7 Tensors and free modules

Throughout this section let $(R, +, \cdot)$ always be a commutative ring with a multiplicative identity element.

7.1 Lemma. *Let M and N be R-modules. Let M be a free R-module with basis $E = \{e_i\}_{i \in I} \subseteq M$, and let $e_1, \ldots, e_k \in E$ be pairwise different elements for some $k \in \mathbb{N}_{>0}$. Let $n_1, \ldots, n_k \in N$ be elements such that $\sum_{i=1}^k e_i \otimes n_i = 0$ in $M \otimes_R N$. Then $n_1 = \ldots = n_k = 0$.*

Proof. For any $i \in I$, the i-th coordinate map

$$\begin{array}{rccc} p_i : & M & \to & R \\ & m = \sum_{i \in I} r_i e_i & \mapsto & r_i \end{array}$$

is a well-defined R-linear map. The map

$$\begin{array}{rccc} \varphi_i : & M \times N & \to & N \\ & (n, m) & \mapsto & p_i(m) \cdot n \end{array}$$

is bilinear. Hence, by the universal property of the tensor product, there exists a unique R-linear map $\tilde{\varphi}_i : M \otimes N \to N$ such that for all $(m, n) \in M \times N$ holds $\tilde{\varphi}_i(m \otimes n) = p_i(m) \cdot n$. In particular, for all $i \in \{1, \ldots, k\} \subseteq I$ we compute

$$0 = \tilde{\varphi}_i(0) = \tilde{\varphi}_i(\sum_{j=1}^k m_j \otimes n_j) = \sum_{j=1}^k \tilde{\varphi}_i(m_j \otimes n_j) = \sum_{j=1}^k p_i(m_j) \cdot n_j = n_i$$

as claimed. □

7.2 Proposition. *Let M be a free R-module with basis $(e_i)_{i \in I}$. Let N be an R-module. Then for any $t \in M \otimes N$ there exists a unique family $(n_i)_{i \in I}$ with $|\{i \in I : n_i \neq 0\}| < \infty$ such that*

$$t = \sum_{i \in I} e_i \otimes n_i.$$

Proof. The existence of such a family follows since for the tensor product $\tau : M \times N \to M \otimes N$ holds

$$\begin{aligned} \operatorname{im}(\tau) &= \operatorname{span}_R\{\tau(M \times N)\} \\ &= \operatorname{span}_R\{m \otimes n : m \in M, n \in N\} \\ &= \operatorname{span}_R\{\textstyle\sum_{i \in I} r_i e_i \otimes n : \sum_{i \in I} r_i e_i \in M, n \in N\}. \end{aligned}$$

The uniqueness follows from lemma 7.1. □

7.3 Corollary. *Let M and N be free R-modules with bases $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$, respectively. Then $(e_i \otimes f_j)_{(i,j) \in I \times J}$ is a basis of $M \otimes_R N$. Moreover, if M and N are finitely generated and free, then*

$$\operatorname{rank}(M \otimes_R N) = \operatorname{rank}(M) \cdot \operatorname{rank}(N).$$

In particular, if $(R, +, \cdot)$ is a field, then $\dim(M \otimes_R N) = \dim(M) \cdot \dim(N)$.

Proof. Straightforward. □

7.4 Example. Consider $M = N := \mathbb{C}$ as a free module (i.e. vector space) over $\mathbb{R}$. Clearly, an $\mathbb{R}$-basis of $\mathbb{C}$ is given by $\{1, i\} \subset \mathbb{C}$. Thus

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{span}_{\mathbb{R}}\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\},$$

and $\dim_{\mathbb{R}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = 4$.

7.5 Example. Consider $M := \mathbb{R}$ and $N := \mathbb{C}$ as free modules over $\mathbb{R}$. Let $\alpha : \mathbb{R} \to \mathbb{C}$ be the inclusion map, and let $\beta : \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Note that β is $\mathbb{R}$-linear.

On the basis $\{1 \otimes 1, 1 \otimes i\} \subset \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ we compute for the induced map $\alpha \otimes \beta(1 \otimes i) = \alpha(1) \otimes \beta(i) = 1 \otimes (-i) = -1 \otimes i$, as well as $\alpha \otimes \beta(1 \otimes 1) = 1 \otimes 1$. Using the isomorphism given by the choice of the basis $\{1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i\} \subset \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ as in example 7.4, we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\alpha \otimes \beta} & \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{R}^2 & \xrightarrow[\varphi_A]{} & \mathbb{R}^4 \end{array}$$

where the $\mathbb{R}$-linear map $\varphi_A : \mathbb{R}^2 \to \mathbb{R}^4$ is represented with respect to the standard bases by the matrix

