8 Base extensions

Up to this point, we were discussing modules over a given commutative ring and their tensor products. For a complete picture, we need to understand what happens, when we vary the underlying rings.

8.1 Remark. Let $(R, +_R, \cdot_R)$ and $(S, +_S, \cdot_S)$ be commutative rings with multiplicative units. Let $j : R \to S$ be a homomorphism of commutative rings with units, that is, we have $j(1_R) = 1_S$. We define an operation of R on S by

$$\begin{array}{rccc} \lambda : & R \times S & \to & S \\ & (r, s) & \mapsto & r \cdot_\lambda s := j(r) \cdot_S s. \end{array}$$

8.2 Lemma. *The triple $(S, +_S, \lambda)$ from remark 8.1 is an R-module.*

Proof. Straightforward, by direct verification of the axioms. □

8.3 Definition. Let $j : R \to S$ be a homomorphism of commutative rings with units. Let $(M, +, \cdot)$ be an R-module. Then the tensor product

$$M_S := S \otimes_R M$$

is called the *extension of M with respect to j*.

8.4 Remark. By lemma 8.2, we view S as an R-module. The extension of M is a tensor product of R-modules. Hence it has the structure of an R-module, which for the moment shall be denoted by $(M_S, +_\otimes, \cdot_\otimes)$. (Don't worry, once we have established the formal setup, we will omit the subscripts.) Now we define an operation of S on M_S on generators $t = a \otimes m \in M_S$, with $a \in S$ and $m \in M$, by

$$\begin{array}{rccc} \Lambda : & S \times M_S & \to & M_S \\ & (s, t) & \mapsto & s \cdot_\Lambda t := (s \cdot_S a) \otimes m. \end{array}$$

8.5 Lemma. *The triple $(M_S, +_\otimes, \Lambda)$ from remark 8.4 is an S-module.*

Proof. The triple $(M_S, +_\otimes, \cdot_\otimes)$ is an R-module, so $(M_S, +_\otimes)$ is an Abelian group. The axioms of an S-module can be verified by direct computation.

For example, let $s, s' \in S$ and let $t = a \otimes m \in M_S$ be a generating element, with $a \in S$ and $m \in M$. Then one computes

$$\begin{aligned} (s +_S s') \cdot_\Lambda t &= ((s +_S s') \cdot_S a) \otimes m \\ &= (s \cdot_S a +_S s' \cdot_S a) \otimes m \\ &= (s \cdot_S a) \otimes m +_\otimes (s' \cdot_S a) \otimes m \\ &= s \cdot_\lambda t +_\otimes s' \cdot_\lambda t. \end{aligned}$$

This shows axiom ?? of definition ???, and the remaining axioms follow analogously. □

8.6 Example. Consider the inclusion map $j : \mathbb{R} \to \mathbb{C}$ as a homomorphism of commutative rings with units. The operation λ of $\mathbb{R}$ on $\mathbb{C}$ with respect to this homomorphism is just the usual multiplication of complex numbers $x \cdot_\lambda z = j(x) \cdot_\mathbb{C} z := x \cdot z$, for $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

For some $n \in \mathbb{N}_{>0}$, consider $V := \mathbb{R}^n$ as an $\mathbb{R}$-module. Then its extension with respect to j is $V_\mathbb{C} := \mathbb{C} \otimes_\mathbb{R} \mathbb{R}^n$. We have already seen in example **??** that as an $\mathbb{R}$-module, this tensor product is isomorphic to $\mathbb{C}^n$. Obviously, $\mathbb{C}^n$ is a $\mathbb{C}$-module, and $V_\mathbb{C}$ is a $\mathbb{C}$-module, too, by lemma 8.5. We want to see that the two $\mathbb{C}$-module structures are actually “the same”. More precisely, we want to verify that the isomorphism between $V_\mathbb{C}$ and $\mathbb{C}^n$ is an isomorphism of $\mathbb{C}$-modules.

