5 Multilinear Maps

An invaluable feature of vector spaces over the field of real numbers is that they admit *inner products*. Geometric concepts, such as orthogonality or norm, can be introduced with respect to a given inner product. The definition of multilinear maps generalizes both inner products and linear maps in a natural way.

Throughout this section, let $(R, +, \cdot)$ always denote a commutative ring with a multiplicative identity element. For an R-module $(M, +, \cdot)$ we will simply write M.

5.1 Definition. Let $p \in \mathbb{N}_{>0}$. Let $M_1, \ldots, M_p$ and N be R-modules. A *p-linear map* from $M_1 \times \ldots \times M_p$ to N is a map

$$\varphi : \quad M_1 \times \ldots \times M_p \to N$$

such that for all $(m_1, \ldots, m_p) \in M_1 \times \ldots \times M_p$ and for all $i = 1, \ldots, p$ the map

$$\begin{array}{cccc} \varphi^i_{m_1,\ldots,m_p} : & M_1 \times \ldots \times M_p & \to & N \\ & m & \mapsto & \varphi(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots, m_p) \end{array}$$

is a homomorphism of R-modules.

5.2 Remark. **a)** An 1-linear map $\varphi : M_1 \to N$ is a homomorphism of R-modules, or equivalently, an R-linear map. A 2-linear map is called *bilinear*. For a general $p \in \mathbb{N}_{>0}$, a p-linear map is called *multilinear*.

b) A map $\varphi : M_1 \times M_2 \to N$ is bilinear, if and only if for $i = 1, 2$, for all $m_i, m'_i \in M_i$ and for all $r \in R$ hold

$$\begin{array}{lccc} (1) & \varphi(m_1 + m'_1, m_2) & = & \varphi(m_1, m_2) + \varphi(m'_1, m_2) \\ (2) & \varphi(r \cdot m_1, m_2) & = & r \cdot \varphi(m_1, m_2) \\ (3) & \varphi(m_1, m_2 + m'_2) & = & \varphi(m_1, m_2) + \varphi(m_1, m'_2) \\ (4) & \varphi(m_1, r \cdot m_2) & = & r \cdot \varphi(m_1, m_2). \end{array}$$

5.3 Example. Let $(R, +, \cdot) = (K, +, \cdot)$ be a field. Consider the vector space $V := K^n$ of dimension $n > 0$ over K. For typographical reasons, we use throughout the "horizontal" notation for a vector $v \in K^n$, so that the corresponding column vector is written as the transpose ${}^t v$.

a) The *standard inner product*

$$\begin{array}{cccc} \langle .,. \rangle : & K^n \times K^n & \to & K \\ & (u,v) & \mapsto & \langle u,v \rangle := u \cdot^t v \end{array}$$

is a bilinear map.

b) The determinant map $\det : \mathrm{Mat}(n,n,K) = (K^n)^n \to K$, considered as a map in the columns of the respective matrices,

$$\begin{array}{cccc} \det : & K^n \times \ldots \times K^n & \to & K \\ & (v_1,\ldots,v_n) & \mapsto & \det(v_1,\ldots,v_n) \end{array}$$

is an n-linear map.

5.4 Notation. Let $p \in \mathbb{N}_{>0}$. Let $M_1,\ldots,M_p$ and N be R-modules. We denote by

$$L_R(M_1,\ldots,M_p;N) := \{\varphi : M_1 \times \ldots \times M_p \to N \ p\text{-linear}\}$$

the set of all p-linear maps from $M_1 \times \ldots \times M_p$ to N. In particular, for $M_1 = \ldots = M_p =: M$, we write for p-linear maps from the p-fold direct product of M to N simply

$$L_R^p(M;N) := L_R(M \times \ldots \times M;N).$$

5.5 Remark. a) The triple $(L_R(M_1,\ldots,M_p;N), +_{p-w}, \cdot_{p-w})$ is again an R-module, with respect to the point-wise defined composition and operation.

b) For $p = 1$, we clearly have $L_R(M;N) = L_R^1(M;N) = \mathrm{Hom}_R(M,N)$. In the special case $N = R$ we obtain the dual module $L_R^1(M;R) = M^*$.

