10 The exterior algebra

Throughout this section let $(R, +, \cdot)$ always be a commutative ring with a multiplicative identity element, and let M be an R-module.

10.1 Definition. Let $p \in \mathbb{N}_{\geq 2}$. Let N be an R-module. A p-linear map $\varphi : M^p \to N$ is called *alternating*, if for all $(m_1, \ldots, m_p) \in M^p$ the following implication holds:

If there exist $i, j \in \{1, \ldots, p\}$ with $i \neq j$ such that $m_i = m_j$ holds, then $\varphi(m_1, \ldots, m_p) = 0$.

10.2 Notation. We define the R-module of alternating p-linear maps by

$$\mathrm{Alt}^p_R(M, N) := \{\varphi : M^p \to N : \ \varphi \text{ alternating } \}.$$

10.3 Example. Let $(R, +, \cdot) = (K, +, \cdot)$ be a field, and let $M = K^n$. As vector spaces over K, we identify $M^n \cong \mathrm{Mat}(n, n, K)$. Then the determinant map $\det : \mathrm{Mat}_K(n, n) \to K$ is alternating.

10.4 Definition. Let M be an R-module. For $p \in \mathbb{N}_{\geq 2}$ we define a submodule of $\bigotimes^p M$ by

$$N^p(M) := \mathrm{span}_R\{m_1 \otimes \ldots \otimes m_p \in \bigotimes{}^p M : \ \exists i \neq j \text{ s.th. } m_i = m_j\}.$$

The R-module quotient

$$\bigwedge{}^p M := \bigotimes{}^p M \,/\, N^p(M)$$

is called the *p-th exterior power of* M, or the *p-th alternating power of* M. For equivalence classes, we use the notation

$$m_1 \wedge \ldots \wedge m_p := [m_1 \otimes \ldots \otimes m_p] \in \bigwedge{}^p M.$$

10.5 Remark. The composed map τ^a, defined by

$$M^p \xrightarrow{\tau} \bigotimes{}^p M \xrightarrow{\pi} \bigwedge{}^p M, \qquad \tau^a = \pi \circ \tau : M^p \to \bigwedge{}^p M$$

is p-linear and alternating. Indeed, for an element $(m_1, \ldots, m_p) \in M^p$ with $m_i = m_j$ for some $i \neq j$, we have $\tau(m_1, \ldots, m_p) \in N^p(M)$, so that $\pi \circ \tau(m_1, \ldots, m_p) = 0 \in \bigwedge^p M$.

10.6 Remark. As a quotient of the p-fold tensor product, the alternating product inherits rules for computation analogous to those listed in **??**. In the case $p = 2$, we have for all $m, m', m'' \in M$ and $r \in R$ the equalities

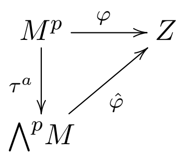
$$\begin{array}{lrcl}
(1) & (rm) \wedge m' & = & m \wedge (rm') \\
(2) & (m + m') \wedge m'' & = & m \wedge m'' + m' \wedge m'' \\
(3) & m \wedge (m' + m'') & = & m \wedge m' + m \wedge m'' \\
(4) & m \wedge 0 & = & 0 \\
(5) & 0 \wedge m & = & 0.
\end{array}$$

Analogous formulae hold for all $p \in \mathbb{N}_{\geq 2}$. We have furthermore

(6) $m_1 \wedge \ldots \wedge m_p = 0$ if $m_i = m_j$ for some $1 \leq i, j \leq p$ with $i \neq j$.

10.7 Proposition. *Let M be an R-module. The p-th exterior power of M is up to isomorphism uniquely determined by the following universal property.*

For any R-module Z, and any alternating p-linear map $\varphi : M^p \to Z$, there exists a unique p-linear map $\hat{\varphi}$ such that the diagram



commutes.

Proof. Follows from the universal property of the tensor product. □

10.8 Remark. As before, the universal property of the p-th exterior power implies the existence of a covariant functor

$$\begin{array}{rccc}
\bigwedge^p : & (R\text{-Mod}) & \to & (R\text{-Mod}) \\
& M & \mapsto & \bigwedge^p \mathrm{M} \\
& M \xrightarrow{\alpha} M' & \mapsto & \bigwedge^p M \xrightarrow{\wedge^p \alpha} \bigwedge^p M'
\end{array}$$

such that the equality $\wedge^p \alpha(m_1 \wedge \ldots \wedge m_p) = \alpha(m_1) \wedge \ldots \wedge \alpha(m_p)$ holds for all $(m_1 \ldots, m_p) \in M^p$.