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

7.6 Lemma. *Let M be a free R-modules with basis $\{e_i\}_{i \in I}$ for some index set I. Let N be an R-module, and let $\{n_i\}_{i \in I} \subseteq N$ be a family of elements indexed by I, too. Then there exists a unique homomorphism of R-modules $\varphi : M \to N$ such that for all $i \in I$ holds $\varphi(e_i) = n_i$.*

Proof. Under the given assumptions, we construct a homomorphism $\varphi : M \to N$ as follows. Let $m \in M$. Since $\{e_i\}_{i\in I}$ is a basis of M, there exists a unique family $\{a_i\}_{i\in I}$ in R, with $|\{i \in I : a_i \neq 0\}| < \infty$, such that $m = \sum_{i\in I} a_i e_i$. We define $\varphi(m) := \sum_{i\in I} a_i n_i \in N$. It is easy to see that φ is R-linear, and satisfies $\varphi(e_i) = n_i$ for all $i \in I$.

Let $\psi : M \to N$ be another homomorphism such that $\psi(e_i) = n_i$ for all $i \in I$. Then for any $m = \sum_{i\in I} a_i e_i \in M$, using the R-linearity of both ψ and φ, we compute $\psi(m) = \sum_{i\in I} a_i \psi(e_i) = \sum_{i\in I} a_i \varphi(e_i) = \varphi(m)$. □

The above lemma 7.6 states, that for a free module, a homomorphism can be uniquely defined by just specifying the images of the elements of a basis. We will frequently make use of this fact.

7.7 Lemma. *Let M and N be free R-modules with bases $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$, respectively. Let M be of finite rank. Then* $\mathrm{Hom}_R(M, N)$ *is a free R-module with basis $\{\varepsilon_{e_i,f_j}\}_{(i,j)\in I\times J}$, where for $(i, j) \in I \times J$ the homomorphism ε_{e_i,f_j} is defined on the basis of M by*

$$\begin{array}{rccl} \varepsilon_{e_i,f_j} : & M & \to & N \\ & e_k & \mapsto & \begin{cases} f_j, & \text{if } k = i \\ 0, & \text{if } k \neq i. \end{cases} \end{array}$$

Proof. By lemma 7.6, the homomorphisms ε_{e_i,f_j} are well-defined by defining them on a basis.

Let $\alpha : M \to N$ be a homomorphism of R-modules. Since $\{f_j\}_{j\in J}$ is a basis of N, there exist for all $k \in I$ families $\{a^k_j\}_{j\in J}$ in R with $|\{j \in J : a^k_j \neq 0\}| < \infty$, such that $\alpha(e_k) = \sum_{j\in J} a^k_j f_j$. From the definition, we obtain $f_j = \varepsilon_{e_i,f_j}(e_k)$, if and only if $i = k$, and thus

$$a^k_j f_j = \varepsilon_{e_k,f_j}(a^k_j e_k) = \sum_{i\in I} \varepsilon_{e_i,f_j}(a^k_j e_k) = \sum_{i\in I} a^i_j \varepsilon_{e_i,f_j}(e_k).$$

Therefore $\alpha(e_k) = \sum_{(i,j)\in I\times J} a^i_j \varepsilon_{e_i,f_j}(e_k)$. Again by lemma 7.6 this implies $\alpha = \sum_{(i,j)\in I\times J} a^i_j \varepsilon_{e_i,f_j}$. Note that this sum is indeed finite, by the choice of the families $\{a^k_j\}_{j\in J}$, together with the fact that $|I| < \infty$. This shows that $\{\varepsilon_{e_i,f_j}\}_{(i,j)\in I\times J}$ is a generating subset for $\mathrm{Hom}_R(M, N)$.

To see that the family $\{\varepsilon_{e_i,f_j}\}_{(i,j)\in I\times J}$ is R-linearly independent, consider a family $\{a^i_j\}_{j\in J}$ in R with $|\{(i, j) \in I \times J : a^i_j \neq 0\}| < \infty$, such that $\alpha := \sum_{(i,j)\in I\times J} a^i_j \varepsilon_{e_i,f_j} = 0$. In particular, for all $i \in I$ we compute

$$0 = \alpha(e_i) = \sum_{(i,j)\in I\times J} a^i_j \varepsilon_{e_i,f_j}(e_i) = \sum_{(i,j)\in I\times J} a^i_j f_j.$$

Since the basis $\{f_j\}_{j\in J}$ is R-linearly independent, we must have $a^i_j = 0$ for all $j \in J$. □

