (A, \mu, \sigma)$ be a homomorphism of commutative rings with multiplicative identity elements, such that $\lambda(1_R) = \lambda(1_A)$. Then $(A, \mu, \lambda, \sigma)$ is an R-algebra.

9.14 Definition. Let $(A, \mu, \lambda, \sigma)$ and $(A', \mu', \lambda', \sigma')$ be R-algebras. A homomorphism $\alpha : A \to A'$ of R-modules is called a *homomorphism of R-algebras*, if for all $a, b \in A$ holds

$$\alpha(\sigma(a, b)) = \sigma'(\alpha(a), \alpha(b)).$$

9.15 Remark. Let M be an R-module. Let $p, q \in \mathbb{N}$. By corollary 9.8, there is a canonical isomorphism $\tilde{\sigma}_{q,p} : \bigotimes^p M \otimes \bigotimes^q M \to \bigotimes^{p+q} M$. We obtain a bilinear map $\sigma_{p,q} : \bigotimes^p M \times \bigotimes^q M \to \bigotimes^{p+q} M$ by composition with the tensor map as in the following diagram:

$$\begin{tikzcd}
\bigotimes^p M \times \bigotimes^q M \arrow[r, "\sigma_{p,q}"] \arrow[d, "\tau"'] & \bigotimes^{p+q} M \\
\bigotimes^p M \otimes \bigotimes^q M \arrow[ur, "\tilde{\sigma}_{q,p}"'] &
\end{tikzcd}$$

By abuse of notation, omitting the isomorphism $\tilde{\sigma}_{q,p}$, we write for a tensor $s \in \bigotimes^p M$ of degree p and a tensor $t \in \bigotimes^q M$ of degree q simply

$$s \otimes t := \sigma_{p,q}(s, t) \in \bigotimes^{p+q} M.$$

9.16 Definition. Let M be an R-module. The *tensor algebra* of M is the R-module

$$\bigotimes M := \bigoplus_{p=0}^{\infty} \left(\bigotimes^p M\right)$$

together with the bilinear map

$$\begin{array}{rccc}
\sigma : & \bigotimes M \times \bigotimes M & \to & \bigotimes M \\
& \left(\sum_{i=0}^{\infty} s_i, \sum_{j=0}^{\infty} t_j\right) & \mapsto & \sum_{k=0}^{\infty} \sum_{i+j=k} s_i \otimes t_j
\end{array}$$

where $s_i \in \bigotimes^i M$ and $t_j \in \bigotimes^j M$ for all $i, j \in \mathbb{N}$.

Recall that by the definition of the direct sum of modules, all sums in the above definition have only finitely many summands, which are not equal to zero.

9.17 Remark. In general, the tensor algebra is not commutative. It is associative, and it hat $1_R \in \bigotimes^0 M$ as its unique unital element.

Moreover it is a *graded algebra*, which means the following. An element $t \in \bigotimes M$ is called *homogeneous*, if $t \in \bigotimes^d M$ for some $d \in \mathbb{N}$. Then d is called the *degree* of t. Any element of the algebra is in a unique way the sum of finitely many homogeneous elements. If $t \in \bigotimes M$ is homogeneous of some degree d, and $t' \in \bigotimes M$ is homogeneous of degree d', then $t \otimes t' \in \bigotimes^{t+t'} M$, so $t \otimes t'$ is homogeneous of degree $t + t'$.

Compare example 9.11: the algebra of polynomials $R[X]$ is graded, too.

9.18 Example. Let M be a free R-module of rank 1. By definition, there exists a basis element $e_1 \in M$, such that any $m \in M$ can be written uniquely as $m = re_1$, for some $r \in R$.

For any $p \in \mathbb{N}_{>0}$, a basis of $\bigotimes^p M$ is given by the p-fold product $e_1 \otimes \ldots \otimes e_1$. Thus by definition, an element $t \in \bigotimes M = \bigoplus_{p=0}^{\infty} \bigotimes^p M$ is given as a unique sum $t = \sum_{p=0}^{\infty} a_p e_1 \otimes \ldots \otimes e_1$, where only finitely many of the coefficients $a_p \in R$ are not equal to zero. Therefore we can construct a well-defined R-linear map into R-module of polynomials by

$$\begin{array}{cccc} \alpha : & \bigotimes M & \to & R[X] \\ & \sum_{p=0}^{\infty} a_p e_1 \otimes \ldots \otimes e_1 & \mapsto & \sum_{p=0}^{\infty} a_p X^p . \end{array}$$

By comparing the definition of the algebra multiplication in $\bigotimes M$ with the rules for multiplying polynomials, one immediately verifies that α is even an isomorphism of R-algebras $\bigotimes M \cong R[X]$.