Indeed, for any homorphism $\alpha : M_1 \to M_2$ of R-modules, the composed map $\tau_2^a \circ (\alpha \times \ldots \times \alpha) : M_1^p \to M_2^p \to \bigwedge^p M_2$ is alternating. The map $\wedge^p \alpha : \bigwedge^p M_1 \to \bigwedge^p M_2$ is defined as the unique R-linear map satisfying $\wedge^p \alpha \circ \tau_1^a = \tau_2^a \circ (\alpha \times \ldots \times \alpha)$ given by the universal property.

10.9 Proposition. *Let M be a free R-module of rank $n < \infty$. Then*

$$\bigwedge^p M = \{0\} \quad \text{for all } p > n.$$

Proof. Let $\{e_1 \ldots, e_n\}$ be a basis of M. Then $\bigotimes^p M = \text{span}_R\{e_{i_1} \otimes \ldots \otimes e_{i_p} : 1 \leq i_1, \ldots, i_p \leq n\}$. By its construction as a quotient,

$$\bigwedge^p M = \text{span}_R\{e_{i_1} \wedge \ldots \wedge e_{i_p} : 1 \leq i_1, \ldots, i_p \leq n\}.$$

If $p > n$, then for any p-tuple $(i_1, \ldots, i_p)$, theres exists at least one pair of indices $1 \leq j, k \leq p$ with $j \neq k$ but $i_j = i_k$. Thus $e_{i_1} \wedge \ldots \wedge e_{i_p} = 0$. □

10.10 Remark. Let $p \in \mathbb{N}_{>0}$. Recall that the group of permutations $(\Sigma_p, \circ)$ is given by the set Σ_p of bijective maps from $\{1, \ldots, p\}$ to itself, together with the composition "$\circ$" of maps. For a permutation $\sigma \in \Sigma_p$, the composed map

$$\begin{array}{ccccc} M^p & \to & M^p & \to & \bigotimes^p M \\ (m_1, \ldots, m_p) & \mapsto & (m_{\sigma(1)}, \ldots, m_{\sigma(p)}) & \mapsto & m_{\sigma(1)} \otimes \ldots \otimes m_{\sigma(p)} \end{array}$$

is p-linear. Thus, by the universal property of the tensor product, it defines a unique R-linear map $\bigotimes^p M \to \bigotimes^p M$ which shall also be denoted by σ, by abuse of notation.

Obviously, for this map holds $\sigma(N^p(M)) \subseteq N^p(M)$. By the universal property of the quotient $\bigwedge^p M = \bigotimes^p M / N^p(M)$ there exists a unique R-linear map $\overline{\sigma}$, which makes the following diagram commutative:

$$\begin{array}{ccc} \bigotimes^p M & \xrightarrow{\sigma} & \bigotimes^p M \\ \downarrow \pi & & \downarrow \pi \\ \bigwedge^p M & \xrightarrow[\overline{\sigma}]{} & \bigwedge^p M \end{array}$$

It is customary to denote the unique homomorphism $\overline{\sigma}$ again by σ. By construction, it is given on generating elements by

$$\begin{array}{cccc} \sigma : & \bigwedge^p M & \to & \bigwedge^p M \\ & m_1 \wedge \ldots \wedge m_p & \mapsto & m_{\sigma(1)} \wedge \ldots \wedge m_{\sigma(p)} \end{array}$$

10.11 Proposition. *Let M be an R-module and let $p \in \mathbb{N}_{\geq 2}$. Then for all $a \in \bigwedge^p M$ and all $\sigma \in \Sigma_p$ holds*

$$\sigma(a) = \text{sign}(\sigma)\, a.$$

Proof. Since the map $\sigma : \bigwedge^p M \to \bigwedge^p M$ is R-linear, it is enough to prove the formula on generating elements $a = m_1 \wedge \ldots \wedge m_p \in \bigwedge^p M$, where $m_1, \ldots, m_p \in M$.