5.6 Exercise. Show that for any $p \in \mathbb{N}_{>0}$ there is an isomorphism of R-modules

$$L_R^p(R;R) \cong R.$$

5.7 Proposition. *Let M, N and L be R-Modules. Then there is an isomorphism of R-modules*

$$L_R(M,N;L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,L)).$$

Proof.

□

5.8 Corollary. *Let M and N be R-modules. Then there is an isomorphism of R-modules*

$$L_R(M, N; R) \cong \mathrm{Hom}_R(M, N^*).$$

Proof. Immediately from proposition 5.7, since $N^* = \mathrm{Hom}_R(N, R)$. □

5.9 Example. To get an intuitive idea of the statement in corollary 5.8, we consider a standard problem from physics.

We think of a physical object as a point in real-world space, so its *position* is given by a vector $\vec{x} \in \mathbb{R}^3 =: V$. Suppose that a constant *force* is present (e.g. gravitation). The force has a magnitude and a direction, so it is also represented by a vector $\vec{f} \in \mathbb{R}^3 =: F$. Note that from a physicist's point of view $V \neq F$ (for a start, one is measured in "meters" m, while the other is measured in "Newton" $N = \frac{kg \cdot m}{s^2}$).

Moving the physical object involves *work* (measuring the change of its potential energy). The amount of work while moving our object from $\vec{x}$ to $\vec{x} + \vec{y}$ is denoted by $W(\vec{f}, \vec{y}) \in \mathbb{R}$. Note that negative work occurs, when energy is released (think of dropping a stone).

Clearly, doubling the force doubles the work involved. Moreover, forces are additive: if they act in different directions, they may cancel each other out. Mathematically, we have an $\mathbb{R}$-linear map

$$\begin{array}{rccc} w_{\vec{y}} : & F & \to & \mathbb{R} \\ & \vec{f} & \mapsto & W(\vec{f}, \vec{y}) \end{array}$$

computing how much work is needed to move the object a fixed distance $\vec{y}$, depending on varying forces acting.

On the other hand, we may wish to compute the work needed to move the object an arbitrary distance $\vec{y}$ in the presence of a constant force $\vec{f}$. Obviously, the longer the distance is, the more work is needed. Again, we have an $\mathbb{R}$-linear map

$$\begin{array}{rccc} w_{\vec{f}} : & V & \to & \mathbb{R} \\ & \vec{y} & \mapsto & W(\vec{f}, \vec{y}). \end{array}$$

In summary, we found a bilinear map $W : F \times V \to \mathbb{R}$. The formula for computing the work in physics is simply

$$W = \langle \vec{f}, \vec{y} \rangle$$

where $\langle \vec{f}, \vec{y} \rangle$ denotes the standard inner product on $\mathbb{R}^3$, up to physical measuring units.

Let us relate this to corollary 5.8. Note that for any $\vec{f} \in F$ holds $w_{\vec{f}} \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) = V^*$. As a bilinear map, we have $W \in L_{\mathbb{R}}(F, V; \mathbb{R})$. It corresponds uniquely to the map $\vec{f} \mapsto w_f$ in $\mathrm{Hom}(F, V^*)$.

5.10 Lemma. *Let M be a free R-module of dimension $n < \infty$, together with a basis $E = \{e_1, \dots, e_n\} \subset M$. Then there is a canonical isomorphism of R-modules*

$$L^2_R(M; R) \cong \mathrm{Mat}(n, n, R).$$

Proof. Let $\varphi \in L^2_R(M; R)$. We define a matrix $A_\varphi := (a_{ij})_{1 \le i,j \le n} \in \mathrm{Mat}(n, n, R)$ by

$$a_{ij} := \varphi(e_i, e_j).$$

With this notation, we obtain a map

$$\begin{array}{cccc} \alpha: & L^2_R(M; R) & \to & \mathrm{Mat}(n, n, R) \\ & \varphi & \mapsto & A_\varphi \end{array}$$