7.8 Remark. Let M and N be free R-modules of finite ranks $r, s \in \mathbb{N}_{<0}$ with bases $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_s\}$, respectively. Then, analogously to the theory of vector spaces, we may use lemma 7.7 to identify homomorphisms $\alpha \in \mathrm{Hom}_R(M,N)$ with matrices $A_\alpha \in \mathrm{Mat}(m,n,R)$. Using the notation from the proof of 7.7, an isomorphism of R-modules is given by

$$\begin{array}{rccc} A: & \mathrm{Hom}_R(M,N) & \to & \mathrm{Mat}(m,n,R) \\ & \alpha & \mapsto & A_\alpha := (a^i_j)_{1\le i\le r, 1\le j\le s} \end{array}$$

Recall that $a^i_j \in R$ has been defined as the j-th coordinate of the image of the i-th basis vector $\alpha(e_i)$.

With respect to this identification, the homomorphisms ε_{e_i,f_j} correspond precisely to the elementary matrices E^j_i, where all entries are 0, except the entry in the i-th column and j-th line, which equals 1. Obviously, these matrices form a basis of $\mathrm{Mat}(m,n,R)$.

7.9 Remark. In general, the claim of lemma 7.7 is not true, if the R-module M is not of finite rank. There exist examples of free modules of infinite rank, where the dual module is not free, see [**?**, II, §2.6].

7.10 Proposition. *Let M, M', N and N' be free R-modules of finite ranks. Then there is an isomorphism of R-modules*

$$\tilde{T}: \mathrm{Hom}_R(M,M') \otimes_R \mathrm{Hom}_R(N,N') \to \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N')$$

such that for all $\alpha \in \mathrm{Hom}_R(M,M')$, $\beta \in \mathrm{Hom}_R(N,N')$, $m \in M$ and $n \in N$ holds

$$\tilde{T}(\alpha \otimes \beta)(m \otimes n) = \alpha(m) \otimes \beta(n).$$

Proof. Recall that there is a homomorphism of R-modules

$$\begin{array}{rccc} T: & \mathrm{Hom}_R(M,M') \times \mathrm{Hom}_R(N,N') & \to & \mathrm{Hom}_R(M \otimes_R N, M' \otimes_R N') \\ & (\alpha,\beta) & \mapsto & \alpha \otimes \beta := \alpha \otimes \mathrm{id}_{N'} \circ \mathrm{id}_M \otimes \beta \end{array}$$

which is easily seen to be bilinear. Hence the R-linear map $\tilde{T}$ exists as claimed.

Consider bases $\{e_i\}_{i\in I}$, $\{f_j\}_{j\in J}$, $\{e'_k\}_{k\in K}$ and $\{f'_\ell\}_{\ell\in L}$ of M, M', N and N', respectively. By lemma 7.7, they determine a basis $\{\varepsilon_{e_i,f_j}\}_{(i,j)\in I\times J}$ of $\mathrm{Hom}_R(M,M')$, and a basis $\{\varepsilon_{e'_k,f'_\ell}\}_{(k,\ell)\in K\times L}$ of $\mathrm{Hom}_R(N,N')$. Hence by corollary 7.3, a basis of $\mathrm{Hom}_R(M,M')\otimes_R \mathrm{Hom}_R(N,N')$ is given by $\{\varepsilon_{e_i,f_j}\otimes \varepsilon_{e'_k,f'_\ell}\}_{(i,j,k,\ell)\in I\times J\times K\times L}$.
On the other hand, again using corollary 7.3, we have bases $\{e_i\otimes e'_k\}_{(i,k)\in I\times K}$ of $M\otimes_R N$ and $\{f_j\otimes f'_\ell\}_{(j,\ell)\in J\times L}$ of $M'\otimes N'$. By 7.7, they give a basis $\{\varepsilon_{e_i\otimes e'_k, f_j\otimes f'_\ell}\}_{(i,j,k,\ell)\in I\times J\times K\times L}$ of $\mathrm{Hom}_R(M\otimes_R N, M'\otimes_R N')$.
Let an index tuple $(i,j,k,\ell)\in I\times J\times K\times L$ be given. Consider an element $e_s\otimes e'_t$ of the basis of $M\otimes_R N$, for some $s\in I$ and $t\in K$. We compute

$$\tilde{T}(\varepsilon_{e_i,f_j}\otimes\varepsilon_{e'_k,f'_\ell})(e_s\otimes e'_t) = \varepsilon_{e_i,f_j}(e_s)\otimes\varepsilon_{e'_k,f'_\ell}(e'_t) = \begin{cases} f_j\otimes f'_\ell, & \text{if } s=i,\ t=k \\ 0, & \text{otherwise.}\end{cases}$$