Consider a representative $t = m_1 \otimes \ldots \otimes m_p \in \bigotimes^p M$, so that $\pi(t) = a$. It is enough to show $n := \sigma(t) - \mathrm{sign}(\sigma)t \in N^p(M)$. To do this, we write $\sigma = \tau_1 \circ \ldots \circ \tau_k$, where $\tau_1, \ldots, \tau_k$ are transpositions. Note that $\mathrm{sign}(\sigma) = (-1)^k$.

We will prove the claim by induction on k. For $k = 1$, let $\sigma = \tau$ be the transposition interchanging the indices i and j. Without loss of generality we may assume $1 \le i < j \le p$. We compute

$$\begin{array}{rcl} n & = & m_1 \otimes \ldots \otimes m_j \otimes \ldots \otimes m_i \otimes \ldots \otimes m_p - (-1) m_1 \otimes \ldots \otimes m_p \\ & = & m_1 \otimes \ldots \otimes (m_i + m_j) \otimes \ldots \otimes (m_i + m_j) \otimes \ldots \otimes m_p \\ & & - m_1 \otimes \ldots \otimes m_i \otimes \ldots \otimes m_i \otimes \ldots \otimes m_p \\ & & - m_1 \otimes \ldots \otimes m_j \otimes \ldots \otimes m_j \otimes \ldots \otimes m_p \\ & \in & N^p(M). \end{array}$$

Now let $k \ge 2$, and assume that the formula holds up to $k - 1$. We have $\sigma = \tau_1 \circ \sigma'$, where $\sigma' = \tau_2 \circ \ldots \circ \tau_k$. By assumption, we already have $\sigma'(t) - \mathrm{sign}(\sigma')t \in N^p(M)$. Then clearly also $n' := \tau_1(\sigma'(t) - \mathrm{sign}(\sigma')t) \in N^p(M)$. We compute

$$\begin{array}{rcl l l} n' & = & \sigma(t) - \mathrm{sign}(\sigma')\,\tau_1(t) & \in N^p(M), & \text{and} \\ n_1 & := & \tau_1(t) - \mathrm{sign}(\tau_1)t & \in N^p(M) & \text{by step 1.} \end{array}$$

From this we obtain

$$\begin{array}{rcl} \sigma(t) - \mathrm{sign}(\sigma)(t) & = & \sigma(t) - \mathrm{sign}(\sigma')\mathrm{sign}(\tau_1)\, t \\ & = & \sigma(t) - \mathrm{sign}(\sigma')(\tau_1(t) - n_1) \\ & = & \sigma(t) - \mathrm{sign}(\sigma')\,\tau_1(t) + \mathrm{sign}(\sigma')n_1 \\ & = & n' + \mathrm{sign}(\sigma')n_1 \\ & \in & N^p(M). \end{array}$$

Thus $\sigma(a) - \mathrm{sign}(\sigma)a = \pi(\sigma(t) - \mathrm{sign}(\sigma)(t)) = 0$, as claimed. □

10.12 Lemma. *Let M be a free R-module of rank $n < \infty$ with basis $\{e_1, \ldots, e_n\}$. Then there exists a unique alternating p-linear map*

$$\det : M^n \to R$$

called the determinant map*, such that* $\det(e_1, \ldots, e_n) = 1$.

Proof. By construction, the n-th alternating product is given as

$$\bigwedge^n M = \mathrm{span}_R\{e_{i_1} \wedge \ldots \wedge e_{i_n} : 1 \leq i_1, \ldots, i_n \leq n\}.$$

If $\{i_1, \ldots, i_n\} \subsetneq \{1, \ldots, n\}$, we must have $i_j = i_k$ for some $j \neq k$, so that $e_{i_1} \wedge \ldots \wedge e_{i_n} = 0$. We may hence assume that all n indices of the generating elements are pairwise different, and all numbers $1, \ldots, n$ occur as indices. Reordering of the indices changes the element only by a sign $\pm 1_R$, so we get

$$\bigwedge^n M = \mathrm{span}_R\{e_1 \wedge \ldots \wedge e_n\} = R \cdot e_1 \wedge \ldots \wedge e_n.$$

Consider the coordinate map

$$\begin{array}{rccc} j : & \bigwedge^n M & \to & R \\ & r \cdot e_1 \wedge \ldots \wedge e_n & \mapsto & r \end{array}$$

By composition with the map $\tau^a : M^n \to \bigwedge^n M$, we define $\det := j \circ \tau^a$. Clearly, this is p-linear and alternating, and it satisfies $\det(e_1, \ldots, e_n) = 1$.