If $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$ as $\mathbb{R}$-module, then a basis of $V_\mathbb{C}$ over $\mathbb{R}$ is given by $\{1 \otimes e_1, i \otimes e_1, \ldots, 1 \otimes e_n, i \otimes e_n\}$ by proposition **??**. A basis of $\mathbb{C}^n$ over $\mathbb{R}$ is $\{e_1, ie_1, \ldots, e_n, ie_n\}$. An isomorphism $\psi : \mathbb{C} \otimes_\mathbb{R} \mathbb{R}^n \to \mathbb{C}^n$ can be defined on the basis by $\psi(\varepsilon \otimes e_j) := \varepsilon e_j$, for $j = 1, \ldots, n$ and $\varepsilon \in \{1, i\}$. Now let $z \in \mathbb{C}$. We compute for all elements of the basis of $V_\mathbb{C}$

$$\psi(z \cdot_\Lambda (\varepsilon \otimes e_i) = \psi((z\varepsilon) \otimes e_i) = z\varepsilon e_i = z \cdot_\Lambda \psi(\varepsilon \otimes e_i).$$

This shows that $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} \mathbb{R}^n$ and $\mathbb{C}^n$ are isomorphic as $\mathbb{C}$-modules.

8.7 Lemma. *Let $j : R \to S$ be a homomorphism of commutative rings with units. Let $\alpha : M \to M'$ be a homomorphism of R-modules. Then*

$$\alpha_S := \mathrm{id}_S \otimes \alpha : \quad M_S \to M'_S$$

is a homomorphism of S-modules.

Proof. By definition, α_S is a homomorphism of R-modules. In particular, it is a group homomorphism of the underlying Abelian groups, independently whether they are considered as R-modules or as S-modules. So it only remains to show that α_S respects the multiplicative structures, too.

Let $s \in S$, and $t \in M_S$. Since we already know that the additive structures are respected, we may assume without loss of generality that t is a generating element. Let $t = a \otimes m$, with $a \in S$ and $m \in M$. Then we compute

$$\begin{array}{rcl} \alpha_S(s \cdot_\Lambda t) & = & \alpha_S((sa) \otimes m) \\ & = & \mathrm{id}_S \otimes \alpha((sa) \otimes m) \\ & = & (sa) \otimes \alpha(m) \\ & = & s \cdot_\Lambda (a \otimes \alpha(m)) \\ & = & s \cdot_\Lambda (\mathrm{id}_S \otimes \alpha(a \otimes m)) \\ & = & s \cdot_\Lambda \alpha_S(t). \end{array}$$

This shows that the map α_S is indeed S-linear. □

8.8 Remark. Let $j : R \to S$ be a homomorphism of commutative rings with units. There is an *extension functor*

$$\begin{array}{rccc} S \otimes_R \bullet : & (R\text{-Mod}) & \to & (S\text{-Mod}) \\ & M & \mapsto & M_S \\ & M \xrightarrow{\alpha} M' & \mapsto & M_S \xrightarrow{\alpha_S} M'_S \end{array}$$

8.9 Example. A major motivation for dealing with tensor products is the fact that extensions provide a useful tool for dealing with torsion.

Consider a $\mathbb{Z}$-module M. In general, the module M is *not torsion free*, i.e. for the torsion submodule of M

$$T(M) := \{m \in M : \exists a \in \mathbb{Z} \setminus \{0\} \text{ such that } am = 0\}$$

holds $T(M) \neq \{0\}$. We claim that the extension $M_{\mathbb{Q}}$ is always torsion free. Indeed, $M_{\mathbb{Q}}$ is a $\mathbb{Q}$-vector space by lemma 8.5. Since any vector space admits a basis, it is in particular a free $\mathbb{Q}$-module. For any free $\mathbb{Q}$-module holds $T(M_{\mathbb{Q}}) = \{0\}$, compare exercise **??**.

For example, let $n \in \mathbb{N}_{>0}$, and consider the $\mathbb{Z}$-module $M := \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}$. Then the extension of M is given by $M_{\mathbb{Q}} \cong \mathbb{Q}$.

8.10 Lemma. *Let $j : R \to S$ and $j' : S \to T$ be homomorphisms of rings with units. In particular, S is an R-module via j, and T is an S-module via j', as well as an R-module via the composition $j' \circ j$. Then for all R-modules M there is a natural isomorphism of T-modules*

$$M_T \cong (M_S)_T.$$

Proof. The proof of the claim is straightforward and left as an exercise to the reader. □