It is easy to verify for two bilinear maps $\varphi, \psi : M \times M \to R$ and elements $r \in R$ the equations

$$A_{\varphi+\psi} = A_\varphi + A_\psi \quad \text{and} \quad A_{r\varphi} = rA_\varphi.$$

Thus α is a homomorphism of R-modules. It is even an isomorphism, where for a matrix $A \in \mathrm{Mat}(n, n, R)$ the image $\varphi_A := \alpha^{-1}(A)$ under the inverse homomorphism is given by

$$\begin{array}{cccc} \varphi_A: & M \times M & \to & R \\ & (m_1, m_2) & \mapsto & m_1 \cdot A \cdot^t m_2 \end{array}$$

□

Note that the isomorphism of lemma 5.10 is canonical only because a basis E is given a priory. In general, for a free R-module of finite dimension, all we can say is that such an isomophism always exists.

5.11 Lemma. *Let $p \in \mathbb{N}_{>0}$. Let $M_1, \dots, M_p$ and N, N' be R-modules, and let $\varphi : M_1 \times \ldots \times M_p \to N$ be a p-linear map. Let $\beta : N \to N'$ be a homomorphism of R-modules. Then $\beta \circ \varphi : M_1 \times \ldots \times M_p \to N'$ is p-linear.*

Proof. Straightforward. □

5.12 Remark. In the special case $M_1 = \ldots = M_p =: M$, lemma 5.11 implies that for any homomorphism $\beta : N \to N'$ of R-modules, there is a map

$$\begin{array}{rccc} \beta_* : & L_R^p(M;N) & \to & L_R^p(M;N') \\ & \varphi & \mapsto & \beta \circ \varphi \end{array}$$

which is in fact a homomorphism of R-modules. We thus obtain for any given R-module M a (covariant) functor

$$\begin{array}{rccc} L_R^p(M,\bullet) : & (R\text{-Mod}) & \to & (R\text{-Mod}) \\ & N & \mapsto & L_R^p(M;N) \\ & \beta & \mapsto & \beta_* \end{array}$$

The verification of the details is left to the reader.

5.13 Exercises. a) Let $p \in \mathbb{N}_{>0}$, and let $\sigma \in \Sigma_p$ be a permutation of the set $\{1, \ldots, p\}$. Let $M_1, \ldots, M_p$ and N be R-modules. Then there exists a natural isomorphism of R-modules

$$L_R(M_1, \ldots, M_p; N) \cong L_R(M_{\sigma(1)}, \ldots, M_{\sigma(p)}; N).$$

b) Let $M_1, \ldots, M_p$ and M_1' and N be R-modules. Then there exists a natural isomorphism of R-modules

$$L_R(M_1 \oplus M_1', M_2, \ldots, M_p; N) \cong L_R(M_1, \ldots, M_p; N) \oplus L_R(M_1', \ldots, M_p; N).$$

c) Let M_1, M_2 and N be R-modules, with submodules $M_1' \subseteq M_1$ and $N' \subseteq N$. Let $\varphi \in L_R(M_1, M_2; N)$ be a bilinear map. Suppose that for all $m_1' \in M_1'$ and all $m_2 \in M_2$ holds $\varphi(m_1', m_2) \in N'$. Then the map

$$\begin{array}{rccc} \overline{\varphi} : & M_1/M_1' \times M_2 & \to & N/N' \\ & ([m_1], m_2) & \mapsto & [\varphi(m_1, m_2)] \end{array}$$

is well-defined and bilinear.

5.14 Remark. Let $\varphi : M_1 \times \ldots \times M_p \to N$ be a p-linear map of R-modules. By looking a examples, it is easy to see that in general neither is $\varphi^{-1}(\{0\})$ a submodule of $M_1 \times \ldots \times M_p$, nor is $\varphi(M_1 \times \ldots \times M_p)$ a submodule of N.

5.15 Definition. Let $\varphi : M_1 \times \ldots \times M_p \to N$ be a p-linear map of R-modules. The *image of* φ is the smallest submodule of N, which contains the set-theoretic image of φ

$$\mathrm{im}(\varphi) := \mathrm{span}(\varphi(M_1 \times \ldots \times M_p)).$$