To prove uniqueness, consider another alternating p-linear map $d : M^n \to R$ with $d(e_1, \ldots, e_n) = 1$. By the universal property of the alternating product, there is a unique R-linear map $\hat{d} : \bigwedge^n M \to R$ such that $d = \hat{d} \circ \tau^a$.

Let $a \in \bigwedge^n M$. Then there exists an $r \in R$ such that $a = r \cdot e_1 \wedge \ldots \wedge e_n$. We compute

$$\begin{array}{rcl} \hat{d}(a) & = & r \cdot \hat{d}(e_1 \wedge \ldots \wedge e_n) = r \cdot d(e_1, \ldots, e_n) \\ & = & r \cdot 1_R \\ & = & r \cdot \det(e_1, \ldots, e_n) = r \cdot j(e_1 \wedge \ldots \wedge e_n) \quad = j(a) \end{array}$$

Hence $\hat{d} = j$, and thus $d = \tau^a \circ \hat{d} = \tau^a \circ j = \det$. □

10.13 Proposition. *Let M be a free R-module of rank $n < \infty$ with basis $\{e_1, \ldots, e_n\}$. Let $p \in \mathbb{N}_{\geq 2}$. Then the p-th exterior power $\bigwedge^p M$ is a free R-module with basis $(e_{i_1} \wedge \ldots \wedge e_{i_p})_{1 \leq i_1 < \ldots < i_p \leq n}$. In particular, for its rank holds*

$$\mathrm{rank}(\bigwedge^p M) = \binom{n}{p}.$$

Proof. Clearly, $\{e_{i_1} \wedge \ldots \wedge e_{i_p}\}_{1 \leq i_1, \ldots, i_p \leq n}$ is a generating system of $\bigwedge^p M$. By proposition 10.11, we may assume that the indices are ordered as $1 \leq i_1 \leq \ldots \leq i_p \leq n$. We may furthermore confine ourselves to strict inequalities, since otherwise $e_{i_1} \wedge \ldots \wedge e_{i_p} = 0$.

It remains to prove the R-linear independence of the generating family. This needs some preparation.

We denote by $\mathcal{I}$ the set of all tuples $I := (i_1, \ldots, i_p)$ with $1 \leq i_1 < \ldots < i_p \leq n$. For $I \in \mathcal{I}$ we define the projection map

$$\begin{array}{rccc} \pi_I : & M & \to & R^p \\ & m = \sum_{i=1}^n r_i e_i & \mapsto & (r_{i_1}, \ldots, r_{i_p}) \end{array}$$

which is clearly R-linear. Consider the unique p-linear determinant map $\det : (R^p)^p \to R$ from lemma 10.12 with respect to the standard basis $\{s_i\}_{i=1,\ldots,p}$ of R^p. Its composition with the p-fold direct product of π_I defines an alternating p-linear map

$$\begin{array}{rccc} \varphi_I : & M^p & \to & R \\ & (m_1, \ldots, m_p) & \mapsto & \det(\pi_I(m_1), \ldots, \pi_I(m_p)) \end{array}$$

The universal property of the alternating power gives a unique R-linear map $\hat{\varphi}_I : \bigwedge^p M \to R$ such that for all generating elements $m_1 \wedge \ldots \wedge m_p \in \bigwedge^p M$ holds $\hat{\varphi}(m_1 \wedge \ldots \wedge m_p) = \det(\pi_I(m_1), \ldots, \pi_I(m_p))$.