By definition, we have

$$\varepsilon_{e_i\otimes e'_k, f_j\otimes f'_\ell}(e_s\otimes e'_t) = \begin{cases} f_j\otimes f'_\ell, & \text{if } s=i,\ t=k \\ 0, & \text{otherwise.}\end{cases}$$

Thus the two homomorphism agree on a basis, and hence we have an identity $\tilde{T}(\varepsilon_{e_i,f_j}\otimes\varepsilon_{e'_k,f'_\ell}) = \varepsilon_{e_i\otimes e'_k, f_j\otimes f'_\ell}$. In particular, the homomorphism $\tilde{T}$ is surjective. Moreover, since $\tilde{T}$ is bijectively mapping a basis to a basis, it is also injective, and thus an isomorphism as claimed. □

7.11 Corollary. *Let M and N be free R-modules of finite ranks. Then there are isomorphisms*

$$\begin{array}{lrcl} a) & M^*\otimes_R N & \cong & \mathrm{Hom}_R(M,N); \\ b) & (M\otimes_R N)^* & \cong & M^*\otimes_R N^*. \end{array}$$

Proof. Note that for any R-module M, there is a canonical isomorphism $\mathrm{Hom}_R(R,M)\cong M$. Using this, together with **??**, we compute immediately from proposition 7.10

$$\begin{aligned} \mathrm{Hom}_R(M,N) &\cong \mathrm{Hom}_R(M\otimes_R R, R\otimes_R N) \\ &\cong \mathrm{Hom}_R(M,R)\otimes_R \mathrm{Hom}_R(R,N) \\ &\cong M^*\otimes_R N \end{aligned}$$

as well as

$$\begin{aligned} (M\otimes_R N)^* &= \mathrm{Hom}_R(M\otimes_R N, R) \\ &\cong \mathrm{Hom}_R(M\otimes_R N, R\otimes_R R) \\ &\cong \mathrm{Hom}_R(M,R)\otimes_R \mathrm{Hom}_R(N,R) \\ &= M^*\otimes_R N^*. \end{aligned}$$

This proves the claims. □

7.12 Corollary. *Let M, N and L be free R-modules of finite ranks. Then there is an isomorphism*

$$\mathrm{Hom}_R(M, N \otimes_R L) \cong \mathrm{Hom}_R(M, N) \otimes_R L.$$

Proof. We obtain $\mathrm{Hom}_R(M, N \otimes_R L) \cong \mathrm{Hom}_R(M \otimes_R R, N \otimes_R L) \cong \mathrm{Hom}_R(M, N) \otimes_R \mathrm{Hom}_R(R, L) \cong \mathrm{Hom}_R(M, N) \otimes_R L$ directly from proposition 7.10. □

7.13 Proposition. *Let M, M', N and N' be free R-modules of finite ranks. Let $\alpha : M \to M'$ and $\beta : N \to N'$ be both injective homomorphisms of R-modules. Then $\alpha \otimes \beta : M \otimes_R N \to M' \otimes_R N'$ is injective, too.*

Proof. Recall from **??** the identity $\alpha \otimes \beta = \alpha \otimes \mathrm{id}_{N'} \circ \mathrm{id}_M \otimes \beta$. We will only show that $\mathrm{id}_M \otimes \beta$ is injective, if β is injective. The proof for $\alpha \otimes \mathrm{id}_{N'}$ is completely analogous, and taken together this implies the injectivity of $\alpha \otimes \beta$.

Let $t \in \ker(\mathrm{id}_M \otimes \beta)$. The modules M and N are free, so there exist bases $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$, respectively. Thus there is a unique family $\{r_{ij}\}_{(i,j) \in I \times J}$ in R, such that $t = \sum_{(i,j) \in I \times J} r_{ij} e_i \otimes f_j$. We compute

$$0 = \mathrm{id}_M \otimes \beta(t) = \mathrm{id}_M \otimes \beta\Big(\sum_{(i,j) \in I \times J} r_{ij} e_i \otimes f_j\Big) = \sum_{(i,j) \in I \times J} e_i \otimes \beta(r_{ij} f_j).$$

For all $i \in I$, lemma 7.1 now implies $\beta(\sum_{j \in J} r_{ij} f_j) = 0$. Since β is injective by assumption, we must have $\sum_{j \in J} r_{ij} f_j = 0$. But $\{f_j\}_{j \in J}$ is a basis, so we obtain $r_{ij} = 0$ for all $(i, j) \in I \times J$. Therefore $t = 0$. □

7.14 Remark. We have seen in **??** that for a given R-module M, the functor $M \otimes_R \bullet : (R\text{-Mod}) \to (R\text{-Mod})$ is right-exact. The functor is left-exact, if the module M is free.