9.19 Remark. All R-algebras, together with their homomorphisms, form a category (R-Alg). The construction of the tensor algebra is functorial.

Indeed, let M and M' be R-modules, and let $\alpha : M \to M'$ be a homomorphism of R-modules. Let $p \in \mathbb{N}_{\geq 2}$. The composed map

$$M \times \ldots \times M \overset{\alpha \times \ldots \times \alpha}{\longrightarrow} M' \times \ldots \times M' \overset{\tau}{\longrightarrow} \bigotimes^p M'$$

is p-linear, so by the universal property of the tensor product, it defines an R-linear map $\otimes^p \alpha : \bigotimes^p M \to \bigotimes^p M'$. This induces an R-linear map on the direct sums $\otimes\alpha : \bigotimes M \to \bigotimes M'$. It is straightforward to verify that this construction defines a (covariant) functor

$$\begin{array}{cccc} \bigotimes : & (R\text{-Mod}) & \to & (R\text{-Alg}) \\ & M & \mapsto & \bigotimes M \\ & \alpha : M \to M' & \mapsto & \otimes\alpha : \bigotimes M \to \bigotimes M' \end{array}$$

9.20 Proposition. *Let M be an R-module. The tensor algebra $\bigotimes M$ has the following universal property.*

For any associative R-algebra A with a unital element 1_A, and any homomorphism $\varphi : M \to A$ of R-modules, there exists a unique homomorphism of R-algebras $\tilde{\varphi} : \bigotimes M \to A$, such that $\tilde{\varphi}(1_R) = 1_A$ and the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "i"'] & A \\ \bigotimes M \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

commutes.

Proof. For all $p \in \mathbb{N}_{>0}$ consider the p-linear map

$$\begin{array}{cccc} \varphi_p : & M \times \ldots \times M & \to & A \\ & (m_1, \ldots, m_p) & \mapsto & \varphi(m_1) \cdot \ldots \cdot \varphi(m_p) \end{array}$$

By the universal property of the tensor product, there exists a unique R-linear map $\tilde{\varphi}_p : \bigotimes^p M \to A$ such that the diagram

$$\begin{tikzcd} M^p \arrow[r, "\varphi_p"] \arrow[d, "\tau_p"'] & A \\ \bigotimes^p M \arrow[ur, "\tilde{\varphi}_p"'] & \end{tikzcd}$$

commutes. For $p = 0$, using that A is an R-algebra with unital element $1_A \in A$, we define $\tilde{\varphi}_0 : R \to A$ by $\tilde{\varphi}_0(r) := r \cdot 1_A$. Putting everything together, we define

$$\begin{array}{cccc} \tilde{\varphi} : & \bigotimes M & \to & A \\ & \sum_{p=0}^{\infty} t_p & \mapsto & \sum_{p=0}^{\infty} \tilde{\varphi}_p(t_p) \end{array}$$

It is clear from the construction that for all $m \in M$ holds $\tilde{\varphi}(m) = \tilde{\varphi}_1(m) = \varphi(m)$, and also $\tilde{\varphi}(1_R) = 1_A$. We leave it to the reader to verify that $\tilde{\varphi}$ is a homomorphism of R-algebras.

It only remains to prove the uniqueness of $\tilde{\varphi}$. Let us consider a homomorphism of R-algebras $\psi : \bigotimes M \to A$ which satisfies $\psi(m) = \varphi(m)$ for all $m \in M$, and $\psi(1_R) = 1_A$. Because of their R-linearity, it is enough to prove the equality of ψ and $\tilde{\varphi}$ on decomposable elements $m_1 \otimes \ldots \otimes m_p \in \bigotimes^p M$ for $p \in \mathbb{N}$. In the case $p = 0$ we have $\bigoplus^0 M = R$, so for $r \in R$ we find $\psi(r) = r\psi(1_R) = r \cdot 1_A = r\tilde{\varphi}(1_R) = \tilde{\varphi}(r)$. From the fact that ψ is compatible with the algebra multiplication "$\otimes$" of $\bigotimes M$ we compute

$$\begin{aligned}
\psi(m_1 \otimes \ldots \otimes m_p) &= \psi(m_1) \cdot_A \ldots \cdot_A \psi(m_p) \\
&= \varphi(m_1) \cdot_A \ldots \cdot_A \varphi(m_p) \\
&= \varphi_p(m_1, \ldots, m_p) \\
&= \tilde{\varphi}_p(m_1 \otimes \ldots \otimes m_p) \\
&= \tilde{\varphi}(m_1 \otimes \ldots \otimes m_p)
\end{aligned}$$

Hence we obtain $\psi = \tilde{\varphi}$, as claimed. $\square$