Consider another tuple $J \in \mathcal{I}$. For $J = I$ we compute

$$\hat{\varphi}_I(e_{j_1} \wedge \ldots \wedge e_{j_p}) = \det(\pi_I(e_{i_1}), \ldots, \pi_I(e_{i_p})) = \det(s_1, \ldots, s_p) = 1_R.$$

However, if $I \neq J$ there must exist some $\ell \in \{1, \ldots, p\}$ with $j_\ell \notin \{i_1, \ldots, i_p\}$. Hence $\pi_I(e_{j_\ell}) = 0_R$. Thus

$$\hat{\varphi}_I(e_{j_1} \wedge \ldots \wedge e_{j_p}) = \det(\pi_I(e_{j_1}), \ldots, \pi_I(e_{j_p})) = 0_R.$$

Consider now an R-linear combination $a = \sum_{(j_1,\ldots,j_p)\in\mathcal{I}} r^{j_1,\ldots,j_p} e_{j_1} \wedge \ldots \wedge e_{j_p} \in \bigwedge^p M$ with all $r^{j_1,\ldots,j_p} \in R$, and suppose $a = 0$. By the properties of the R-linear map, we compute

$$0_R = \hat{\varphi}_I(a) = r^{i_1,\ldots,i_p}$$

for all $I = (i_1, \ldots, i_p) \in \mathcal{I}$. □

10.14 Exercise. Let M be a vector space over a field K. Let $m_1, \ldots, m_p \in M$. Then $m_1 \wedge \ldots \wedge m_p \neq 0$ if and only if $m_1, \ldots, m_p$ are K-linearly independent.

10.15 Notation. Let M be a module over a commutative ring $(R, +, \cdot)$ with multiplicative unit. We define

$$\bigwedge M := \bigoplus_{p \in \mathbb{N}} \bigwedge^p M$$

as an R-module, where $\bigwedge^0 M := R$ and $\bigwedge^1 M := M$. By taking direct sums, there is a canonical R-linear map

$$\pi^a : \bigotimes M \to \bigwedge M.$$

10.16 Proposition. *There exists a unique R-algebra structure $(\bigwedge M, +, \cdot, \wedge)$, with respect to which π^a is a homomorphism of R-algebras.*

Proof. By construction, π^a is a homomorphism of R-modules. It is surjective, so for any $a, a' \in \bigwedge M$, there exist elements $t, t' \in \bigotimes M$, such that $\pi^a(t) = a$ and $\pi^a(t') = a'$. We then define

$$a \wedge a' := \pi(t \otimes t').$$

By a straightforward computation, one verifies that this gives a well-defined bilinear map, which is unique. □

10.17 Remark. In particular, proposition 10.16 implies that there is a unique well-defined bilinear map

$$\begin{array}{cccc} \wedge : & \bigwedge^p M \times \bigwedge^q M & \to & \bigwedge^{p+q} M \\ & (a_1, a_2) & \mapsto & a_1 \wedge a_2 \end{array}$$

for all $p, q \in \mathbb{N}$, such that for all $t_1 \in \bigotimes^p M$ and $t_2 \in \bigotimes^q M$ holds

$$\pi^a(t_1) \wedge \pi^a(t_2) = \pi^a(t_1 \otimes t_2).$$

10.18 Definition. Let M be a module over a commutative ring $(R, +, \cdot)$ with multiplicative unit. Then $(M, +, \cdot, \wedge)$ is called the *exterior algebra* of M.

10.19 Remark. a) The exterior algebra $(M, +, \cdot, \wedge)$ is an associative algebra with multiplicative unit $1_R \in \bigwedge M$. In general, it is not commutative.

b) As before, there is a functor

$$\begin{array}{cccc} \bigwedge : & (R\text{-Mod}) & \to & (R\text{-Alg}) \\ & M & \mapsto & \bigwedge M \\ & \varphi & \mapsto & \wedge\varphi \end{array}$$

10.20 Lemma. *Let $a_1 \in \bigwedge^p M$ and $a_2 \in \bigwedge^q M$ with $p, q \in \mathbb{N}$. Then the following formula holds:*

$$a_2 \wedge a_1 \;=\; (-1)^{pq}\, a_1 \wedge a_2.$$

Proof. By linearity, it is enough to prove the claim on decomposable elements. Let $a_1 = m_1 \wedge \ldots \wedge m_p$ and $a_2 = m_{p+1} \wedge \ldots \wedge m_{p+q}$, with $m_1, \ldots, m_{p+q} \in M$. Let $\sigma \in \Sigma_{p+q}$ be the permutation mapping the tuple $(1, \ldots, p+q)$ to $(p+1, \ldots, p+q, 1, \ldots, p)$. One easily verifies $\mathrm{sign}(\sigma) = (-1)^{pq}$. Then

$$\begin{aligned} a_2 \wedge a_1 &= m_{p+1} \wedge \ldots \wedge m_{p+q} \wedge m_1 \wedge \ldots \wedge m_p \\ &= \sigma(m_1 \wedge \ldots \wedge m_{p+q}) \\ &= \mathrm{sign}(\sigma) \cdot m_1 \wedge \ldots \wedge m_{p+q} \\ &= (-1)^{pq}\, a_1 \wedge a_2 \end{aligned}$$

by proposition 10.11. □

10.21 Example. Let $R = \mathbb{R}$ and $M := \mathbb{R}^3$, with standard basis $\{e_1, e_2, e_3\}$. Then we find

$$\bigwedge \mathbb{R}^3 \cong \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}.$$

For the element $a := e_2 + e_1 \wedge e_3$ one computes $a \wedge a = -2e_1 \wedge e_2 \wedge e_3 \neq 0$.

10.22 Example. Let M be a free R-module with a finite basis $\{e_1, \ldots, e_n\}$. Let $\varphi : M \to M$ be an R-linear map, which is given with respect to the chosen basis by a matrix

$$A_\varphi = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{1n} & \ldots & a_{nn} \end{pmatrix}.$$

Let $p \in \mathbb{N}_{\geq 2}$. For an element $e_{i_1} \wedge \ldots \wedge e_{i_p}$ with $1 \leq i_1 < \ldots < i_p \leq n$ of the induced basis of $\bigwedge^p M$ we compute

$$\begin{aligned} \wedge^p \varphi(e_{i_1} \wedge \ldots \wedge e_{i_p}) &= \varphi(e_{i_1}) \wedge \ldots \wedge \varphi(e_{i_p}) \\ &= \textstyle\sum_{j_1, \ldots, j_p = 1}^{n} a_{i_1, j_1} e_{j_1} \wedge \ldots \wedge a_{i_p, j_p} e_{j_p} \\ &= \textstyle\sum_{1 \leq j_1 < \ldots < j_n \leq n} \sum_{\sigma \in \Sigma_p} \mathrm{sign}(\sigma) a_{i_1, j_1} \ldots a_{i_p, j_p} e_{j_1} \wedge \ldots \wedge e_{j_p} \\ &= \textstyle\sum_{1 \leq j_1 < \ldots < j_n \leq n} \det(A^{i_1, \ldots, i_p}_{j_1, \ldots, j_p}). \end{aligned}$$

In particular, for $p = n = \text{rang}(M)$ we find on the generating element

$$\begin{array}{rccc} \wedge^n\varphi : & \bigwedge^n M & \to & \bigwedge^n M \\ & e_1 \wedge \ldots \wedge e_n & \mapsto & \det(A) \cdot e_1 \wedge \ldots \wedge e_n \end{array}$$

For example, let $n = 2$, and $A_\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then one computes

$$\begin{aligned} \wedge^2\varphi(e_1, e_2) &= \varphi(e_1) \wedge \varphi(e_2) = (ae_1 + ce_2) \wedge (be_1 + de_2) \\ &= abe_1 \wedge e_1 + ade_1 \wedge e_2 + cbe_2 \wedge e_1 + cde_2 \wedge e_2 \\ &= (ad - bc)e_1 \wedge e_2. \end{aligned}$$

10.23 Proposition. *Let M be a free R-module of rank $n < \infty$. Then $\bigwedge M$ is a free R-module of rank*

$$\text{rank}(\bigwedge M) = 2^n.$$

Proof. By proposition 10.13, we have $\text{rank}(\bigwedge^p M) = \binom{n}{p}$ for $0 \leq p \leq n$, and $\bigwedge^p M = \{0\}$ for $p > n$ by proposition 10.9. We thus compute

$$\text{rank}(\bigwedge M) = \sum_{i=0}^{n} \binom{n}{p} = \sum_{i=0}^{n} \binom{n}{p} 1^p 1^{n-p} = (1+1)^n = 2^n$$

using the binomial formula. □